Algebraic Geometry — Compactified Jacobians of Néron type, by Lucia Caporaso, communicated on 12 November 2010.

Abstract. — We characterize stable curves $X$ whose compactified degree-$d$ Jacobian is of Néron type. This means the following: for any one-parameter regular smoothing of $X$, the special fiber of the Néron model of the Jacobian is isomorphic to a dense open subset of the degree-$d$ compactified Jacobian of $X$. It is well known that compactified Jacobians of Néron type have the best modular properties, and that they are endowed with a mapping property useful for applications.

Key words: Stable curve, Picard scheme, Néron model, compactification, balanced line bundle.

Mathematic Subject Classification (2000): 14K30, 14H10.

1. Introduction and preliminaries

Let $X$ be a stable curve and $f : \mathcal{X} \to B$ a one-parameter smoothing of $X$ with $\mathcal{X}$ a nonsingular surface; $X$ is thus identified with the special fiber of $f$ and all other fibers are smooth curves. Let $N^d_f \to B$ be the Néron model of the degree-$d$ Jacobian of the generic fiber of $f$; its existence was proved by A. Néron in [N], and its connection with the Picard functor was established by M. Raynaud in [R]. So, $N^d_f \to B$ is a smooth and separated morphism, whose generic fiber is the degree-$d$ Jacobian of the generic fiber of $f$; the special fiber, denoted $N^d_X$, is isomorphic to a disjoint union of copies of the generalized Jacobian of $X$. $N^d_f \to B$ can be interpreted as the maximal separated quotient of the degree-$d$ Picard scheme $\text{Pic}^d_f \to B$. In particular, if $\text{Pic}^d_f \to B$ is separated, which happens if and only if $X$ is irreducible, then $N^d_f = \text{Pic}^d_f$ (we refer to [R], [BLR] or [Ar] for details).

The Néron model has a universal property, the Néron Mapping Property ([BLR, def. 1]), which determines it uniquely. Moreover, as $d$ varies in $\mathbb{Z}$, the special fibers, $N^d_X$, of $N^d_f \to B$ are all isomorphic.

By contrast, the compactified degree-$d$ Jacobian of a reducible curve $X$, denoted $P^d_X$, has a structure which varies with $d$. For example, the number of irreducible components, and the modular properties, depend on $d$; see Section 2 for details and references.

$P^d_X$ will be called of Néron type if its smooth locus is isomorphic to $N^d_X$. Compactified Jacobians of Néron type have the best modular properties. Moreover they inherit a mapping property from the universal property of the Néron model which provides a very useful tool; see for example [CE] for applications to Abel maps.
The purpose of this paper is to classify, for every $d$, those stable curves $X \in \overline{M}_g$ such that $P^d_X$ is on Néron type. The question is interesting if $g \geq 2$, for otherwise $P^d_X$ is always irreducible, and hence of Néron type.

Before stating our main result, we need a few words about compactified Jacobians. $P^d_X$ parametrizes certain line bundles on quasistable curves having $X$ as stabilization. These are the so-called “balanced” line bundles; among balanced line bundles there are some distinguished ones, called “strictly balanced”, which have better modular properties. In fact, to every balanced line bundle there corresponds a unique point in $P^d_X$, but different balanced line bundles may determine the same point. On the other hand every point of $P^d_X$ corresponds to a unique class of strictly balanced line bundles.

The curve $X$ is called $d$-general if every balanced line bundle of degree $d$ is strictly balanced. This is equivalent to the fact that $P^d_X$ is a geometric GIT-quotient.

The property of being $d$-general depends only on the weighted dual graph of $X$, and the locus of $d$-general curves in $\overline{M}_g$ has been precisely described by M. Melo in [M].

Now, the degree-$d$ compactified Jacobian of a $d$-general curve is of Néron type, by [C2, Thm. 6.1]. But, as we will prove, the converse does not hold.

More precisely, a stable curve $X$ is called weakly $d$-general if a curve obtained by smoothing every separating node of $X$, and maintaining all the non separating nodes, is $d$-general; see Definition 1.13.

Our main result, Theorem 2.9, states that $P^d_X$ is of Néron type if and only if $X$ is weakly $d$-general. The locus of weakly $d$-general curves in $\overline{M}_g$ is precisely described in section 2.11; its complement turns out to have codimension at least 2.

I am grateful to M. Melo and F. Viviani for their precious comments.

1.1. Notations and conventions

(1) We work over an algebraically closed field $k$. The word “curve” means projective scheme of pure dimension one. The genus of a curve will be the arithmetic genus, unless otherwise specified.

(2) By $X$ we will always denote a nodal curve of genus $g \geq 2$. For any subcurve $Z \subset X$ we denote by $g_Z$ its arithmetic genus, by $Z^c := X \setminus \overline{Z}$ and by $\delta_Z := \#(Z \cap \overline{Z})$. We set $w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z$.

(3) A node $n$ of a connected curve $X$ is called separating if $X \setminus \{n\}$ is not connected. The set of all separating nodes of $X$ is denoted by $X_{\text{sep}}$ and the set of all nodes of $X$ by $X_{\text{sing}}$.

(4) A nodal curve $X$ of genus $g \geq 2$ is called stable if it is connected and if every component $E \subset X$ such that $E \cong \mathbb{P}^1$ satisfies $\delta_E \geq 3$. $X$ is called quasistable if it is connected, if every $E \subset X$ such that $E \cong \mathbb{P}^1$ satisfies $\delta_E \geq 2$, and if two exceptional components never intersect, where an exceptional component is defined as an $E \cong \mathbb{P}^1$ such that $\delta_E = 2$. We denote by $X_{\text{exc}}$ the union of the exceptional components of $X$. 
(5) Let $S \subset \mathcal{S}_{\text{sing}}$ we denote by $v_S : X^\vee_S \to X$ the normalization of $X$ at $S$, and by $\mathcal{X}_S$ the quasistable curve obtained by “blowing-up” all the nodes in $S$, so that there is a natural surjective map

$$\mathcal{X}_S = \bigcup_{i=1}^{#S} E_i \cup X^\vee_S \to X$$

restricting to $v_S$ on $X^\vee_S$ and contracting all the exceptional components $E_i$ of $\mathcal{X}_S$. $\mathcal{X}_S$ is also called a quasistable curve of $X$.

(6) Let $C_1, \ldots, C_\gamma$ be the irreducible components of $X$. Every line bundle $L \in \text{Pic} X$ has a multidegree $\deg L = (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L) \in \mathbb{Z}^\gamma$. Let $d = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$, we set $|d| = \sum |d_i|$; for any subcurve $Z \subset X$ we abuse notation slightly and denote

$$d_Z := \sum_{C_i \subset Z} d_i.$$

1.2. Compactified Jacobs of Néron type. Let $X$ be any nodal connected curve and $f : \mathcal{X} \to B$ a one-parameter regular smoothing for $X$, i.e. $B$ is a smooth connected one-dimensional scheme with a marked point $b_0 \in B$, $\mathcal{X}$ is a regular surface, and $f$ is a projective morphism whose fiber over $b_0$ is $X$ and whose remaining fibers are smooth curves. We set $U := B\{b_0\}$ and let $f_U : \mathcal{X}_U \to U$ be the family of smooth curves obtained by restricting $f$ to $U$. Consider the relative degree $d$ Picard scheme over $U$, denoted $\text{Pic}^d_{f_U} \to U$. Its Néron model over $B$ will be denoted by

$$(1.1) \quad N^d_f := N(\text{Pic}^d_{f_U}) \to B,$$

and its fiber over $b_0$ will be denoted by $N^d_X$; $N^d_X$ is isomorphic to a finite number of copies of the generalized Jacobian of $X$. The number of copies is independent of $d$; to compute it we introduce the so-called “degree class group”.

Let $\gamma$ be the number of irreducible components of $X$. For every component $C_i$ of $X$ set $k_{i,j} := \#(C_i \cap C_j)$ if $j \neq i$, and $k_{i,i} = -\#(C_i \cap C \setminus C_i)$ so that the matrix $(k_{i,j})$ is symmetric matrix. Notice that for every regular smoothing $f : \mathcal{X} \to B$ of $X$ as above, we have $\deg_{C_j} \circ f(C_i) = k_{i,j}$. Hence this matrix is also related to $f$, although it does not depend on the choice of $f$ (as long as $\mathcal{X}$ is regular).

We have $\sum_{j=1}^\gamma k_{i,j} = 0$ for every $i$. Now, for every $i = 1, \ldots, \gamma$ set $c_i := (k_{1,i}, \ldots, k_{\gamma,i}) \in \mathbb{Z}^\gamma$ and $Z := \{d \in \mathbb{Z}^\gamma : |d| = 0\}$ so that $c_i \in Z$. We can now define the sublattice $\Lambda_X := \langle c_1, \ldots, c_\gamma \rangle \subset Z$.

The degree class group of $X$ is the group $\Delta_X := Z/\Lambda_X$. It is not hard to prove that $\Delta_X$ is a finite group.

Let $d$ and $d'$ be in $\mathbb{Z}^\gamma$; we say that they are equivalent if $d - d' \in \Lambda_X$. We denote by $\Delta^d_X$ the set of equivalence classes of multidegrees of total degree $d$; for a multidegree $d$ we write $[d]$ for its class. It is clear that $\Delta_X = \Delta_X^0$ and that

$$\#\Delta_X = \#\Delta^d_X.$$
Now back to $N^d_X$, the special fiber of (1.1); as we said it is a smooth, possibly non connected scheme of pure dimension $g$.

**Fact 1.3.** Under the above assumptions, the number of irreducible (i.e. connected) components of $N^d_X$ is equal to $\#\Delta_X$.

This is well known; see [R, 8.1.2] (where $\Delta_X$ is the same as ker $\beta$/Im $\alpha$) or [BLR, thm. 9.6.1]. Using the standard notation of Néron models theory we have $\Delta_X = \Phi_{N^d_X}$, i.e. $\Delta_X$ is the “component group” of $N^d_f$.

For every stable curve $X$ and every $d$ we denote by $P^d_X$ the degree $d$ compactified Jacobian (or degree-$d$ compactified Picard scheme). $P^d_X$ has been constructed in [OS] for a fixed curve, and independently for families in [S] and in [C1] (the constructions of [OS] and [S] are here considered with respect to the canonical polarization); these three constructions give the same scheme by [Al], see also [LM]. We mention that another compactified Jacobian is constructed in [E], whose connection with the others is under investigation; see [MV]. An explicit description of $P^d_X$ will be recalled in 2.2. We here anticipate the fact that $P^d_X$ is a connected, projective scheme of pure dimension $g$. As we said in the introduction, several geometric and modular properties of $P^d_X$ depend on $d$.

**Definition 1.4.** Let $X$ be a stable curve and $P^d_X$ its degree-$d$ compactified Jacobian. We say that $P^d_X$ is of Néron type if the number of irreducible components of $P^d_X$ is equal to the number of irreducible components of $N^d_X$.

**Example 1.5.** A curve $X$ is called tree-like if every node of $X$ lying in two different irreducible components is a separating node.

The compactified Jacobian of a tree-like curve $X$ is easily seen to be always of Néron type. Indeed, $P^d_X$ is irreducible for every $d$; on the other hand $\#\Delta_X = 1$ so that $N^d_X$ is also irreducible.

Let now $\pi : \overline{P}^d_f \to B$ be the compactified degree-$d$ Picard scheme of a regular smoothing $f : \mathcal{X} \to B$ of a stable curve $X$, as defined in 1.2. So the fiber of $\pi$ over $b_0$ is $P^d_X$, and the restriction of $\pi$ over $U = B \setminus \{b_0\} \subset B$ is Pic$^d_{f_U}$. We denote $P^d_f \to B$ the smooth locus of $\pi$. By the Néron Mapping Property there exists a canonical $B$-morphism, $\chi_f$, from $P^d_f$ to the Néron model of Pic$^d_{f_U}$:

$$\chi_f : P^d_f \to N^d_f$$

extending the identity map from the generic fiber of $\pi$ to the generic fiber of $N^d_f \to B$.

**Proposition 1.6.** With the above set up, $P^d_X$ is of Néron type if and only if the map $\chi_f : P^d_f \to N^d_f$ is an isomorphism for every $f : \mathcal{X} \to B$ as above.

The proof, requiring a description of $\overline{P}^d_f$, will be given in subsection 2.6.
1.7. **Smoothing separating nodes.** A stable weighted graph of genus \( g \geq 2 \) is a pair \((\Gamma, w)\), where \( \Gamma \) is a graph and \( w : V(\Gamma) \to \mathbb{Z}_{\geq 0} \) a weight function. The genus of \((\Gamma, w)\) is the number \( g(\Gamma, w) \) defined as follows:

\[
g(\Gamma, w) = \sum_{v \in V(\Gamma)} w(v) + b_1(\Gamma).
\]

A weighted graph will be called *stable* if every \( v \in V(\Gamma) \) such that \( w(v) = 0 \) has valency at least 3.

Let \( X \) be a nodal curve of genus \( g \), the weighted dual graph of \( X \) is the weighted graph \((\Gamma_X, w_X)\) such that \( \Gamma_X \) is the usual dual graph of \( X \) (the vertices of \( \Gamma_X \) are identified with the irreducible components of \( X \) and the edges are identified with the nodes of \( X \); an edge joins two, possibly equal, vertices if the corresponding node is in the intersection of the corresponding irreducible components), and \( w_X \) is the weight function on the set of irreducible components of \( X \), \( V(\Gamma_X) \), assigning to a vertex the geometric genus of the corresponding component. Hence

\[
g = \sum_{v \in V(\Gamma_X)} w_X(v) + b_1(\Gamma_X) = g(\Gamma_X, w_X).
\]

\( X \) is a stable curve if and only if \((\Gamma_X, w_X)\) is a stable weighted graph.

Now we ask: What happens to the weighted dual graph of \( X \) if we smooth all the separating nodes of \( X \)?

To answer this question, we introduce a new weighted graph, denoted by \((\Gamma^2, w^2)\), associated to a weighted graph \((\Gamma, w)\). \((\Gamma^2, w^2)\) is defined as follows. \( \Gamma^2 \) is the graph obtained by contracting every separating edge of \( \Gamma \) to a point. Therefore \( \Gamma^2 \) is 2-edge-connected, i.e. free from separating edges (this explains the notation). To define the weight function \( w^2 \), notice that there is a natural surjective map contracting the separating edges of \( \Gamma \)

\[
\sigma : \Gamma \to \Gamma^2,
\]

and an induced surjection on the set of vertices

\[
\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v).
\]

Now we define \( w^2 \) as follows. For every \( v^2 \in V(\Gamma^2) \)

\[
w^2(v^2) = \sum_{v \in \phi^{-1}(v^2)} w(v).
\]

As \( \sigma \) does not contract any cycle, \( b_1(\Gamma) = b_1(\Gamma^2) \) and \( g(\Gamma, w) = g(\Gamma^2, w^2) \).

**Remark 1.8.** If \((\Gamma, w)\) is the weighted dual graph of a curve \( X \), \((\Gamma^2, w^2)\) is the weighted dual graph of any curve obtained by smoothing every separating node.
of \( X \). We shall usually denote by \( X^2 \) such a curve. Of course \( X \) and \( X^2 \) have the same genus.

1.9. \textit{d-general and weakly d-general curves}. Let us recall the definitions of balanced and strictly balanced multidegrees.

**Definition 1.10.** Let \( X \) be a quasistable curve of genus \( g \geq 2 \) and \( L \in \text{Pic}^d X \). Let \( d \) be the multidegree of \( L \).

(1) We say that \( L \), or \( d \), is \textit{balanced} if for any subcurve (equivalently, for any connected subcurve) \( Z \subseteq Y \) we have (notation in 1.1(2))

\[
\deg_Z L \geq m_Z(d) := \frac{d w_Z}{2g - 2} - \delta_Z, \tag{1.2}
\]

and \( \deg_Z L = 1 \) if \( Z \) is an exceptional component.

(2) We say that \( L \), or \( d \), is \textit{strictly balanced} if it is balanced and if strict inequality holds in (1.2) for every \( Z \subseteq X \) such that \( Z \subsetneq X \) and \( Z \cap Z^c \neq X_{\text{exc}} \).

(3) We denote

\[
\overline{B}_d(X) = \{ d : |d| = d \text{ balanced on } X \} \supseteq B_d(X) = \{ d : \text{strictly balanced} \}.
\]

The following trivial observations are useful.

**Remark 1.11.** (A) Let \( Z = Z_1 \sqcup Z_2 \subset X \) be a disconnected subcurve. Then

\[
m_Z(d) = m_{Z_1}(d) + m_{Z_2}(d).
\]

(B) Suppose \( X \) stable and \( d \in B_d(X) \). Then \( d \) is not strictly balanced if and only if there exists a subcurve \( Z \subsetneq X \) such that \( d_Z = m_Z(d) \), or equivalently, \( d_{Z^c} = m_{Z^c}(d) + \delta_Z \).

**Remark 1.12.** Let \( X \) be stable. By [C2, Prop. 4.12], every multidegree class in \( \Delta^d_X \) has a balanced representative, which is unique if and only if it is strictly balanced. Therefore

\[
\#B_d(X) \leq \#\Delta_X \leq \#\overline{B}_d(X).
\]

The terminology “strictly balanced” is not to be confused with “stably balanced” (used elsewhere and unnecessary here). The two coincide for stable curves; in general, a stably balanced line bundle is strictly balanced, but the converse may fail. Let us explain the difference. The compactified Picard scheme of \( X \), \( \overline{\text{Pic}}^d X \), is a GIT-quotient of a certain scheme by a certain reductive group \( G \). Strictly balanced line bundles correspond to the GIT-semistable orbits that are closed in the GIT-semistable locus. Stably balanced line bundles correspond to GIT-stable points and balanced line bundles correspond to GIT-semistable points. As every point in \( \overline{\text{Pic}}^d X \) parametrizes a unique closed orbit, strictly balanced
line bundles of degree $d$ on quasistable curves of $X$ are bijectively parametrized by $P^d_X$. See Fact 2.2 below.

**Definition 1.13.** Let $X$ be a stable curve. We will say that $X$, or its weighted dual graph $(\Gamma_X, w_X)$, is $d$-general if $B_d(X) = B_d(\overline{X})$ (cf. [C2, 4.13]). (Equivalently, $X$ is $d$-general if the inequalities in Remark 1.12 are both equalities.)

We will say that $X$ is weakly $d$-general if $(\Gamma^2_X, w^2_X)$ is $d$-general.

**Remark 1.14.** The following facts are well known (see loc.cit.).

1. The set of $d$-general stable curves is a nonempty open subset of $\overline{M}_g$.
2. $(d - g + 1, 2g - 1) = 1$ if and only if every stable curve of genus $g$ is $d$-general.
3. The property of being $d$-general depends only on the weighted dual graph (obvious).

**Example 1.15.** If $X_{\text{sep}} = \emptyset$, then $X$ is $d$-general if and only if it is weakly general.

If $X$ is tree-like, then $(\Gamma^2_X, w^2_X)$ has only one vertex, hence it is $d$-general for every $d$. Therefore tree-like curves are weakly $d$-general for every $d$.

## 2. Irreducible Components of Compactified Jacobians

### 2.1. Compactified degree-$d$ Jacobians

Let us describe the compactified Jacobian $P^d_X$ for any degree $d$. We use the set up of [C1] and [C2]; in these papers there is the assumption $g \geq 3$, but by [OS], [S] and [Al] we can extend our results to $g \geq 2$. A synthetic account of the modular properties of the compactified Jacobian for a curve or for a family can be found in [CE, 3.8 and 5.10].

**Fact 2.2.** Let $X$ be a stable curve of genus $g \geq 2$. Then $P^d_X$ is a connected, reduced, projective scheme of pure dimension $g$, admitting a canonical decomposition (notation in 1.1(5))

$$P^d_X = \bigsqcup_{S \subset X_{\text{sing}}} P^d_S$$

such that for every $S \subset X_{\text{sing}}$ and $d \in B_d(X_S)$ there is a natural isomorphism

$$P^d_S \cong \text{Pic}^{d^v}_S X^v_S$$

where $d^v$ denotes the multidegree on $X^v_S \subset X_S$ defined by restricting $d$.

Let $i(P^d_X)$ be the number of irreducible components of $P^d_X$; then

$$B_d(X) \leq i(P^d_X) \leq \#\Delta_X.$$
Corollary 2.3. Let $X$ be a stable curve.

(1) The decomposition of $\overline{P_X^d}$ in irreducible components is

$$\overline{P_X^d} = \bigcup_{(S,d) \in I_X^d} P_S^d, \quad \text{where } I_X^d := \{(S,d) : S \subset X_{\text{sep}}, d \in B_d(X_S)\}.$$

(2) Suppose that $X$ is $d$-general; then $P_X^d$ is of Néron type, and for every nonempty $S \subset X_{\text{sep}}$ we have $B_d(X_S) = \emptyset$.

Proof. From Fact 2.2 we have that the irreducible components of $\overline{P_X^d}$ are the closures of subsets $P_S^d \cong \text{Pic}^{d^*} X_S^v$ where $S$ is such that $\dim \text{Pic}^{d^*} X_S^v = g$. Now, it is clear that

$$\dim \text{Pic}^{d^*} X_S^v = \dim J(X_S^v) = g \quad \text{if and only if } S \subset X_{\text{sep}}.$$

Therefore the irreducible components of $\overline{P_X^d}$ correspond bijectively to pairs $(S,d)$ with $S \subset X_{\text{sep}}$ and $d \in B_d(X_S)$.

Now part (2). It is clear that the set $I_X^d$ contains a subset identifiable with $B_d(X)$, namely the subset $\{(\emptyset,d) : d \in B_d(X)\}$. If $X$ is $d$-general then $\#B_d(X) = \#\Delta_X$, hence by (2.1) we must have that $I_X^d$ contains no pairs other than those of type $(\emptyset,d)$. This concludes the proof. \qed

Lemma 2.4. Let $X$ be a stable curve and let $\mu \in \Delta_X^d$ be a multidegree class. Then there exists a unique $S(\mu) \subset X_{\text{sing}}$ and a unique $d(\mu) \in B_d(X_{S(\mu)})$ such that for every $d \in B_d(X)$ with $|d| = \mu$ the following properties hold.

(1) There is a canonical surjection

$$\text{Pic}^d X \twoheadrightarrow P_S^{d(\mu)} = \text{Pic}^{d(\mu)^*} X_S^{v(\mu)},$$

(2) We have

$$S(\mu) = \bigcup_{Z \subset X^d \mu = m_Z(d)} Z \cap Z^e.$$

Proof. The proof is routine. Let us sketch it using the combinatorial results [C1, Lemma 5.1 and Lemma 6.1]. The terminology used in that paper differs from ours as follows: what we here call a “strictly balanced multidegree $d$ on a quasistable curve $X$” is there called an “extremal pair $(X,d)$”; cf. subsection 5.2 p. 631.

So, the pair $(X_{S(\mu)}, d(\mu))$ is the “extremal pair” associated to $\mu$. This means the following. For every balanced line bundle $L$ on $X$ such that $[\deg L] = \mu$ the point in $\overline{P_X^d}$ associated to $L$ parametrizes a line bundle $\hat{L} \in \text{Pic}^d X_{S(\mu)}$, and the restriction of $\hat{L}$ to $X_S^v$ is uniquely determined by $L$. Conversely every line bundle in $\text{Pic}^{d(\mu)^*} X_S^{v(\mu)}$ is obtained in this way.
More precisely, as we said, $\overline{P}_d^X$ is a GIT quotient; let us denote it by $\overline{P}_d^X = V_X/G$, so that $V_X$ is made of GIT-semistable points. Let $O_G(L) \subset V_X$ be the orbit of $L$. Then the semistable closure of $O_G(L)$ contains a unique closed orbit $O_G(\overline{L})$ as above. Moreover for every $d' \in B_d(X)$ having class $\mu$ there exists $L' \in \text{Pic}^d X$ such that the above $O_G(\overline{L})$ lies in the closure of $O_G(L')$. Hence the maps $\text{Pic}^d X \to \overline{P}_d^X$ and $\text{Pic}^{d'} X \to \overline{P}_d^X$ have the same image.

Using the notation of Fact 2.2, we have that for every balanced $d$ of class $\mu$ the canonical map $\text{Pic}^d X \to \overline{P}_d^X$ has image $\text{Pic}^{d(\mu)} S(\mu)$, so that the first part is proved.

Now (2). The previously mentioned Lemma 5.1 implies that for every $d \in \overline{B}_d(X)$ and every $Z$ such that $d_Z = m_Z(d)$ we have $Z \cap Z^c \subset S(\mu)$. By the above Lemma 6.1 each $n \in S(\mu)$ is obtained in this way.

**Proposition 2.5.** Let $X$ be a stable curve. $\overline{P}_d^X$ is of Néron type if and only if for every $d \in \overline{B}_d(X)$ and every connected $Z \subset X$ such that $d_Z = m_Z(d)$ we have

$$Z \cap Z^c \subset X_{\text{sep}}. \tag{2.2}$$

**Proof.** We begin by observing that, with the notation of Corollary 2.3 and Lemma 2.4, we have

$$I_d^X = \{(d(\mu), S(\mu)), \forall \mu \in \Delta_d^X \text{ such that } \dim \text{Pic}^{d(\mu)} S(\mu) = g\}.$$

Indeed, by Fact 2.2 the set on the right is clearly included in $I_d^X$. On the other hand let $(S, d) \in I_d^X$. To show that there exists $\mu \in \Delta_d^X$ such that $d = d(\mu)$ we can assume that $S \neq \emptyset$ (otherwise it is obvious). So, $d$ is a strictly balanced multidegree of total degree $d$ on $\dot{X}_S$. Let $n \in S$; by Corollary 2.3 the node $n$ is separating for $X$; let $X = Z \cup Z^c$ with $Z \cap Z^c = \{n\}$. Then $Z$ and $Z^c$ can be viewed as subcurves of $\dot{X}_S$, where they do not intersect since the node $n$ is replaced by an exceptional component $E$. Now, $d_E = 1$, therefore $d_Z = m_Z(d)$ and $d_{Z^c} = m_{Z^c}(d)$. Let $C_S \subset Z \subset X$ be the irreducible component intersecting $Z^c$ (so that $C_S \subset \dot{X}_S$ intersects $E$). Let $d^X$ be the multidegree on $X$ defined as follows: for every irreducible component $C \subset X$

$$d^X_C = \begin{cases} d_C + 1 & \text{if } C = C_Z, \\ d_C & \text{otherwise.} \end{cases}$$

As $d$ is balanced on $\dot{X}_S$ one easily checks that $d^X$ is balanced on $X$. Note that $d^X$ is not strictly balanced, since $d^X_{Z^c} = m_{Z^c}(d)$ (see Remark 1.11). By iterating the above procedure for every node in $S$ we arrive at a balanced multidegree on $X$ whose class we denote by $\mu \in \Delta_d^X$. By Lemma 2.4 we have that $d = d(\mu)$.

Suppose that $\overline{P}_d^X$ is of Néron type. By the previous discussion there is a natural bijection between $\Delta_d^X$ and $I_d^X$, mapping $\mu \in \Delta_d^X$ to $(S(\mu), d(\mu))$. By Corollary 2.3 we have $S(\mu) \subset X_{\text{sep}}$. Hence for every multidegree $d \in \overline{B}_d(X)$ such that $[d] = \mu$ we have that condition (2) of that lemma holds. In particular every $Z$ as in our statement is such that $Z \cap Z^c \subset S(\mu) \subset X_{\text{sep}}$. 


Conversely, if $P^d_X$ is not of Néron type there is a class $\mu \in \Delta^d_X$ such that

$$g > \dim P^d_{S(\mu)} = \dim J(X^v_{S(\mu)}).$$

But then $S(\mu)$ contains some non separating node of $X$. Hence, by Lemma 2.4(2), there exists a connected subcurve $Z \subset X$ such that $d_Z = m_Z(d)$ and such that $Z \cap Z^c$ contains some non separating node. 

2.6. Proof of Proposition 1.6. We generalize the proof of [C2, Thm. 6.1]. Let $f : \mathcal{X} \to B$ be a regular smoothing of $X$ as defined in subsection 1.2, and $\pi : \overline{P}^d_f \to B$ be the compactified degree-$d$ Picard scheme. Its smooth locus $P^d_f \to B$ is such that its fiber over $b_0$, denoted $P^d_X$, satisfies

$$(2.3) \quad P^d_X = \coprod_{(S,d) \in I^d_X} P^d_{S}$$

(note in 2.3) where each $P^d_{S}$ is irreducible of dimension $g$. If the morphism $\chi_f : P^d_f \to N^d_f$ is an isomorphism, then $P^d_X$ has as many irreducible components as $N^d_X$, hence the same holds for $P^d_X$. So $P^d_X$ is of Néron type.

Conversely, if $P^d_X$ is of Néron type, then $P^d_X$ has an irreducible component for every $\mu \in \Delta^d_X$ so that (2.3) takes the form

$$P^d_X = \coprod_{\mu \in \Delta^d_X} P^d_{S(\mu)}.$$ 

Let us construct the inverse of $\chi_f$. We pick a balanced representative $d^u_\mu$ for every multidegree class $\mu \in \Delta^d_X$ (it exists by Remark 1.12). By [C2, Lemma 3.10] we have

$$N^d_f \cong \coprod_{\mu \in \Delta^d_X} \text{Pic}^d_f \sim_U$$

where $\sim_U$ denotes the gluing of the Picard schemes $\text{Pic}^d_f$ along their restrictions over $U$ (as $\text{Pic}^d_{f_U} = \text{Pic}^d_{f_0}$ for every $\mu$). Now, the Picard scheme $\text{Pic}^d_f$ is endowed with a Poincaré bundle, which is a relatively balanced line bundle on $\mathcal{X} \times_B \text{Pic}^d_f$. By the modular property of $\overline{P}^d_f$ the Poincaré bundle induces a canonical $B$-morphism

$$\psi^\mu_f : \text{Pic}^d_f \to P^d_{S(\mu)} \subseteq \overline{P}^d_f.$$ 

As $\mu$ varies, the restrictions of these morphisms over $U$ all coincide with the identity map $\text{Pic}^d_{f_U} \to \text{Pic}^d_{f_U} \subseteq \overline{P}^d_f$. Therefore the $\psi^\mu_f$ can be glued together to a morphism

$$\psi_f : N^d_f \to P^d_f \subseteq \overline{P}^d_f.$$ 

It is clear that $\psi_f$ is the inverse of $\chi_f$. Proposition 1.6 is proved. 

\[\square\]
2.7. The main result. From Proposition 2.5 we derive the following.

**Corollary 2.8.** Let \( X \) be a stable curve free from separating nodes. Then \( \overline{P^d_X} \) is of \( \text{Néron} \) type if and only if \( X \) is \( d \)-general.

**Proof.** By Corollary 2.3(2) there is only one implication to prove. Namely, suppose that \( X \) is not \( d \)-general. Then there exists \( d \in B_d(X) \setminus B_d(X) \), and hence a subcurve \( Z \subset X \) such that \( d_Z = m_Z(d) \) (see Remark 1.11). As \( X_{\text{sep}} = \emptyset \), condition (2.2) of Proposition 2.5 cannot be satisfied. Therefore \( \overline{P^d_X} \) is not of \( \text{Néron} \) type. \( \square \)

We are ready to prove our main result.

**Theorem 2.9.** Let \( X \) be a stable curve. Then \( \overline{P^d_X} \) is of \( \text{Néron} \) type if and only if \( X \) is weakly \( d \)-general.

**Proof.** Observe that if \( X \) is free from separating nodes we are done by Corollary 2.8. Let \( (\Gamma, w) \) be the weighted graph of \( X \) and consider the weighted graph \( (\Gamma^2, w^2) \) defined in subsection 1.7. We denote by \( X^2 \) a stable curve whose weighted graph is \( (\Gamma^2, w^2) \). By Remark 1.8 the curve \( X^2 \) can be viewed as a smoothing of \( X \) at \( X_{\text{sep}} \).

Recall that we denote by \( \sigma : \Gamma \to \Gamma^2 \) the contraction map and by

\[
\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v)
\]

the induced map on the vertices, i.e. on the irreducible components. The subcurves of \( X \) naturally correspond to the so-called “induced” subgraphs of \( \Gamma \), i.e. those subgraphs \( \Gamma' \) such that if two vertices \( v, w \) of \( \Gamma \) are in \( \Gamma' \), then every edge of \( \Gamma \) joining \( v \) with \( w \) lies in \( \Gamma' \). Similarly, the induced subgraphs of \( \Gamma^2 \) correspond to subcurves of \( X^2 \). If \( Z^2 \) is a subcurve of \( X^2 \), and \( \Gamma_{Z^2} \subset \Gamma_{X^2} \) its corresponding subgraph, we denote by \( Z \subset X \) the subcurve associated to \( \sigma^{-1}(\Gamma_{Z^2}) \) (it is obvious that the subgraph \( \sigma^{-1}(\Gamma_{Z^2}) \) is induced if so is \( \Gamma_{Z^2} \)); we refer to \( Z \) as the “preimage” of \( Z^2 \). Of course \( \sigma(\Gamma_Z) = \Gamma_{Z^2} \).

For any \( Z \subset X \) which is the preimage of a subcurve \( Z^2 \subset X^2 \) we have

\[
Z \cap X_{\text{sep}} \subset Z_{\text{sep}}
\]

or, equivalently, \( Z \cap Z^c \cap X_{\text{sep}} = \emptyset \). Conversely, every \( Z \subset X \) satisfying (2.4) is the preimage of some \( Z^2 \subset X^2 \).

Hence \( Z^2 \) can be viewed as a smoothing of \( Z \) at its separating nodes that are also separating nodes of \( X \), i.e. at \( Z_{\text{sep}} \cap X_{\text{sep}} \). Thus, for every \( Z^2 \) with preimage \( Z \) we have \( g_Z = g_{Z^2} \) and \( \delta_Z = \delta_{Z^2} \); hence for every \( d \in \mathbb{Z} \)

\[
m_{Z^2}(d) = m_Z(d).
\]
We shall now view multidegrees as an integer valued functions on the vertices. We claim that we have a surjection

\[ \alpha : B_d(X) \to B_d(X^2) \]

defined as follows: for every vertex \( v^2 \in V(\Gamma^2) \) we set

\[ \alpha(d)(v^2) := \sum_{v \in \phi^{-1}(v^2)} d(v). \]

Let us first show that if \( d \) is balanced, so is \( \alpha(d) \). For every subcurve \( Z^2 \subset X^2 \) we have \( \alpha(d)|_{Z^2} = d_Z \) where \( Z \subset X \) is the preimage of \( Z^2 \); by (2.5) the inequality (1.2) is satisfied on \( Z^2 \) if (and only if) it is satisfied on \( Z \).

Let us now show that \( \alpha \) is surjective. Let \( d^2 \) be a balanced multidegree on \( X^2 \); we know that \( X^2 \) can be chosen to be a smoothing of \( X \) at \( X_{\text{sep}} \). In other words there exists a family of curves \( X_t \), all having \( (\Gamma^2, w^2) \) as weighted graph, specializing to \( X \). But then there also exists a family of line bundles \( L_t \) on \( X_t \), having degree \( d^2 \), specializing to a balanced line bundle of some degree \( d \) on \( X \) (this follows from the construction of the universal compactified Picard scheme \( \overline{P}_{d, g} \to \overline{M}_g \), see [C2, subsection 5.2]). By the definition of \( \alpha \), it is clear that the multidegree \( d \) is such that \( \alpha(d) = d^2 \).

We are ready to prove the Theorem. Assume that \( \overline{P}_X^d \) is of Néron type. Our goal is to prove that \( X^2 \) is \( d \)-general. By contradiction, let \( Z^2 \subset X^2 \) be a connected subcurve such that for some \( d^2 \in B_d(X^2) \) we have \( d^2_{Z^2} = m_{Z^2}(d) \). Let \( Z \) be the preimage of \( Z^2 \), and let \( d \in B_d(X) \) be such that \( \alpha(d) = d^2 \). Then

\[ d_Z = d^2_{Z^2} = m_{Z^2}(d) = m_{Z}(d). \]

By Proposition 2.5 we obtain that \( Z \cap Z^c \subset X_{\text{sep}} \). This is in contradiction with (2.4); so we are done.

Conversely, let \( X \) be weakly \( d \)-general; i.e. \( B_d(X^2) = B_d(X^2) \). To show that \( P_X^d \) is of Néron type we use again Proposition 2.5, according to which it suffices to show that for every \( d \in B_d(X) \) and for every \( Z \subset X \) such that \( Z \cap Z^c \neq X_{\text{sep}} \) we have \( d_Z > m_Z(d) \).

By contradiction. Let \( Z \) be a connected subcurve such that \( Z \cap Z^c \neq X_{\text{sep}} \), and \( d_Z = m_Z(d) \) for some balanced multidegree \( d \) on \( X \). We choose \( Z \) maximal with respect to this properties. This choice yields

\[ Z \cap Z^c \cap X_{\text{sep}} = \emptyset. \]

Indeed, if \( Z \cap Z^c \) contains some \( n \in X_{\text{sep}} \), there exists a connected component \( Z' \) of \( Z^c \) such that \( Z \cap Z' = \{n\} \). Let \( W := Z \cup Z' \); then \( W \) is a connected curve containing \( Z \). Now, \( W \cap W^c = Z \cap Z^c \setminus \{n\} \), hence \( W \cap W^c \neq X_{\text{sep}} \); moreover, using Remark 1.11 one easily checks that \( d_W = m_W(d) \). This contradicts the maximality of \( Z \).
By (2.6) we have that $Z \cap X_{\text{sep}}$ is all contained in $Z_{\text{sep}}$ therefore, as observed immediately after (2.4), the curve $Z$ is the preimage of a subcurve $Z^2 \subset X^2$. Now let $d^2 = \alpha(d)$; so $d^2 \in B_d(X^2) = B_d(X^2)$ by hypothesis. We have
\[ d^2_{Z^2} = d_Z = m_Z(d) = m_{Z^2}(d). \]
This contradicts the fact that $d^2$ is strictly balanced.

**Corollary 2.10.** Let $X$ be a stable curve of genus $g$, and let $d = g - 1$. Then $P^d_X$ is of Néron type if and only if $X$ is a tree-like curve.

**Proof.** As $d = g - 1$, by [M, Remark 2.3] $X$ is $d$-general if and only if $X$ is irreducible. Hence $X$ is weakly $d$-general if and only if $X$ is tree-like. 

**2.11.** The locus of weakly $d$-general curves in $M_g$. The locus of $d$-general curves in $M_g$ has been studied in details in [M] (also in [CE] if $d = 1$ for applications to Abel maps). A stable curve $X$ which is not $d$-general is called $d$-special. The locus of $d$-special curves is a closed subscheme denoted $\Sigma_d^g \subset M_g$. By [M, Lemma 2.10], $\Sigma_d^g$ is the closure of the locus of $d$-special curves made of two smooth components. Curves made of two smooth components are called vine curves.

We are going to exhibit a precise description of $D^d_g$, the complement in $M_g$ of the locus of weakly $d$-general curves:
\[ D^d_g := \{ X \in M_g : P^d_X \text{ not of Néron type} \}. \]
In the following statement by codim $D^d_g$ we mean the codimension of an irreducible component of maximal dimension.

**Proposition 2.12.** $D^d_g$ is the closure of the locus of $d$-special vine curves with at least 2 nodes. Moreover
\[ \text{codim } D^d_g = \begin{cases} + \infty & \text{(i.e. } D^d_g = \emptyset) \text{ if } (d - g + 1, 2g - 2) = 1, \\ 3 & \text{if } (d - g + 1, 2g - 2) = 2 \text{ and } g \text{ is even}, \\ 2 & \text{otherwise}. \end{cases} \]

**Proof.** By Theorem 2.9, we have that $X \in D^d_g$ if and only if $X$ is not weakly $d$-general, if and only if $X^2$ is not $d$-general (where $X^2$ is as in 1.8). This is equivalent to the fact that there exists $d \in B_d(X^2)$ and a subcurve $Z \subseteq X^2$ such that $d_Z = m_Z(d)$; as $X^2$ has no separating nodes, for every subcurve $Z \subseteq X^2$ we have $\delta_Z \geq 2$. This observation added to the proof of [M, Lemma 2.10] gives that $X^2$ (and every curve with the same weighted graph) lies in the closure of the locus of $d$-special vine curves with at least two nodes. Therefore the same holds for $X$, since $X$ is a specialization of curve with the same weighted graph as $X^2$.

Conversely, let $X$ be in the closure of the locus of $d$-special vine curves with at least two nodes. Then $X^2$ is also in this closure, as such vine curves are obviously free from separating nodes. By [M, Lemma 2.10] the curve $X^2$ is $d$-special, hence $X$ is not weakly $d$-general.
Let us turn to the codimension of $D^d_g$. The fact that if $(d - g + 1, 2g - 2) = 1$ then $D^d_g$ is empty is well known ([C2]). Conversely, assume $D^d_g = \emptyset$. By the previous part, the locus of $d$-special vine curves with at least two nodes is also empty. Now the proof of the numerical Lemma 6.3 in [C1] shows that this implies that $(d - g + 1, 2g - 2) = 1$. In fact, the proof of that Lemma shows that if there are no $d$-special vine curves with two or three nodes then $(d - g + 1, 2g - 2) = 1$.

Next, recall that the locus, $V_\delta$, of vine curves with $\delta$ nodes has pure codimension $\delta$, and notice that the sublocus of $d$-special curves is a union of irreducible components of $V_\delta$.

Now, again by the proof of the above Lemma 6.3, if $(d - g + 1, 2g - 2) \neq 1$ and if there are no $d$-special vine curves with two nodes, then $(d - g + 1, 2g - 2) = 2$, $g$ is even and every vine curve with three nodes, having one component of genus $g/2 - 1$, is $d$-special. This completes the proof of the Proposition.

A precise description of the locus of $d$-special vine curves is given in [M, Prop. 2.13]. Her result combined with the previous proposition yields a more precise description of the locus of stable curves whose compactified degree-$d$ Jacobian is of Néron type, for every fixed $d$.

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Received 2 March 2010,
and in revised form 2 November 2010.
For editorial reason the publication has been postponed.

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