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The periodic Ambrosetti–Prodi problem for nonlinear perturbations of the \( p \)-Laplacian

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Abstract. We prove an Ambrosetti–Prodi type result for the periodic solutions of the equation
\[
(|u'|^{p-2}u')' + f(u)u' + g(x, u) = t,
\]
when \( f \) is arbitrary and \( g(x, u) \to +\infty \) or \( g(x, u) \to -\infty \) when \( |u| \to \infty \). The proof uses upper and lower solutions and the Leray–Schauder degree.

Keywords. Ambrosetti–Prodi problem, periodic solutions, upper and lower solutions, topological degree

1. Introduction

Let \( \Omega \subset \mathbb{R}^N \) be open, bounded and smooth, and let us denote by \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) the eigenvalues of \(-\Delta\) with Dirichlet boundary conditions on \( \partial \Omega \), and by \( \phi > 0 \) the principal eigenfunction. Consider the semilinear Dirichlet problem
\[
\Delta u + f(u) = v(x) \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega,
\]
where \( v \in C_0^{\alpha, \alpha}(\Omega) \) and \( f \in C^2(\mathbb{R}) \). The following seminal result was proved by Ambrosetti–Prodi in 1972 [2].

Theorem 1. Assume that \( f \) satisfies the following conditions:
\[
f''(s) > 0 \quad \text{for all} \; s \in \mathbb{R}
\]
and
\[
0 < \lim_{s \to -\infty} \frac{f(s)}{s} < \lambda_1 < \lim_{s \to +\infty} \frac{f(s)}{s} < \lambda_2.
\]

Then there exists a closed connected manifold \( A_1 \subset C_0^{\alpha, \alpha}(\Omega) \) of codimension 1 such that \( C_0^{\alpha, \alpha}(\Omega) \setminus A_1 = A_0 \cup A_2 \) and [1] has exactly zero, one or two solutions according as \( v \) is in \( A_0, A_1 \) or \( A_2 \).

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The proof of Theorem 1 is based upon an extension of Caccioppoli’s mapping theorem to some singular case. Conditions (3) mean that the nonlinearity $f$ crosses the first eigenvalue $\lambda_1$ of $-\Delta$ when $s$ goes from $-\infty$ to $+\infty$.

It is convenient to write (1) in an equivalent way. Let

$$Lu := \Delta u + \lambda_1 u, \quad g(u) := f(u) - \lambda_1 u,$$

$$v(x) = t\phi(x) + h(x) \quad \text{with} \quad \int_{\Omega} h(x)\phi(x) \, dx = 0,$$

so that problem (1) is equivalent to

$$Lu + g(u) = t\phi(x) + h(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

(4)

condition (2) is equivalent to

$$g''(s) > 0 \quad \text{for all} \quad s \in \mathbb{R},$$

(5)

and condition (3) is equivalent to

$$-\lambda_1 < \lim_{s \to -\infty} \frac{g(s)}{s} < 0 < \lim_{s \to +\infty} \frac{g(s)}{s} < \lambda_2 - \lambda_1.$$  

(6)

A cartesian representation of $A_1$ was given by Berger–Podolak in 1975 [4].

**Theorem 2.** If conditions (5) and (6) hold, then there exists $t_1$ such that (4) has exactly zero, one or two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.

The proof of Theorem 2 is based upon a global Lyapunov–Schmidt reduction. The same year, using upper and lower solutions, Kazdan–Warner [9] weakened the assumptions (and the conclusions) of Berger–Podolak.

**Theorem 3.** If

$$-\infty \leq \limsup_{s \to -\infty} \frac{g(s)}{s} < 0 < \liminf_{s \to +\infty} \frac{g(s)}{s} \leq +\infty,$$

(7)

then there exists $t_1$ such that (4) has zero or at least one solution according as $t < t_1$ or $t > t_1$.

The multiplicity conclusion of Ambrosetti–Prodi (without exactness) was obtained independently by Dancer in 1978 [6] and Amann–Hess in 1979 [1] under the Kazdan–Warner condition (7), when $g$ satisfies a suitable growth condition at $+\infty$. We state the more general result of Dancer.

**Theorem 4.** If condition (7) holds and

$$\lim_{s \to +\infty} \frac{g(s)}{s^\sigma} = 0, \quad \sigma = \frac{N + 1}{N - 1},$$

(8)

then there exists $t_1$ such that (4) has zero, at least one or at least two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.

The proof of Theorem 4 is a combination of the method of upper and lower solutions and of degree theory.
Condition (7) implies that
\[ \lim_{|u| \to \infty} g(u) = +\infty. \] (9)

Can we replace (7) by (9) in the Ambrosetti–Prodi problem?

In 1986, a positive answer was given in [7] for a second ordinary differential equation with periodic boundary conditions. We describe the result in the special case
\[ u'' + cu' + g(u) = t + h(x), \quad u(0) - u(T) = u'(0) - u'(T) = 0, \] (10)

where \( c \in \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R}, h : [0, T] \to \mathbb{R} \) are continuous and \( \int_0^T h(x) \, dx = 0 \). Notice that 0 is the principal eigenvalue of \(-d^2/dx^2 - cd/dx\) with the \( T \)-periodic boundary conditions.

**Theorem 5.** If condition (9) holds, then there exists \( t_1 \) such that (10) has zero, at least one or at least two solutions according as \( t < t_1, t = t_1 \) or \( t > t_1 \).

The nonlinearities
\[ g(u) = |u|^{1/2}, \quad g(u) = \log(1 + |u|) \]
satisfy condition (9) but are such that
\[ \lim_{u \to -\infty} \frac{g(u)}{u} = \lim_{u \to +\infty} \frac{g(u)}{u} = 0. \]

There is no crossing of the zero eigenvalue!

A similar conclusion holds for the Neumann problem
\[ \Delta u + g(u) = t + h(x) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \] (11)

with \( g : \mathbb{R} \to \mathbb{R} \) and \( h : \Omega \to \mathbb{R} \) Hölder continuous, and \( \int_\Omega h(x) \, dx = 0 \), as shown in 1987 in [11], with the following result.

**Theorem 6.** Assume that condition (9) holds and
\[ \lim_{u \to +\infty} \frac{g(u)}{u^\sigma} = 0, \quad \sigma = \frac{N}{N - 2} \quad \text{when } N \geq 3. \] (12)

Then there exists \( t_1 \) such that (11) has zero, at least one or at least two solutions according as \( t < t_1, t = t_1 \) or \( t > t_1 \).

A natural question was to know if condition (9) could also replace condition (6) in the Dirichlet problem. In the case of dimension \( N = 1 \),
\[ u'' + u + g(u) = t(2/\pi)^{1/2} \sin x + h(x), \quad u(0) = u(\pi) = 0, \] (13)

with \( g : \mathbb{R} \to \mathbb{R} \) and \( h : [0, \pi] \to \mathbb{R} \), continuous, and \( \int_0^\pi h(x) \sin x \, dx = 0 \), the following result was proved in 1987 in [5].
Theorem 7. If condition (9) holds, then there exists \( t_1 \leq t_2 \) such that (13) has zero, at least one or at least two solutions according as \( t < t_1 \), \( t \in [t_1, t_2] \) or \( t > t_2 \). If \( u \mapsto Mu + g(u) \) is nondecreasing in a neighborhood of 0 for some \( M \), then \( t_1 = t_2 \).

The problems of knowing if \( t_1 = t_2 \) without an extra condition upon \( g \) (even if \( N = 1 \)) and of extending Theorem 7 to higher dimensions are still open. A partial answer to the second question for the Dirichlet problem can be found in a 1987 paper of Kannan–Ortega [8], for sufficiently smooth \( g \) and \( h \).

Theorem 8. If
\[
|g(u)| \leq \gamma |u|^{\sigma} + \beta, \quad \sigma < \frac{N+1}{N-1} \quad \text{when} \quad N > 2,
\]
\[
\lim_{s \to -\infty} [\lambda_1 s + g(s)] = +\infty, \quad \lim_{s \to +\infty} g(s) = +\infty,
\]
then there exists \( t_1 \) such that (3) has zero, at least one or at least two solutions according as \( t < t_1 \), \( t = t_1 \) or \( t > t_1 \).

The stability of \( T \)-periodic solutions obtained in [7] was considered by Ortega in 1989 [14, 15].

Theorem 9. Assume that \( c > 0 \), \( g \in C^1(\mathbb{R}) \) is strictly convex and satisfies condition (9), and
\[
0 < g'(\infty) \leq \left( \frac{\pi}{T} \right)^2 + \frac{c^2}{4}.
\]
Then, for each \( t > t_1 \), one solution of (10) is asymptotically stable and the other unstable.

The proof is based upon the use of Poincaré’s operator and Brouwer degree.

The delicate case of almost periodic solutions of (10) was studied by Ortega–Tarallo in 2003 [16].

Theorem 10. Assume that \( h \in C(\mathbb{R}, \mathbb{R}) \) is almost periodic, \( g \in C^1(\mathbb{R}) \) is strictly convex and satisfies
\[
-\infty \leq g'(-\infty) < 0 < g'(\infty) \leq \frac{c^2}{4}.
\]
Then there exists \( t_1 \) such that (10) has zero, at most one or exactly two almost periodic solutions according as \( t < t_1 \), \( t = t_1 \) or \( t > t_1 \).

The proof uses separation conditions, Opial’s method of ordered upper and lower solutions and a special case of a result on nonordered upper and lower solutions given in [13].

Let \( p > 1 \),
\[
\phi : \mathbb{R} \to \mathbb{R}, \quad s \mapsto |s|^{p-1}s,
\]
\( f : \mathbb{R} \to \mathbb{R} \) be continuous, \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be \( T \)-periodic in \( x \) for some \( T > 0 \) and continuous, and let \( t \in \mathbb{R} \). In this paper, we are interested in studying the ‘\( p \)-Laplacified’
Ambrosetti–Prodi problem for the $T$-periodic solutions of the equation
\[
(\phi(u'))' + f(u')u' + g(x, u) = t,
\]
in terms of the value of the forcing term $t$. A $T$-periodic solution of (18) is a periodic function $u \in C^1(\mathbb{R})$ of period $T$ such that $\phi \circ u' \in C^1(\mathbb{R})$ and which satisfies (18). Using an approach similar to that of [7], but with substantial technical differences due to the presence of the $p$-Laplacian, we prove here the following result.

**Theorem 11.** If
\[
\lim_{|s| \to \infty} g(x, s) = +\infty \quad \text{uniformly in } x \in \mathbb{R},
\]
then there exists $t_1$ such that (18) has zero, at least one or at least two $T$-periodic solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.

This theorem is a consequence of Lemmas 4, 6 and 7. Let us mention that, very recently, Arcoya and Ruiz [3] have extended the conditions of Amann–Hess for the Ambrosetti–Prodi problem to perturbations of the $p$-Laplacian in $\Omega \subset \mathbb{R}^N$ with Dirichlet conditions, when $p \geq 2$. It is interesting to notice that, in the case where $1 < p < 2$, their conclusion is similar to the one in [5].

We use the following notations. For $k \geq 0$ integer, let
\[
C^k_T = \{ u : \mathbb{R} \to \mathbb{R} : u \text{ is of class } C^k \text{ and } T \text{-periodic} \}.
\]
If $v \in C^0_T$, and $p \geq 1$, we set
\[
\overline{v} := \frac{1}{T} \int_0^T v(x) \, dx, \quad \underline{v} = v - \overline{v},
\]
\[
\|v\|_\infty = \max_{\mathbb{R}} |v|, \quad \|v\|_p = \left( \frac{1}{T} \int_0^T |v(x)|^p \, dx \right)^{1/p}.
\]
If $\Omega \subset X$ is an open bounded set of a normed space $X$ and if $S : \Omega \subset X \to X$ is compact and such that $0 \not\in (I - S)(\partial \Omega)$, the **Leray–Schauder degree** of $I - S$ with respect to $\Omega$ and 0 is denoted by $d_{LS}(I - S, \Omega, 0)$.

2. Periodic upper and lower solutions and degree

We need the following results on the method of upper and lower solutions.

**Definition 1.** A $T$-periodic lower solution $\alpha$ (resp. $T$-periodic upper solution $\beta$) of (18) is a $C^1$ $T$-periodic function such that $\phi \circ \alpha' \in C^1(\mathbb{R})$ (resp. $\phi \circ \beta' \in C^1(\mathbb{R})$) and
\[
(\phi(\alpha'(x)))' + f(\alpha(x))\alpha'(x) + g(x, \alpha(x)) \geq t \quad \text{(20)}
\]
(resp.
\[
(\phi(\beta'(x)))' + f(\beta(x))\beta'(x) + g(x, \beta(x)) \geq t \quad \text{(21)}
\]
for all $x \in \mathbb{R}$. A lower (resp. upper) solution is strict if the strict inequality holds in (20) (resp. (21)).
If the \( T \)-periodic lower solution \( \alpha \) and the \( T \)-periodic upper solution \( \beta \) of (18) are such that \( \alpha(x) \leq \beta(x) \) for all \( x \in \mathbb{R} \), let us define the bounded continuous map \( r : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
r(x, u) = \begin{cases} 
\alpha(x) & \text{if } u < \alpha(x), \\
u & \text{if } \alpha(x) \leq u \leq \beta(x), \\
\beta(x) & \text{if } u > \beta(x),
\end{cases}
\]

and consider the modified equation

\[
(\phi(u'))' - [\phi(u) - \phi[r(x, u)]] + f[r(x, u)]u' + g[x, r(x, u)] = t.
\]

(22)

The following result is classical. We give its simple proof for completeness.

**Lemma 1.** Each possible \( T \)-periodic solution \( u \) of (22) is such that

\[
\alpha(x) \leq u(x) \leq \beta(x) \quad (x \in \mathbb{R}).
\]

**Proof.** We prove the first inequality, the other case being similar. If the conclusion does not hold, \( u - \alpha \) reaches a negative minimum, say at \( \xi \), so that

\[
u(\xi) < \alpha(\xi), \quad u'(\xi) = \alpha'(\xi).
\]

Hence, \( r(\xi, u(\xi)) = \alpha(\xi) \), and, by (1),

\[
(\phi(u'(\xi)))' - [\phi(u(\xi)) - \phi(\alpha(\xi))] + f(\alpha(\xi))\alpha'(\xi) + g(\xi, \alpha(\xi))
\]

\[
= t < (\phi(\alpha'(\xi)))' + f(\alpha(\xi))\alpha'(\xi) + g(\xi, \alpha(\xi)),
\]

so that

\[
(\phi(u'(\xi)))' - (\phi(\alpha'(\xi)))' < \phi(u(\xi)) - \phi(\alpha(\xi)) < 0.
\]

By continuity, there exists \( \varepsilon > 0 \) such that

\[
(\phi(u'(x)))' - (\phi(\alpha'(x)))' < 0 \quad \text{whenever} \quad x \in [\xi - \varepsilon, \xi + \varepsilon],
\]

and \( \phi \circ u' - \phi \circ \alpha \) is decreasing on \( [\xi - \varepsilon, \xi + \varepsilon] \), and vanishes at \( \xi \). This easily implies that \( u - \alpha \)' \( < 0 \) on \( [\xi, \xi + \varepsilon] \) and \( (u - \alpha)' \) \( > 0 \) on \( [\xi - \varepsilon, \xi] \), a contradiction with \( u - \alpha \) reaching a minimum. \( \square \)

**Remark 1.** If \( \alpha \) and \( \beta \) are respectively \( T \)-periodic lower and upper solutions of (18) such that \( \alpha(x) < \beta(x) \) for all \( x \in \mathbb{R} \), a similar proof shows that each possible \( T \)-periodic solution of (22) is such that

\[
\alpha(x) < \beta(x) \quad (x \in \mathbb{R}).
\]

(24)

The following result will be useful in proving the existence of a \( T \)-periodic solution of the modified equation.
Lemma 2. Given $t^* \in \mathbb{R}$, there exist $R, R' > 0$ such that, for each $\lambda \in [0, 1]$, each $t$ with $|t| \leq t^*$ and each possible $T$-periodic solution of

$$
(\phi(u'))' - \phi(u) - \lambda \phi[r(x, u)] + \lambda f[r(x, u)]u' + \lambda g[x, r(x, u)] = \lambda t
$$

one has

$$
\|u\|_\infty < R, \quad \|u'\|_\infty < R'.
$$

Proof. Let $\lambda \in [0, 1]$ and $u$ be a possible $T$-periodic solution of (25). If we multiply both members of (25) by $u$, integrate over $[0, T]$ and use integration by parts and the $T$-periodicity, we get

$$
-\|u'\|_p^p - \|u\|_p^p - \frac{\lambda}{T} \int_0^T u(x)\phi(r(x, u(x))) \, dx
+ \frac{\lambda}{T} \int_0^T u(x)f(r(x, u(x))u'(x) \, dx + \frac{\lambda}{T} \int_0^T u(x)g(x, r(x, u(x))) \, dx
= \frac{\lambda}{T} \int_0^T u(x)t \, dx.
$$

Hence, for some constants $M, M'$ we have, using the H"older inequality,

$$
\|u'\|_p^p + \|u\|_p^p \leq M\|u\|_p\|u'\|_p + M'\|u\|_p + |t^*| \|u\|_p.
$$

This easily implies the existence of $S = S(t^*)$ and $S' = S'(t^*)$ such that

$$
\|u\|_p + \|u'\|_p < S, \quad \|u\|_\infty < S'.
$$

Now, there exists $\xi$ such that $u'(\xi) = 0$, so that, integrating (25) between $\xi$ and $x$ we obtain

$$
\phi(u'(x)) + \int_\xi^x [\phi(u(s)) - \phi(r(s, u(s))) + \lambda f(r(s, u(s)))u'(s) + \lambda g(s, r(s, u(s))]] \, ds
= \int_\xi^x t \, ds.
$$

Hence, using (27) we get $|u'(x)|^{p-1} < S'$ for all $x \in [0, T]$ and some $S'' = S''(t^*)$. \qed

Lemma 3. For each $h \in C_T$ there exists a unique $T$-periodic solution $u$ of

$$
(\phi(u'))' - \phi(u) = h(x).
$$

Furthermore, the mapping

$$
\mathcal{H} : C_T \rightarrow C_T^1, \quad h \mapsto u,
$$

is completely continuous.
Proof. The existence of at least one \( T \)-periodic solution for \( (28) \) follows from Corollary 4.1 in \([10]\) and the fact that \(-1\) is not an eigenvalue of the \( p \)-Laplacian with periodic boundary conditions, or from Remark 2.1 in \([12]\). For the uniqueness, if \( u \) and \( v \) are \( T \)-periodic and such that
\[
(\phi(u') - \phi(v'))' - [\phi(u) - \phi(v)] = 0.
\]
(30)

Now, it is easily checked that, for all \( r, s \in \mathbb{R} \), one has
\[
[\phi(r) - \phi(s)](r - s) \geq (|r|^{p-1} - |s|^{p-1}) (|r| - |s|)
\]
and hence, integrating (30), we obtain
\[
0 \geq \int_0^T (|u'|^{p-1} - |v'|^{p-1}) (|u'| - |v'|) + \int_0^T (|u|^{p-1} - |v|^{p-1}) (|u| - |v|) \geq 0.
\]
Consequently, for all \( x \in \mathbb{R} \),
\[
|u'(x)| = |v'(x)|, \quad |u(x)| = |v(x)|.
\]
(31)

Hence (30) can be written as
\[
(|u'|^{p-2}(u' - v'))' - |u|^{p-2}(u - v) = 0,
\]
which gives, by integration after multiplication by \( u - v \),
\[
\int_0^T [(|u'|^{p-2}(u' - v'))^2 + |u|^{p-2}(u - v)^2] = 0,
\]
and hence, together with (31), implies that \( u = v \). Now it follows from an argument analogous to the one used in the proof of Lemma \([2]\) that
\[
-\|u''\|_p^p - \|u\|_p^p = \frac{1}{T} \int_0^T u(x) h(x) \, dx,
\]
so that by the Hölder inequality,
\[
\|u''\|_p^p + \|u\|_p^p \leq \|h\|_\infty \|u''\|_p^p + \|u\|_p^p \frac{1}{1/p},
\]
which gives
\[
\|u\|_p \leq \|h\|_\infty^{1/p-1}, \quad \|u''\|_p \leq \|h\|_\infty^{1/p-1},
\]
(32)
and hence, for some constant \( C \) depending only upon \( T \),
\[
\|u\|_\infty \leq C \|h\|_\infty^{1/p-1}.
\]
(33)

Now, there exists \( \xi \) such that \( u'(\xi) = 0 \), so that integrating (28) between \( \xi \) and \( x \) we obtain
\[
\phi(u'(x)) + \int_\xi^x \phi(u(s)) \, ds = \int_\xi^x h(s) \, ds,
\]
and hence, for all $x \in [0, T]$, 
\[
\|u'(x)\|^{p-1} \leq T(C^{p-1} + 1)\|h\|_{\infty},
\]
which gives, for some constant $C'$ only depending upon $T$, 
\[
\|u'\|_{\infty} \leq C'\|h\|^{1/p-1}. \tag{34}
\]

Let $(h_n)$ be a sequence in $C_T$ such that 
\[
\|h_n\|_{\infty} \leq R \tag{35}
\]
for all $n \geq 1$ and some $R > 0$. Let $u_n := \mathcal{H}(h_n)$. From relations $\tag{33}, \tag{34}$ and Ascoli–Arzelà’s theorem, we can assume, passing to a subsequence if necessary, that $u_n \to u \in C_T$ uniformly on $\mathbb{R}$. Now, if $\xi_n \in [0, T]$ is such that $u_n'(\xi_n) = 0$, we have, for all $x \in [0, T]$, 
\[
\phi(u_n'(x)) = -\int_{\xi_n}^{x} \phi(u_n(s)) \, ds + \int_{\xi_n}^{x} h_n(s) \, ds, \tag{36}
\]
and, from relations $\tag{33}, \tag{35}$ and Ascoli–Arzelà’s theorem, we can assume, passing to a subsequence if necessary, that the right-hand member of $\tag{36}$ converges to some $z \in C_T$ uniformly on $[0, T]$. Consequently, $(u_n')$ converges uniformly on $[0, T]$ to $\phi^{-1}(z)$, and so $\mathcal{H}$ is completely continuous. \hfill \qed

Define 
\[
\mathcal{G}_t : C_T^1 \to C_T, \quad u \mapsto -\phi(u) - f(u)u' - g(\cdot, u) + t, \tag{37}
\]
and, for $\alpha, \beta \in C_T$ such that $\alpha(x) < \beta(x)$ for all $x \in \mathbb{R}$, and $R' > 0$, define the open bounded set $\Omega \subset C_T^1$ by 
\[
\Omega := \{ u \in C_T^1 : \alpha(x) < u(x) < \beta(x), \quad -R' < u'(x) < R' \ (x \in \mathbb{R}) \}. \tag{38}
\]

**Proposition 1.** If $\tag{18}$ has a T-periodic lower and upper solutions $\alpha, \beta$ such that $\alpha(x) \leq \beta(x)$ for all $x \in \mathbb{R}$, then it has a $T$-periodic solution $u$ such that $\alpha(x) \leq u(x) \leq \beta(x)$ for all $x \in \mathbb{R}$. Furthermore, if $\alpha$ and $\beta$ are strict and if $\alpha(x) < \beta(x)$ for all $x \in \mathbb{R}$, then 
\[
d_{LS}[I - \mathcal{H}\mathcal{G}_1, \Omega, 0] = 1. \tag{39}
\]

**Proof.** By Lemma $[\cdot]$ the existence conclusion follows from the existence of a $T$-periodic solution to $\tag{22}$. Let 
\[
\hat{\Omega} := \{ u \in C_T^1 : \|u\|_{\infty} < R, \|u'\|_{\infty} < R' \}
\]
where $R$ and $R'$ are given by Lemma $[\cdot]$ and let 
\[
\hat{\mathcal{G}} : C_T^1 \times [0, 1] \to C_T, \quad (u, \lambda) \mapsto -\lambda \phi(r(\cdot, u)) - \lambda f(r(\cdot, u))u' - \lambda g(\cdot, u) + \lambda t.
\]
It is clear from Lemma $[\cdot]$ that the $T$-periodic solutions of $\tag{22}$ are the fixed points of $\hat{\mathcal{H}}\hat{\mathcal{G}}(\cdot, 1)$ in $C_T^1$. The homotopy invariance of the Leray–Schauder degree gives 
\[
d_{LS}[I - \hat{\mathcal{H}}\hat{\mathcal{G}}(\cdot, 1), \hat{\Omega}, 0] = d_{LS}[I - \hat{\mathcal{H}}\hat{\mathcal{G}}(\cdot, 0), \hat{\Omega}, 0] = d_{LS}[I, \hat{\Omega}, 0] = 1,
\]
and the excision property of the Leray–Schauder degree gives 
\[
d_{LS}[I - \hat{\mathcal{H}}\hat{\mathcal{G}}(\cdot, 1), \hat{\Omega}, 0] = d_{LS}[I - \hat{\mathcal{H}}\hat{\mathcal{G}}(\cdot, 1), \Omega, 0] = d_{LS}[I - \hat{\mathcal{H}}\hat{\mathcal{G}}, \Omega, 0]. \hfill \qed
3. Existence of the first solution

Assume now that

\[
g(x, u) \to +\infty \quad \text{as } |u| \to \infty, \text{ uniformly in } x \in \mathbb{R}.
\]

(40)

Let

\[
\sigma := \min_{u \in \mathbb{R}, x \in \mathbb{R}} g(x, u).
\]

(41)

Lemma 4. If condition (40) holds, then there exists \( t_1 \geq \sigma \) such that (18) has no \( T \)-periodic solution if \( t < t_1 \) and at least one \( T \)-periodic solution if \( t > t_1 \).

Proof. We first notice that, for \( t \geq t^* := \max_{x \in \mathbb{R}} g(x, 0) \), \( 0 \) is an upper solution for (18) (a strict upper solution if \( t > t^* \)). Given \( t \geq t^* \), it follows from condition (40) that there exists \( R_t > 0 \) such that

\[
g(x, u) > t \quad \text{whenever } |u| \geq R_t, x \in \mathbb{R},
\]

so that \( -R_t \) (or any smaller number) is a strict lower solution for (18). Hence, from Proposition 1, for each \( t \geq t^* \), this equation has at least one \( T \)-periodic solution such that \( -R_t < u(x) < 0 \) for all \( x \in \mathbb{R} \). Let us now show that if (18) has a \( T \)-periodic solution \( \tilde{u} \) for some \( \tilde{t} < t^* \), then it has a \( T \)-periodic solution for all \( t \in [\tilde{t}, t^*] \). Indeed, for such a \( t \), we have

\[
(\phi(\tilde{u}'(x)))' + f(\tilde{u}(x))\tilde{u}'(x) + g(x, \tilde{u}(x)) = \tilde{t} \leq t,
\]

which shows that \( \tilde{u} \) is an upper solution for (18). Furthermore, by the reasoning above, there exists \( R_t > -\min_{x \in \mathbb{R}} \tilde{u} \) such that \( \min_{x \in \mathbb{R}} g(x, -R_t) > t \) so that \( -R_t < \min_{x \in \mathbb{R}} \tilde{u} \) is a lower solution for (18). Again, this implies the existence of a \( T \)-periodic solution for (18). Consequently, the set of \( t \in \mathbb{R} \) such that (18) has a \( T \)-periodic solution is an interval unbounded from above. Let

\[
t_1 = \inf \{t \in \mathbb{R} : (18) \text{ has a } T \text{-periodic solution}\}.
\]

(42)

We now show that (18) has no \( T \)-periodic solution for \( t < \sigma \). Indeed, if \( u \) were a \( T \)-periodic solution of (18) for some \( t < \sigma \), and if \( u(\xi) = \min_{x \in \mathbb{R}} u \), then \( u'(\xi) = 0 \), and

\[
(\phi(u'(\xi)))' = t - g(\xi, u(\xi)) < \sigma - g(\xi, u(\xi)) \leq 0.
\]

By continuity, there exists \( \varepsilon > 0 \) such that

\[
(\phi(u'(x)))' < 0 \quad \text{for } x \in [\xi - \varepsilon, \xi + \varepsilon],
\]

so that \( \phi \circ u' \) is decreasing on \( [\xi - \varepsilon, \xi + \varepsilon] \), and the same is true for \( u' \). This contradicts the fact that \( u \) reaches its minimum at \( \xi \). Consequently, \( t_1 \geq \sigma \). \( \Box \)
4. A priori estimates

We now prove an a priori estimate for the possible $T$-periodic solutions of (18) when $t$ is bounded from above.

**Lemma 5.** For each $t_2 > t_1$, there exist $M(t_2) > 0$ and $N(t_2) > 0$ such that, for each $t \in [t_1, t_2]$ and each possible $T$-periodic solution $u$ of (18), one has

\[
\|u\|_{\infty} < M(t_2), \\
\|u'|_{\infty} < N(t_2).
\]

**Proof.** Let $t \in [t_1, t_2]$ and let $u$ be a $T$-periodic solution of (18). Integrating both members of the equation over $[0, T]$ gives

\[
\frac{1}{T} \int_0^T g(x, u(x)) \, dx = t.
\]

We deduce from (18) that

\[
\tilde{u}(\phi(u'))' + \tilde{u}f(u)' + \tilde{u}g(x, u) = \tilde{u}t,
\]

which, integrated over $[0, T]$, gives, by the $T$-periodicity of $u$,

\[
-\|u''\|_p^p + \frac{1}{T} \int_0^T g(x, u(x)) \tilde{u}(x) \, dx = 0,
\]

and hence, using (45), with $\sigma$ defined in (41),

\[
\|u''\|_p^p = \frac{1}{T} \int_0^T [g(x, u(x)) - \sigma \tilde{u}(x)] \, dx \\
\leq \frac{1}{T} \int_0^T [g(x, u(x)) - \sigma \tilde{u}(x)] \, dx \\
\leq \|\tilde{u}\|_{\infty}(t - \sigma) \leq \|\tilde{u}\|_{\infty}(t_2 - \sigma).
\]

Now, if $\xi$ is such that $\tilde{u}(\xi) = 0$, we have for each $x \in \mathbb{R}$, using the Hölder inequality,

\[
|\tilde{u}(x)| = \left| \int_{\xi}^x u'(s) \, ds \right| \leq T \|u'\|_p,
\]

so that (46) implies that

\[
\|u''\|_p \leq [T(t_2 - \sigma)]^{1/(p-1)}.
\]

Now, there exists $R_2 > 0$ such that $g(x, u) > t_2$ whenever $|u| \geq R_2$ and $x \in \mathbb{R}$. Consequently, if $|u(x)| \geq R_2$ for all $x \in \mathbb{R}$, we have, by (45),

\[
t = \frac{1}{T} \int_0^T g(x, u(x)) \, dx > t_2.
\]
which is impossible. Hence \(|u(\xi)| < R_2\) for some \(\xi \in \mathbb{R}\), which implies
\[
|u(x)| \leq |u(\xi)| + \left| \int_{\xi}^{x} u'(s) \, ds \right| < R_2 + T \|u'\|_p
\leq R_2 + [T(t_2 - \sigma)]^{1/(p-1)} =: M(t_2).
\] (48)

Now, there exists \(\xi \in \mathbb{R}\) such that \(u'(\xi) = 0\). If we set
\[
F(u) = \int_{0}^{u} f(s) \, ds,
\] (49)
we can write (18) in the form
\[
\phi(u'(x)) + F(u(x)) = F(u(\xi)) + \int_{\xi}^{x} [t - g(s, u(s))] \, ds,
\]
which gives, by (48), for each \(x \in \mathbb{R}\),
\[
|\phi(u'(x))| \leq 2 \max_{|u| \leq M(t_2)} |F(u)| + \int_{0}^{T} |g(s, u(s)) - t| \, ds
\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + \int_{0}^{T} [|g(s, u(s)) - \sigma| + |\sigma - t|] \, ds
\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + T[|t - \sigma| + |\sigma - t|]
\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + T(t_2 - \sigma) := S(t_2),
\]
and this immediately yields (44) for any \(N(t_2) > [S(t_2)]^{1/(p-1)}\).

This result allows us to prove the existence of at least one solution for \(t = t_1\).

Lemma 6. If condition (40) holds, then (18) has at least one \(T\)-periodic solution for \(t = t_1\).

Proof. Let \((\tau_k)\) be a sequence in \([t_1, +\infty[\) which converges to \(t_1\), and let \(u_k\) be a \(T\)-periodic solution of (18) with \(t = \tau_k\) given by Lemma 4. From Lemma 5, we know that, for all \(k \geq 1\),
\[
\|u_k\|_{\infty} < M(t_2), \quad \|u_k'\|_{\infty} < N(t_2),
\] (50)
and from Lemma 3 that, for all \(k \geq 1\),
\[
u_k = \mathcal{H}G_{t_1}(u_k).
\] (51)

Conditions (50) and the complete continuity of \(\mathcal{H}\) imply that, up to a subsequence, the right-hand member of (51) converges in \(C_T^1\), and then \((u_k)\) converges to some \(u \in C_T^1\) such that \(u = \mathcal{H}G_{t_1}(u)\), i.e. to a \(T\)-periodic solution of (18). \(\Box\)
5. Existence of two solutions

Define
\[ B(R, R') := \{ u \in C^1_T : \| u \|_\infty < R, \| u' \|_\infty < R' \} \]

Lemma 7. If condition (40) holds, then, for each \( t > t_1 \), (18) has at least two \( T \)-periodic solutions.

Proof. Let \( t_2 > t_1 \) and let \( t \in [t_1, t_2] \). As (18) has no \( T \)-periodic solution for \( t < t_1 \), we have, for all \( t \leq t_2 \), using Lemma 5,
\[ d_{LS}[I - \mathcal{H}G_t, B(M(t_2), N(t_2)), 0] = 0 \] (52)

By the reasoning in the proof of Lemma 4, there exists \( R_{t_2} > 0 \) such that
\[ \min_{x \in \mathbb{R}} g(x, -R_{t_2}) > t_2, \]
and hence
\[ \min_{x \in \mathbb{R}} g(x, -R_{t_2}) > t \quad \text{for all} \quad t \leq t_2. \]

Thus, \( -R_{t_2} \) is a strict \( T \)-periodic lower solution for (18) whenever \( t \leq t_2 \). On the other hand, a \( T \)-periodic solution \( u_1 \) of (18) with \( t = t_1 \) is such that
\[ (\phi(u_1'(x)))' + f(u_1(x))u_1'(x) + g(x, u_1(x)) = t_1 < t, \]
and is a strict \( T \)-periodic upper solution of (18). We can of course always increase \( M(t_2) \) in such a way that
\[ -M(t_2) < -R_{t_2} < u_1(x) < M(t_2) \]
for all \( x \in \mathbb{R} \). Hence, if \( \Omega_1 \) is the open bounded subset of \( B(M(t_2), N(t_2)) \) defined by
\[ \{ u \in C^1_T : -R_{t_2} < u(x) < u_1(x), -N(t_2) < u'(x) < N(t_2) \ (x \in \mathbb{R}) \}, \]
it follows from Proposition 4 that (18) has at least one \( T \)-periodic solution in \( \Omega_1 \), and that
\[ d_{LS}[I - \mathcal{H}G_t, \Omega_1, 0] = 1. \] The excision property of the Leray–Schauder degree and (52) give, for \( t \in [t_1, t_2] \),
\[ d_{LS}[I - \mathcal{H}G_t, B(M(t_2), N(t_2)) \setminus \overline{\Omega}_1, 0] = d_{LS}[I - \mathcal{H}G_t, B(M(t_2), N(t_2)), 0] - d_{LS}[I - \mathcal{H}G_t, \Omega_1, 0] = -1, \]
which implies the existence of a \( T \)-periodic solution of equation (18) contained in \( B(M(t_2), N(t_2)) \setminus \overline{\Omega}_1 \). As \( t_2 > t_1 \) is arbitrary, the proof is complete. \( \square \)
References


