Geometry. — *Configuration spaces of tori*, by *Yoel Feler*, communicated by F. Catanese.

Abstract. — The $n$-point configuration spaces $C^n(T^2) = \{(q_1, \ldots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \forall i \neq j\}$ and $C^n(\mathbb{T}^2) = \{Q \subset \mathbb{T}^2 \mid \#Q = n\}$ of a complex torus $\mathbb{T}^2$ are complex manifolds. We prove that for $n > 4$ any holomorphic self-map $F$ of $C^n(\mathbb{T}^2)$ either carries the whole of $C^n(\mathbb{T}^2)$ into an orbit of the diagonal $(\text{Aut } \mathbb{T}^2)$-action in $C^n(\mathbb{T}^2)$ or is of the form $F(Q) = T(Q)Q$, where $T : C^n(\mathbb{T}^2) \to \text{Aut } \mathbb{T}^2$ is a holomorphic map. We also prove that for $n > 4$ any endomorphism of the torus braid group $B_n(\mathbb{T}^2) = \pi_1(C^n(\mathbb{T}^2))$ with a non-abelian image preserves the pure torus braid group $P_n(\mathbb{T}^2) = \pi_1(C^n(\mathbb{T}^2))$.

Key words: Configuration space; torus braid group; holomorphic endomorphism.

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1. Introduction

The configuration space $C^n(X)$ of a complex space $X$ consists of all $n$-point subsets ("configurations") $Q = \{q_1, \ldots, q_n\} \subset X$, $\#Q = n$. The automorphism group $\text{Aut } X$ acts in $C^n(X)$ by $X \ni Q \mapsto AQ = \{Aq_1, \ldots, Aq_n\}$. If $\text{Aut } X$ is a complex Lie group, any holomorphic map $T : C^n(X) \to \text{Aut } X$ produces the holomorphic self-map ("endomorphism") $F_T$ of $C^n(X)$, $F_T(Q) = T(Q)Q$; such a map $F_T$ is called tame. Choosing a base point $Q^0 \in C^n(X)$, define an endomorphism $F_TQ^0$ by $F_TQ^0(Q) = T(Q)Q^0$; it maps the whole configuration space into one orbit $(\text{Aut } X)Q^0$ of the diagonal $(\text{Aut } X)$-action in $C^n(X)$; maps that have the latter property are said to be orbit-like.

V. Lin [13][15][17] proved that when $n > 4$ and $X$ is $\mathbb{C}$ or $\mathbb{C}P^1$, an endomorphism $F$ of $C^n(X)$ is either tame or orbit-like. The latter happens if and only if $F$ is abelian, i.e. the image $F_*(\pi_1(C^n(X)))$ under the induced endomorphism $F_*$ of the fundamental group $\pi_1(C^n(X))$ is abelian. (Recall that $\pi_1(C^n(X))$ is the braid group $B_n(X)$ of $X$; it is non-abelian whenever $n \geq 3$.) Similar results were obtained by V. Zinde (see [22][26]) for $X = \mathbb{C}P^1$.

Here we treat the endomorphisms of the configuration spaces of a torus $T^2$, which completes the story for all non-hyperbolic Riemann surfaces.

Throughout the paper, $\text{Aut } T^2$ stands for the group of all biholomorphic (≡ biregular) self-mappings of $T^2$.

Definition 1.1. A group homomorphism $\varphi : G \to H$ is called abelian if its image is abelian. A continuous map $F : X \to Y$ of path connected spaces is called abelian if the induced homomorphism $F_* : \pi_1(X) \to \pi_1(Y)$ is abelian.
Theorem 1.2. For \( n > 4 \), each holomorphic map \( F : \mathcal{C}^n(\mathbb{T}^2) \to \mathcal{C}^n(\mathbb{T}^2) \) is either tame or orbit-like; the latter happens exactly when \( F \) is abelian. Any automorphism of \( \mathcal{C}^n(\mathbb{T}^2) \) is tame.

Corollary 1.3. For \( n > 4 \), the set \( \mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \mathcal{C}^n(\mathbb{T}^2)) \) of all holomorphic homotopy classes of non-abelian holomorphic endomorphisms of \( \mathcal{C}^n(\mathbb{T}^2) \) is in natural one-to-one correspondence with the set \( \mathcal{H}(\mathcal{C}^n(\mathbb{T}^2), \text{Aut } \mathbb{T}^2) \) of all holomorphic homotopy classes of holomorphic maps \( \mathcal{C}^n(\mathbb{T}^2) \to \text{Aut } \mathbb{T}^2 \).

Corollary 1.4. For \( n > 4 \), the orbits of the natural \( (\text{Aut } \mathcal{C}^n(\mathbb{T}^2)) \)-action in \( \mathcal{C}^n(\mathbb{T}^2) \) coincide with the orbits of the diagonal \( (\text{Aut } \mathbb{T}^2) \)-action in \( \mathcal{C}^n(\mathbb{T}^2) \).

Artin [11] proved that automorphisms of the braid group \( B_n = B_n(\mathbb{C}) \) preserve the pure braid group \( P_n \). V. Lin [14,15] generalized this to non-abelian endomorphisms of \( B_n(\mathbb{C}) \) and \( B_n(\mathbb{CP}^1) \); the case of \( B_n(\mathbb{CP}^1) \) was handled by V. Zinde [16,17]. N. Ivanov [10,11] proved an analogue of Artin’s theorem for automorphisms of braid groups of all Riemann surfaces of finite type but \( \mathbb{CP}^1 \). Our next theorem states an analogue of Lin’s theorem for the torus braid group \( B_n(\mathbb{T}^2) = \pi_1(\mathcal{C}^n(\mathbb{T}^2)) \) and the pure torus braid group \( P_n(\mathbb{T}^2) \), which is the fundamental group of the ordered configuration space \( \mathcal{E}^n(\mathbb{T}^2) = \{(q_1, \ldots, q_n) \in (\mathbb{T}^2)^n \mid q_i \neq q_j \forall i \neq j \} \). Part (b) of the next theorem is similar to results obtained in [14,26] for the braid groups of \( \mathbb{C}, \mathbb{CP}^1 \) and \( \mathbb{C}^* \).

Theorem 1.5. (a) Let \( n > 4 \) and \( \varphi \) be a non-abelian endomorphism of \( B_n(\mathbb{T}^2) \). Then \( \varphi(P_n(\mathbb{T}^2)) \subseteq P_n(\mathbb{T}^2) \).

(b) For \( n > \max\{m, 4\} \), any homomorphism \( \varphi : B_n(\mathbb{T}^2) \to B_m(\mathbb{T}^2) \) is abelian.

Let us outline the plan of the proof of Theorem 1.2. By Theorem 1.5(a), a non-abelian holomorphic self-map \( F \) of \( \mathcal{C}^n(\mathbb{T}^2) \) fits into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}^n(\mathbb{T}^2) & \xrightarrow{f} & \mathcal{E}^n(\mathbb{T}^2) \\
\downarrow p & & \downarrow p \\
\mathcal{C}^n(\mathbb{T}^2) & \xrightarrow{F} & \mathcal{C}^n(\mathbb{T}^2)
\end{array}
\]

where \( p : \mathcal{E}^n(\mathbb{T}^2) \ni q = (q_1, \ldots, q_n) \mapsto \{q_1, \ldots, q_n\} = Q \in \mathcal{C}^n(\mathbb{T}^2) \) is a Galois covering with Galois group \( S(n) \). The map \( f \) is non-constant, holomorphic and strictly equivariant with respect to the standard action of the symmetric group \( S(n) \) in \( \mathcal{E}^n(\mathbb{T}^2) \), meaning that there is an automorphism \( \alpha \) of \( S(n) \) such that \( f(q \sigma) = \alpha(\sigma) f(q) \) for all \( q \in \mathcal{E}^n(\mathbb{T}^2) \) and \( \sigma \in S(n) \). To study such maps \( f \), we start with an explicit description of all non-constant holomorphic maps \( \lambda : \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\} \). The set \( L \) of all such maps is finite and separates points of a certain submanifold \( M \subset \mathcal{E}^4(\mathbb{T}^2) \) with \( \text{codim } M = 1 \); we endow \( L \) with a special simplicial structure. The action of \( S(n) \) in \( \mathcal{E}^n(\mathbb{T}^2) \) induces a simplicial \( S(n) \)-action in the complex \( L \); the orbits of this action may be exhibited explicitly. A map \( f \) as above induces a simplicial self-map \( f^* \) of \( L \), defined by \( f^* \lambda = \lambda \circ f \) for \( \lambda \in L \), which carries important information about \( f \). Since \( f \) is strictly equivariant, \( f^* \) is nicely related to the \( S(n) \)-action on \( L \). Studying all these things together, we come to the desired conclusion.
The main goal of this section is to prove Theorem 1.5.

By O. Zariski [21] (cf. J. Birman [2]), the torus braid group $B_n(T^2)$ admits a presentation with $n+1$ generators $\sigma_1, \ldots, \sigma_{n-1}, a_1, a_2$ and the defining system of relations

\[
\begin{align*}
(2.1) & \quad \sigma_j \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2, \ i, j = 1, \ldots, n - 3; \\
(2.2) & \quad \sigma_j \sigma_{i+1} = \sigma_{i+1} \sigma_i \quad \text{for } i = 1, \ldots, n - 2; \\
(2.3) & \quad \sigma_k a_k = a_k \sigma_k \quad \text{for } k = 1, 2 \text{ and } i = 2, \ldots, n - 1; \\
(2.4) & \quad (\sigma_1^{-1} a_k)^2 = (a_k \sigma_1^{-1})^2 \quad \text{for } k = 1, 2; \\
(2.5) & \quad \sigma_1 \cdots \sigma_n^{-2} \sigma_n^{-2} \cdots \sigma_1 = a_1 a_2^{-1} a_1^{-1} a_2; \\
(2.6) & \quad a_2 \sigma_1^{-1} \sigma_1 a_2^{-1} \sigma_1 a_1 \sigma_1 = \sigma_1^2.
\end{align*}
\]

For $(a_1, \ldots, a_n) \in E^n(T^2)$ and $m = 1, \ldots, n - 1$, set $E^{n-m}(T^2 \setminus \{a_1, \ldots, a_m\}) = \{(q_{m+1}, \ldots, q_n) \in (T^2 \setminus \{a_1, \ldots, a_m\})^{n-m} : q_i \neq q_j\}$. For $m \leq n - 2$, the maps $\iota_{m+1} : E^{n-m}(T^2 \setminus \{a_1, \ldots, a_m\}) \ni (q_{m+1}, \ldots, q_n) \mapsto q_{m+1} \in T^2 \setminus \{a_1, \ldots, a_m\}$ and $\iota_1 : E^n(T^2) \ni (q_1, \ldots, q_n) \mapsto q_1 \in T^2$ define smooth locally trivial fibrations (see [6]) with fibres isomorphic respectively to $E^{n-1}(T^2 \setminus \{a_1\})$ and to $E^{n-m-1}(T^2 \setminus \{a_1, \ldots, a_m\})$. These spaces are aspherical and the final segments of the exact homotopy sequences of the above fibrations look as $1 \to P_{n-1,1} \to P_n(T^2) \to \mathbb{Z}^2 \to 1$ and $1 \to P_{n-m-1,m+1} \to P_{n,m} \to \mathbb{F}_m \to 1$, where $P_{n-m,m} = \pi_1(E^{n-m}(T^2 \setminus \{a_1, \ldots, a_m\}))$ and $\mathbb{F}_m$ is a free group of rank $m$. This leads to the following well-known statement.

**Proposition 2.1.** The subgroups $P_{n-s,1}$ fit into the normal series $\{1\} \subset P_{1,n-1} \subset \cdots \subset P_{n-m-1,m+1} \subset P_{n,m} \subset \cdots \subset P_{n-1,1} \subset P_{n,0} \cong P_n(T^2)$ with $P_{1,n-1} \cong \mathbb{F}_{n-1}$, $P_{n-m,m} \cong \mathbb{F}_m$, $P_{n-1,1}/P_{n-2,2} \cong \mathbb{F}_2$, $P_n(T^2)/P_{n,1} \cong \mathbb{Z}^2$.

**Corollary 2.2.** Any non-trivial subgroup $H \subset P_n(T^2)$ admits non-trivial homomorphisms to $\mathbb{Z}$. In particular, a group $G$ with the finiteabelianization $G/G' = G/[G, G]$ cannot have non-trivial homomorphisms to $P_n(T^2)$.

The exact homotopy sequence of the covering $p : E^n(T^2) \to E^n(T^2)$ looks as $1 \to P_n(T^2) \overset{E^n}{\longrightarrow} B_n(T^2) \overset{i}{\longrightarrow} S(n) \to 1$, where $\delta(\sigma_i) = (i, i+1)$ for $i = 1, \ldots, n - 1$ and $\delta(a_1) = \delta(a_2) = 1$. Let $i$ be the homomorphism of the Artin braid group $B_n = \pi_1(C^n(C))$ to the torus braid group $B_n(T^2)$ sending the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ to the eponymous generators of $B_n(T^2)$.

**Lemma 2.3.** Let $n > 4$ and let $\varphi : B_n(T^2) \to B_m(T^2)$ be a homomorphism such that the composition $\Phi = \delta \circ \varphi \circ i : B_n \overset{i}{\longrightarrow} B_n(T^2) \overset{\varphi}{\longrightarrow} B_m(T^2) \overset{\delta}{\longrightarrow} S(m)$ is abelian. Then $\varphi$ is abelian. In particular, $\varphi$ is abelian whenever $\delta \circ \varphi$ is.

**Proof.** Let $\Phi' : B_n' \to S(m)$ be the restriction of $\Phi$ to the commutator subgroup $B_n' = [B_n, B_n]$. Since $\Phi$ is abelian, $\Phi'$ is trivial and hence $\varphi(i(B_n')) \subset \text{Ker}\delta = P_n(T^2)$. By the Gorn–Lin theorem [8], $B_n' = [B_n', B_n']$ for $n > 4$, and Corollary 2.2 shows that $\varphi(i(B_n')) = 1$. Hence $\varphi \circ i$ is abelian and (2.2) implies that...
\[ \psi(i(\sigma_1)) = \cdots = \psi(i(\sigma_{n-1})); \] thus, \((a)\) \(\psi(\sigma_1) = \cdots = \psi(\sigma_{n-1})\). By \((2.6)\), \((a)\) and \((2.3)\), we obtain \((b)\) \(\psi(\sigma_2) = \psi(\sigma_2)^{-1}\psi(\sigma_1)\psi(\sigma_2)\psi(\sigma_1)^{-1}\). By \((a)\) and \((2.5)\), we get \((c)\) \((\psi(\sigma_2))^{2n-1} = \psi(\sigma_1)\psi(\sigma_2)^{-1}\psi(\sigma_1)^{-1}\psi(\sigma_2)\). Multiplying the relations \((b)\) and \((c)\) we see that \((\psi(\sigma_2))^{2n} = 1\). Since \(B_m(T^2)\) is torsion free (see \([6, \text{Theorem }8]\)), \(\psi(\sigma_2) = 1\) and, by \((a)\) and \((b)\), \(\psi\) is abelian. \(\square\)

**Proof of Theorem 1.5.** Let \(n > 4\) and let \(\psi\) be a non-abelian endomorphism of \(B_n(T^2)\). By Lemma \((2.3)\), the homomorphism \(\Phi = \delta \circ \psi \circ i : B_n \to S(n)\) is non-abelian. By V. Lin's theorem (see \([17, \text{Sec. }4]\) or \([15, 16, 18]\)), \(\Phi\) coincides with the standard epimorphism \(B_n \to S(n)\) up to an automorphism of \(S(n)\); thus, the homomorphism \(\delta \circ \psi\) is surjective. N. Ivanov (see \([10, \text{Theorem }1]\)) proved that for \(n > 4\) any non-abelian homomorphism \(B_n(T^2) \to S(n)\) whose image is a primitive permutation group on \(n\) letters coincides with the standard epimorphism \(\delta\) up to an automorphism of \(S(n)\). Therefore, \(\text{Ker}(\delta \circ \psi) = P_n(T^2) = \text{Ker} \delta\), \(\text{Ker}(\delta \circ \psi) = \text{Ker} \delta = P_n(T^2)\), which implies that \(\psi^{-1}(P_n(T^2)) = P_n(T^2)\) and a fortiori \(\psi(P_n(T^2)) \subseteq P_n(T^2)\).

To prove \((b)\), we use another theorem of Lin (\([17, \text{Theorem }4.4]\)), which says that for \(n > \max(m, 4)\) any homomorphism \(B_n \to S(m)\) is abelian. By Lemma \((2.3)\), \(\psi\) is abelian. \(\square\)

### 3. Ordered Configuration Spaces

#### 3.1. Holomorphic mappings \(E^m(T^2) \to T^2 \setminus \{0\}\)

**Definition 3.1.** We denote by \(\mathfrak{M}\) the finite cyclic subgroup of \(\text{Aut }T^2\) consisting of \(\pm \text{id}\) and all automorphisms of \(T^2\) induced by multiplication on the complex plane by non-integer complex numbers. \(\mathfrak{M}\) is isomorphic to \(\mathbb{Z}_2, \mathbb{Z}_4\) or \(\mathbb{Z}_6\). Let \(\mathfrak{M}_+\) consist of all \(m \in \mathfrak{M}\) with \(0 \leq \text{Arg} m < \pi\), i.e. \(\mathfrak{M}_+\) consists of 1, 2 or 3 elements (see \([7, \text{Chap. V, Sec. V.4.7}]\)).

Let \(\mathfrak{M}\) be a minimal generating set of the \(\mathbb{Z}\)-module of group endomorphisms of \(T^2\) (any endomorphism of the group \(T^2\) is induced by multiplication on the complex plane by a complex number); \# \(\mathfrak{M} = 2\) (see \([20, \text{Chap. VI, Sec. 5}]\) and \([12, \text{Chap. 10}]\)). Moreover, either \(\# \mathfrak{M}_+ = 1\) or we may assume that \(\mathbb{R} \subseteq \mathfrak{M}_+\).

**Theorem 3.2.** Any non-constant holomorphic map \(f : E^m(T^2) \to T^2 \setminus \{0\}\) is of the form \(f(q_1, \ldots, q_n) = \mathfrak{m}(q_i - q_j)\) with certain \(\mathfrak{m} \in \mathfrak{M}_+\) and \(i \neq j\).

To prove the theorem we need some preparation.

**Definition 3.3.** A configuration \((a_1, \ldots, a_m) \in E^m(T^2)\) is called exceptional if there exist \(i \neq j\) and an endomorphism \(\lambda\) of \(T^2\) such that \(\lambda(a_i) = \lambda(a_j)\) and \(\lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \ldots, a_m\}\).

**Lemma 3.4.** \((a)\) The set \(A\) of all exceptional configurations \(a \in E^m(T^2)\) is contained in a subvariety \(M \subseteq E^m(T^2)\) of codimension 1.

\((b)\) For any non-exceptional configuration \((a_1, \ldots, a_m) \in E^m(T^2)\), every non-constant holomorphic map \(\lambda : T^2 \setminus \{a_1, \ldots, a_m\} \to T^2 \setminus \{0\}\) extends to a biregular automorphism of \(T^2\) sending a certain \(a_i\) to 0.
PROOF. (a) Let $N$ denote the union of all finite subgroups of order $\leq m$ in $\mathbb{T}^2$; this set is finite. Set $M = \{(a_1, \ldots, a_m) \in \mathcal{E}^m(\mathbb{T}^2) \mid a_j - a_i \in N \text{ for some } i \neq j\}$; then $M$ is a subvariety in $\mathcal{E}^m(\mathbb{T}^2)$ of codimension 1. We show that $A \subseteq M$.

Let $a = (a_1, \ldots, a_m) \in A$ and let $i,j$, and $\lambda$ be as in Definition 3.3. Set $\mu(t) = \lambda(t + a_i) - \lambda(a_i)$, $t \in \mathbb{T}^2$. Then $\mu(0) = 0$ and $\mu$ is a group homomorphism with finite kernel $\ker \mu$ (see [3] Chap. 3, Sec. 3.1). If $t \in \ker \mu$, then $\lambda(t + a_i) = \lambda(a_i)$, $t + a_i \in \lambda^{-1}(\lambda(a_i)) \subseteq \{a_1, \ldots, a_m\}$ and $t \in \{a_1 - a_i, \ldots, 0, \ldots, a_m - a_i\}$, that is, $\ker \mu \subseteq \{a_1 - a_i, \ldots, 0, \ldots, a_m - a_i\}$. In particular, $\# \ker \mu \leq m$ and hence $\ker \mu \subseteq N$. Since $\mu(a_j - a_i) = 0$, we have $a_j - a_i \in N$ and $a \in M$.

(b) Let $a = (a_1, \ldots, a_m) \notin A$. The map $\lambda$ extends to a holomorphic self-map $\tilde{\lambda}$ of $\mathbb{T}^2$ (see H. Huber [9] §6, Satz 2; also [11] Chap. VI, Sec. 2, remarks after Cor. 2.6)). By the Riemann–Hurwitz relation (see [7] Chap. I, Sec. I.2.7)), $\tilde{\lambda}$ is an unbranched regular covering map of degree $k < \infty$. Clearly $\tilde{\lambda}^{-1}(0) \subseteq \{a_1, \ldots, a_m\}$ and $\tilde{\lambda}(a_i) = 0$ for a certain $i$. Since $a \notin A$, for all $j \neq i$ we have $\tilde{\lambda}(a_j) \neq 0$, i.e. $\tilde{\lambda}^{-1}(0) = \{a_i\}$ and $\deg \tilde{\lambda} = 1$; thus, $\tilde{\lambda}$ is biregular. □

PROOF OF THEOREM 3.2. The proof is by induction on $n$. Since any holomorphic map $\mathcal{E}^1(\mathbb{T}^2) \cong \mathbb{T}^2 \to \mathbb{T}^2 \setminus \{0\}$ is constant, the base of induction is proved.

Assume that the assertion is already proved for some $n = m - 1 \geq 1$. For $a = (a_2, \ldots, a_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$, denote by $\lambda_a = \lambda(\cdot, a_2, \ldots, a_m)$ the restriction of $\lambda$ to the fibre $p^{-1}(a) = \mathbb{T}^2 \setminus \{a_2, \ldots, a_m\}$ of the map $p: \mathcal{E}^m(\mathbb{T}^2) \ni (q_1, q_2, \ldots, q_m) \mapsto (q_2, \ldots, q_m) \in \mathcal{E}^{m-1}(\mathbb{T}^2)$. It is clear that $S := \{a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \mid \lambda_a = \text{const}\}$ is an analytic subset of $\mathcal{E}^{m-1}(\mathbb{T}^2)$, and either (i) $S = \mathcal{E}^{m-1}(\mathbb{T}^2)$ or (ii) $\dim S \leq m - 2$. In case (i), $\lambda = \lambda(q_1, \ldots, q_m)$ does not depend on $q_1$ and may be considered as a holomorphic map $\mathcal{E}^{m-1}(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$; by the induction hypothesis, $\lambda$ is of the desired form. Let us consider case (ii). By Lemma 3.4(a), the set $A$ of all exceptional configurations is contained in a subvariety $M \subset \mathcal{E}^{m-1}(\mathbb{T}^2)$ of dimension $m - 2$. Let $a \in \mathcal{E}^{m-1}(\mathbb{T}^2) \setminus (S \cup M)$. Then $\lambda_a: \mathbb{T}^2 \setminus \{a_2, \ldots, a_m\}$ is a non-constant map. By Lemma 3.4(b), $\lambda_a$ extends to an automorphism $\tilde{\lambda}_a$ of $\mathbb{T}^2$. Clearly, $\tilde{\lambda}_a(t) = m(t - a_i)$ with some $m = m_a \in \mathbb{N}$ and $i = i_a$ (see [7] Chap. V, Sec. V.4.7)). Thus, for all $q = (q_1, \ldots, q_m)$ in the connected, everywhere dense set $\mathcal{E}^m(\mathbb{T}^2) \setminus p^{-1}(S \cup M)$ we have $(\ast) \lambda(q) = m(q_1 - q_i)$ with certain $m = m_q \in \mathbb{N}$ and $i = i_q$. Since $\mathbb{N}$ is finite, $m$ and $i$ do not depend on $q$, and $(\ast)$ holds true for all $q \in \mathcal{E}^m(\mathbb{T}^2)$, which completes the induction step, thus proving the theorem. □

DEFINITION 3.5. For any $m \in \mathbb{N}_+$ and $i \neq j \in \{1, \ldots, n\}$, the map $e_{m;j}: \mathcal{E}^n(\mathbb{T}^2) \ni (q_1, \ldots, q_n) \mapsto m(q_i - q_j) \in \mathbb{T}^2 \setminus \{0\}$ is called a difference. For $\mu = e_{m;j}$, the pair $(q_i, q_j)$ is called the support of $\mu$ and the automorphism $m \in \mathbb{N}_+$ is called the marker of $\mu$. We denote them by $supp \mu$ and $m_\mu$, respectively.

By Theorem 3.2 any non-constant holomorphic map $\mu: \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \setminus \{0\}$ admits a unique representation in the form of a difference, i.e. $\mu = e_{m;j}$ for some uniquely defined $m \in \mathbb{N}_+$ and $i, j \in \{1, \ldots, n\}$.

3.2. A simplicial structure on the set of differences

For any connected complex space $Y$, V. Lin [17] introduced a natural simplicial structure on the set of all non-constant holomorphic functions $Y \to \mathbb{C} \setminus \{0, 1\}$. He used this structure
in order to study $S(n)$-equivariant endomorphisms of the ordered $n$-point configuration spaces of $C$ and $\mathbb{C}P^1$. We modify this idea and define a similar simplicial structure on the set of all non-constant holomorphic maps $Y \to \mathbb{T}^2 \setminus \{0\}$. (V. Lin pointed out that the same construction applies to the set of all non-constant holomorphic maps $Y \to G \setminus \{e\}$, where $e$ is the unity element of a complex Lie group $G$.)

**Definition 3.6.** For a connected complex space $Y$, let $L(Y)$ denote the set of all non-constant holomorphic maps $\mu : Y \to \mathbb{T}^2 \setminus \{0\}$. For $\mu, v \in L(Y)$, we say that $v$ is a proper remainder of $\mu$ and write $v \mid \mu$ if $\mu - v \in L(Y)$. This relation is symmetric, i.e. $v \mid \mu \iff \mu \mid v$.

A subset $\Delta^n = \{\mu_0, \ldots, \mu_m\} \subseteq L(Y)$ is said to be an $m$-simplex if $\mu_i \mid \mu_j$ for all $i \neq j$. Since a subset of a simplex is also a simplex, we obtain a well-defined simplicial complex $L_\Delta(Y)$ with the set of vertices $L(Y)$.

**Lemma 3.7.** Let $f : Z \to Y$ be a holomorphic map of connected complex spaces. Suppose that for each $\lambda \in L(Y)$ the map $f^*(\lambda) := \lambda \circ f : Z \to \mathbb{T}^2 \setminus \{0\}$ is non-constant. Then $f^* : L(Y) \ni \lambda \mapsto f^*(\lambda) \in L(Z)$ is a simplicial map and the restriction of $f^*$ to $\Delta \subseteq L_\Delta(Y)$ is injective. In particular, $\dim f^*(\Delta) = \dim \Delta$.

**Proof.** For any $\lambda \in L(Y)$, the map $f^*(\lambda) : Z \to \mathbb{T}^2 \setminus \{0\}$ is holomorphic and non-constant; hence $f^*(\lambda) \in L(Z)$. If $\mu, v \in L(Y)$ and $\mu \mid v$, then $\lambda = \mu - v \in L(Y)$ and $f^*(\mu) - f^*(v) = f^*(\mu - v) = f^*(\lambda) \in L(Z)$; consequently, $f^*(\mu) \mid f^*(v)$. This implies that $f^*$ is simplicial and injective on any simplex. \qed

**Remark 3.8.** Clearly, for any regular dominant map $f : Y \to Z$ of non-singular irreducible algebraic varieties, we have $f^* (\lambda) \neq \text{const}$ for all $\lambda \in L(Y)$.

Notice that by Theorem 3.2, \(L(\mathcal{E}^n(\mathbb{T}^2))\) is the set of all differences on $\mathcal{E}^n(\mathbb{T}^2)$.

**Lemma 3.9.** Suppose that either $\# \mathcal{M}_+ < 3$ or $s > 1$. Let $\{\mu_0, \ldots, \mu_s\} \in L_\Delta(\mathcal{E}^n(\mathbb{T}^2))$ be an $s$-simplex. Then $m_{\mu_i} = m_{\mu_j}$, $\#(\text{supp } \mu_i \cap \text{supp } \mu_j) = 1$ for all $i \neq j$, and $\#(\text{supp } \mu_0 \cap \cdots \cap \text{supp } \mu_s) = 1$.

**Proof.** Let $\# \mathcal{M}_+ < 3$, $i \neq j$ and let $\mu_i = m_i (q_{i'l'} - q_{i''})$ and $\mu_j = m_j (q_{j'l'} - q_{j''})$. Since $\mu_i \mid \mu_j$, we must have $\mu_i - \mu_j = m_i (q_{i'l'} - q_{i''})$ for some $\mu \in \mathcal{M}_+$ and $k' \neq k''$. Thus, $m_i (q_{i'l'} - q_{i''}) - m_j (q_{j'l'} - q_{j''}) = m(q_{k'l'} - q_{k''})$. Since $\# \mathcal{M}_+ < 3$, the latter relation can be fulfilled only if either $m_i q_{i'l'} - m_i q_{i''} = 0$ or $m_i q_{j'l'} - m_j q_{j''} = 0$. This implies $m_i = m_j$ and we have (*) either $l' = j'$ or $l'' = j''$. If $s = 1$ we have finished the proof. If $s > 2$, then the property $\#(\text{supp } \mu_i \cap \text{supp } \mu_j) = 1$ implies immediately that $(\text{supp } \mu_0 \cap \cdots \cap \text{supp } \mu_s) = 1$. For $s = 2$ we have $\mu_0 = m_i (q_{i'l'} - q_{i''})$, $\mu_1 = m_j (q_{j'l'} - q_{j''})$ and $\mu_2 = m(q_{k'l'} - q_{k''})$. Since $\mu_0 \mid \mu_1$, $\mu_0 \mid \mu_2$ and $\mu_2 \mid \mu_0$, we obtain $(\text{supp } \mu_0 \cap \text{supp } \mu_1) = (\text{supp } \mu_0 \cap \text{supp } \mu_2) = (\text{supp } \mu_2 \cap \text{supp } \mu_0) = 1$. Let $N = (\text{supp } \mu_0 \cap \text{supp } \mu_1 \cap \text{supp } \mu_2)$. Clearly $N \leq 1$; let us show that $N = 0$. Suppose to the contrary that $N = 0$. Relations (*) apply to $\mu_0$ and $\mu_1$, and without loss of generality we can assume that $i' = j'$. For $\mu_1$ and $\mu_2$ the same relations tell us that either $j' = k'$ or $j'' = k''$; since $N = 0$, the first case is impossible and we are left with $j'' = k''$. Finally, we apply (*) to $\mu_0$ and $\mu_2$ and see that either $i' = k'$ or $i'' = k''$, which leads to a contradiction and completes the proof in the case $\# \mathcal{M}_+ < 3$. By
similar straightforward combinatorial computations, one can prove the lemma in the case \( \# \mathfrak{M}_+ = 3 \). □

The \( S(n) \)-action in \( \mathcal{E}^n(\mathbb{T}^2) \) induces an \( S(n) \)-action in \( L(\mathcal{E}^n(\mathbb{T}^2)) \), defined by \( (\sigma \lambda)(q) = \lambda(\sigma^{-1}q) \), which, in turn, induces a simplicial \( S(n) \)-action in \( L_\Delta(\mathcal{E}^n(\mathbb{T}^2)) \) which preserves dimension of simplices; let us describe the orbits of this action.

**Definition 3.10.** We define the following normal forms of simplices of dimension \( s > 1 \): \( \Delta'_m = \{ e_{m,1,2}, \ldots, e_{m,1,s+2} \} \), \( \nabla'_m = \{ e_{m,2,1}, \ldots, e_{m,s+2,1} \} \), where \( m \in \mathfrak{M}_+ \); these simplices are called normal.

**Lemma 3.11.** For \( s > 1 \), there are exactly \( \# \mathfrak{M} \) orbits of the \( S(n) \)-action on the set of all \( s \)-simplices. Every orbit contains exactly one normal simplex.

**Proof.** Since \( e_{m,a,b} \nmid e_{m,b,c} \), Lemma 3.9 shows that for any \( s \)-simplex \( \Delta \in L_\Delta(\mathcal{E}^n(\mathbb{T}^2)) \) there exist \( m \in \mathfrak{M}_+ \) and distinct indices \( a, b_0, \ldots, b_s \) such that \( \Delta \) equals either \( e_{m,a,b_0}, \ldots, e_{m,a,b_s} \) or \( e_{m,b_0,a}, \ldots, e_{m,b_s,a} \). An appropriate permutation \( \sigma \in S(n) \) carries \( \Delta \) to a normal form. □

### 3.3. Regular mappings \( \mathcal{E}^n(\mathbb{T}^2) \to \mathbb{T}^2 \)

**Lemma 3.12.** Any rational map \( \lambda: (\mathbb{T}^2)^n \to \mathbb{T}^2 \) is of the form

\[
\lambda(q_1, \ldots, q_n) = \sum_{i=1}^{n} k_{i,m} q_i + c,
\]

where \( k_{i,m} \in \mathbb{Z} \) and \( c \in \mathbb{T}^2 \). In particular, it is regular.

**Proof.** The proof is by induction on \( n \). Let \( n = 1 \). Since \( \lambda: \mathbb{T}^2 \to \mathbb{T}^2 \) is rational, it extends to a regular map (see [19, Chap. II, Sec. 3.1, Cor. 1]). Any regular self-map of \( \mathbb{T}^2 \) is of the desired form (see Definition 3.1).

Assume that the theorem has already been proved for some \( n = m - 1 \geq 1 \). There is a subset \( \Sigma \subset (\mathbb{T}^2)^m \) of codimension 1 such that \( \lambda \) is regular on \( (\mathbb{T}^2)^m \setminus \Sigma \). Let \( (t_0, z_0) \in (\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma \) and \( D \) be a small neighbourhood of \( z_0 \) in \( \mathbb{T}^2 \). Without loss of generality, we may assume that \( t_0 = 0 \) and \( (0, z) \notin \Sigma \) for all \( z \in D \). For \( (t, z) \in (\mathbb{T}^2 \times D) \setminus \Sigma \), set \( \mu(t, z) = \lambda(t, z) - \lambda(0, z) \) and \( v(t, z) = \mu(t, z) - \mu(t, z_0) \). For any \( z \in D \), we have \( v(0, z) = 0 \) and the map \( t \mapsto v(t, z) \) extends to a holomorphic endomorphism \( v_z \) of \( \mathbb{T}^1 \); moreover, \( v_{z_0}(\mathbb{T}^2) = 0 \). One can find a neighbourhood \( D' \subseteq D \) of \( z_0 \) and a compact subset \( K \subset \mathbb{T}^2 \times D \) such that for all \( z \in D' \) the set \( K \cap (\mathbb{T}^2 \times \{ z \}) \) is a union of two loops that do not meet \( \Sigma \) and generate \( \pi_1(\mathbb{T}^2 \times \{ z \}) \). Moreover, since \( v(\mathbb{T}^2 \times \{ z_0 \}) = 0 \), we may assume that \( v(K) \) is contained in a small contractible neighbourhood of \( 0 \in \mathbb{T}^2 \). Therefore, for any \( z \in D' \) the map \( v_z \) is contractible and trivial. Thus, \( \mu(t, z) - \mu(t, z_0) \equiv 0 \) and \( \lambda(t, z) \equiv \lambda(0, z) + \lambda(t, z_0) - \lambda(0, z_0) \) for all \( z \in D' \). By the uniqueness theorem, the latter identity holds true for all \( (t, z) \in (\mathbb{T}^2 \times (\mathbb{T}^2)^{m-1}) \setminus \Sigma \); the inductive hypothesis applies to \( \lambda(0, z) \) and \( \lambda(t, z_0) \), and completes the proof. □
4. HOLOMORPHIC MAPPINGS OF CONFIGURATION SPACES

The main goal of this section is to prove Theorem 4.2.

**Theorem 4.1.** (a) For $n > 4$ any non-abelian continuous map $F : C^n(T^2) \to C^n(T^2)$ admits a continuous lifting $\tilde{f} : E^n(T^2) \to E^n(T^2)$ (see diagram (1.1)).

(b) For $n > 4$ any continuous lifting $\tilde{f} : E^n(T^2) \to E^n(T^2)$ of a non-abelian continuous map $F : C^n(T^2) \to C^n(T^2)$ is strictly equivariant.

**Proof.** By the covering mapping theorem, (a) follows from Theorem 4.1. Let us prove (b). The diagram (1.1) for $f$ and $F$ implies that there is an epimorphism $\alpha$ of $S(n)$ such that $\delta \circ F_\ast = \alpha \circ \delta$. Clearly, $f(\sigma q) = \alpha(\sigma) f(q)$ for all $q \in E^n(T^2)$ and $\sigma \in S(n)$; moreover, $\alpha$ is an automorphism, otherwise its image is a non-trivial quotient of $S(n)$, which must be abelian since $n > 4$. Then the homomorphism $\delta \circ F_\ast = \alpha \circ \delta$ is also abelian and, by Lemma 2.3, $F_\ast$ is abelian, a contradiction. \[\square\]

Let us show that every strictly equivariant map induces a simplicial map.

**Lemma 4.2.** Let $n > 2$ and $f = (f_1, \ldots, f_k) : E^n(T^2) \to E^n(T^2)$ be a strictly equivariant holomorphic map. Then $f^* : L(E^n(T^2)) \ni \lambda \mapsto \lambda \circ f \in L(E^n(T^2))$ is a well-defined simplicial map; moreover, it preserves dimension of simplices.

**Proof.** By Lemma 3.7, we only must prove that $\mu \circ f = \text{const}$ for any $\mu \in L(E^n(T^2))$. Suppose to the contrary that $\mu \circ f = c \in T^2$. Then $(\mu \circ f)(\sigma q) = c$ for all $\sigma \in S(n)$. Since $f$ is strictly equivariant, there is $\alpha \in \text{Aut} S(n)$ such that $f(\sigma q) = \alpha(\sigma) f(q)$ for all $\sigma \in S(n)$ and $q \in E^n(X)$; thus $c = \mu(f(\sigma q)) = \mu(\alpha(\sigma) f(q))$. By Theorem 3.2, $\mu = m(q_i - q_j)$ for some distinct $i, j$ and $m \in \mathbb{M}$; hence $c = \mu(f(q)) = m(f_i(q) - f_j(q))$. Since $\alpha$ is an automorphism and $n > 2$, there is $\sigma \in S(n)$ such that $\alpha(\sigma^{-1}(i)) = i$ and $\alpha(\sigma^{-1}(j)) = k \neq j$; thus, $c = \mu(\alpha(\sigma) f(q)) = m(f_{\alpha(\sigma^{-1})(i)}(q) - f_{\alpha(\sigma^{-1})(j)}(q)) = m(f_i(q) - f_k(q))$. Therefore, $m(f_i(q) - f_j(q)) = c = m(f_i(q) - f_k(q))$ and $f_j(q) = f_k(q)$, a contradiction. \[\square\]

4.1. **Proof of Theorem 1.2**

We shall prove two theorems, which together yield Theorem 1.2.

**Theorem 4.3.** For $n > 4$, any non-abelian endomorphism $F$ of $C^n(T^2)$ is tame.

**Proof.** By Theorems 1.5 and 4.1, the map $F$ lifts to a strictly equivariant holomorphic map $\tilde{f}$ that fits into the commutative diagram (1.1). Let $\alpha$ be the automorphism of $S(n)$ corresponding to a strictly equivariant map $\tilde{f}$.

By Lemma 4.2, $f^*$ is a dimension preserving simplicial self-map of $L_\Lambda(E^n(T^2))$. Let $\Delta_1 = \{q_1 - q_2, \ldots, q_1 - q_n\}$ and $\Lambda = f^*(\Delta_1)$. By Lemma 3.11, there is $\sigma \in S(n)$ that brings $\Lambda$ to its normal form; without loss of generality, we may assume that this normal form is $\nabla_m = \{m(q_2 - q_1), \ldots, m(q_n - q_1)\}$, where $m \in \mathbb{M}_{+}$. Set $\tilde{f} = f \circ \sigma$; then (**) $\tilde{f}_j = f_1 + m q_1$ for any $j = 1, \ldots, n$. Define the holomorphic map $\tau : E^n(T^2) \to \text{Aut}(T^2)$ by the condition $\tau(q)(z) = \tau(q_1, \ldots, q_n)(z) = -mz + (\tilde{f}_1(q) + mq_1)$, where...
In view of Theorem 4.3, \( (a) \) shows that for \( F_{B} \) trivial, where \( \pi \) for all \( j = q_{1}, \ldots, q_{n} \in E^{n}(T^{2}) \); therefore \( \tau(q_{j})q = f(\sigma q_{j}) = \alpha(\sigma^{-1})f(q_{j}) \), or, what is the same, \( f(q_{j}) = \alpha(\sigma^{-1})\tau(q_{j})q \) for all \( q_{j} \in E^{n}(T^{2}) \). To complete the proof, we must check that \( \tau \) is \( S(n) \)-invariant; that is, we must prove that \( \tau(sq) = \tau(q) \) for all \( q \in E^{n}(T^{2}) \) and all \( s \in S(n) \). For every \( s \in S(n) \) and \( q \in E^{n}(T^{2}) \), we have \( \tau(sq)q = f(\sigma sq) = f(\sigma s^{-1}\sigma q) = \alpha(\sigma s^{-1})f(\sigma q) = \alpha(\sigma^{-1})\tau(q)q \). Thus, \( \tau(sq)sq = \alpha(\sigma^{-1}q)sq \), where \( ((\tau(sq))^{-1} \cdot \tau(q)) \) in \( \text{Aut} \; T^{2} \) is the product in the group \( \text{Aut} \; T^{2} \). Let us notice that for \( n > 1 \) there is a non-empty Zariski open subset \( A \subset E^{n}(T^{2}) \) such that if \( \theta q = \rho q \) for some \( q \in A \), \( \theta \in \text{Aut} \; T^{2} \) and \( \rho \in S(n) \), then \( \theta = \text{id} \) and \( \rho = 1 \). Therefore, equation \( (**) \) implies \( \tau(sq) = \tau(q) \) and \( \alpha(\sigma s^{-1}\sigma^{-1})s = 1 \) for all \( q \in A \) and all \( s \in S(n) \). Clearly, \( \alpha(s) = \sigma^{-1}s\sigma \) and the continuity of \( \tau \) implies that \( \tau(sq) = \tau(q) \) holds true for all \( q \in E^{n}(T^{2}) \) and all \( s \in S(n) \).

**Remark 4.4.** Let \( n = 3 \) or 4. The statement of Theorem 4.3 still holds true if \( F \) is an automorphism. The only changes we need to make in the proof are as follows: instead of our Theorem 4.1, we use Theorem 2 from [10], which states that \( P_{n}(T^{2}) \) is a characteristic subgroup of \( B_{n}(T^{2}) \); moreover, instead of Lemma 4.2, we use Remark 3.8. The rest of the proof is the same.

**Remark 4.5.** (a) Let \( n \geq 2 \) and \( \text{let } F \) be a tame endomorphism of \( C^{n}(T^{2}) \). Then a morphism \( T : C^{n}(T^{2}) \to \text{Aut} \; T^{2} \) in the ‘tame representation’ \( F = F_{T} \) of \( F \) is uniquely determined by \( T \). Indeed, if \( F_{T} = F_{T'} \), then \( T(Q)Q = T'(Q)Q \) and \( (*) |T(Q)|^{-1}T'(Q)Q = Q \) for all \( Q \in C^{n}(T^{2}) \). Furthermore, a torus automorphism is uniquely determined by its values at a generic pair of distinct points; since \( n \geq 2 \), the identity \( (*) \) shows that \( |T(Q)|^{-1}T'(Q) = \text{id} \) for any generic point \( Q \in C^{n}(T^{2}) \) and hence \( T = T' \).

(b) In view of Theorem 4.3, \( (a) \) shows that for \( n > 4 \) any holomorphic non-abelian map \( F : C^{n}(T^{2}) \to C^{n}(T^{2}) \) admits a unique tame representation \( F = F_{T} = F_{T'} \) and \( F \) is regular whenever \( F \) is. By Remark 4.4, the same statement still holds true only whenever \( n = 3, 4 \) and \( F \) is a (biregular) automorphism.

**Definition 4.6.** The map \( s : C(T^{2}) \ni Q = (q_{1}, \ldots, q_{n}) \mapsto s(Q) = (q_{1} + \cdots + q_{n}) \in T^{2} \) is a locally trivial holomorphic fibering whose fibre \( M_{0} = s^{-1}(0) \) is an irreducible quasiprojective variety. The presentation of \( \pi_{1}(M_{0}) \), found by O. Zariski [21], shows that \( H_{1}(M_{0}, \mathbb{Z}) = \mathbb{Z}_{2n} \).

Let \( \gamma : \mathbb{C} \to T^{2} \) be the universal covering; then there exists a holomorphic covering \( h : M_{0} \times \mathbb{C} \ni (Q, \zeta) \mapsto h(Q, \zeta) = (q_{1} + \gamma(\zeta), \ldots, q_{n} + \gamma(\zeta)) \in C^{n}(T^{2}) \).

The following theorem completes the classification of self-maps of \( C^{n}(T^{2}) \).

**Theorem 4.7.** If \( m > 2 \), then a holomorphic map \( F : C^{n}(T^{2}) \to C^{n}(T^{2}) \) is orbit-like if and only if it is abelian.

**Proof.** Let \( F \) be abelian. Clearly, \( H_{1}(C^{n}(T^{2}), \mathbb{Z}) = B_{n}(T^{2})/B_{n}^{0}(T^{2}) = \mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \). As \( B_{n}(T^{2}) \) has no elements of a finite order, the image \( \text{Im} \; F_{n} \) of the induced homomorphism \( F_{n} : B_{n}(T^{2}) \to B_{n}(T^{2}) \) is a free abelian group. Since \( \pi_{1}(M_{0})/(\pi_{1}(M_{0}))' = \mathbb{Z}_{2n} \), any homomorphism \( \pi_{1}(M_{0}) \to \text{Im} \; F_{n} \) is trivial; in particular, the homomorphism \( (F \circ h)_{e} \) is trivial, where \( h : M_{0} \times \mathbb{C} \to C^{n}(T^{2}) \) is the above-defined covering. This implies that there
is a holomorphic map \( f = (f_1, \ldots, f_m) : M_0 \times \mathbb{C} \to \mathcal{E}^m(\mathbb{T}^2) \) such that \( F \circ h = p \circ f \), where \( p : \mathcal{E}^m(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2) \) is the standard projection. The induced homomorphism \( f_\ast : \pi_1(M_0) \to P_0(\mathbb{T}^2) \) is trivial; thus, for any \( j \), the map \( (q_j - q_1) \circ f : M_0 \times \mathbb{C} \to \mathcal{E}^m(\mathbb{T}^2) \) is contractible and lifts to a holomorphic map \( g_j : M_0 \times \mathbb{C} \to \mathbb{P} \mathcal{E}^m(\mathbb{T}^2) \backslash \{0\} \) is contractible and lifts to a holomorphic map \( g_j : M_0 \times \mathbb{C} \to \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \). Since \( M_0 \times \mathbb{C} \) is algebraic and irreducible, Liouville’s theorem shows that \( g_j \) is constant and, therefore, \( f_j - f_1 = (q_j - q_1) \circ f = c_j \in \mathbb{T}^2 \backslash \{0\} \). Thus, \( f(q) = (0 + f_1(q), c_2 + f_1(q), \ldots, c_m + f_1(q)) \) and \( F \) is orbit-like.

Suppose now that \( F \) is orbit-like. To prove that \( F \) is abelian, it suffices to show that for any point \( q \in \mathcal{C}^m(\mathbb{T}^2) \), the fundamental group of any connected component of the \((\text{Aut} \mathbb{T}^2)\)-orbit \( O_q = (\text{Aut} \mathbb{T}^2)(q) \) is abelian. For \( m > 2 \), any component of \( O_q \) is diffeomorphic to the orbit \( O_q \) of the action of \( \mathbb{T}^2 \) in \( \mathcal{C}^m(\mathbb{T}^2) \) by translations. The latter orbit \( O_q \) is a quotient of \( \mathbb{T}^2 \) by a finite subgroup and hence is homeomorphic to \( \mathbb{T}^2 \). Thus, \( \pi_1(O_q) = \mathbb{Z}^2 \). □

We skip the proof of the next result about abelian maps.

**Proposition 4.8.** (a) Any abelian map \( f : \mathcal{C}^m(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2) \) is homotopically equivalent to a composition \( g \circ s \) of the standard map \( s : \mathcal{C}^2(\mathbb{T}^2) \to \mathbb{T}^2 \) and an appropriate continuous map \( g : \mathbb{T}^2 \to \mathcal{C}^m(\mathbb{T}^2) \).

(b) Any holomorphic map \( F : \mathbb{T}^2 \to \mathcal{C}^m(\mathbb{T}^2) \) is orbit-like.

### 5. Biregular Automorphisms

Here we describe all biregular automorphisms of the algebraic variety \( \mathcal{C}^m(\mathbb{T}^2) \).

**Lemma 5.1.** Any regular map \( R : \mathcal{C}^m(\mathbb{T}^2) \to \mathbb{T}^2 \) is of the form

\[
R([q_1, \ldots, q_n]) = \sum_{m \in \mathbb{N}} k_m m(q_1 + \cdots + q_n) + c,
\]

where \( m \in \mathbb{Z} \) and \( c \in \mathbb{T}^2 \).

**Proof.** Consider the map \( r = R \circ p \), where \( p : \mathcal{E}^m(\mathbb{T}^2) \to \mathcal{C}^m(\mathbb{T}^2) \) is the standard projection. By Lemma 3.12, \( r(q_1, \ldots, q_n) = \sum_{m=1}^{n} \sum_{k \in \mathbb{N}} k_m m(q_1 + \cdots + q_n) + c \). Since \( r \) is \( S(n) \)-invariant, it follows that \( k_{1,m} = \cdots = k_{n,m} = k_m \). Thus, \( r(q_1, \ldots, q_n) = \sum_{m \in \mathbb{N}} k_m m(q_1 + \cdots + q_n) + c \) and \( R([q_1, \ldots, q_n]) = \sum_{m \in \mathbb{N}} k_m m(q_1 + \cdots + q_n) + c \). □

**Theorem 5.2.** For \( n > 2 \), any biregular automorphism \( F \) of \( \mathcal{C}^m(\mathbb{T}^2) \) is of the form \( F(Q) = AQ \), where \( A \in \text{Aut} \mathbb{T}^2 \).

**Proof.** By Theorem 1.2 and Remarks 4.4 and 4.5, there is a unique regular map \( T : \mathcal{C}^m(\mathbb{T}^2) \to \text{Aut} \mathbb{T}^2 \) such that \( F(Q) = T(Q)Q \) for all \( Q = [q_1, \ldots, q_n] \in \mathcal{C}^m(\mathbb{T}^2) \).

Since \( T(Q) \in \text{Aut} \mathbb{T}^2 \), there exist a regular map \( R : \mathcal{C}^m(\mathbb{T}^2) \to \mathbb{T}^2 \) and \( m_0 \in \mathbb{N} \) such that \( T(Q)z = m_0 z + R(Q) \) for all \( z \in \mathbb{T}^2 \) (see [1] Chap. V, Sec. V.4.7). Together with Lemma 5.1, this implies that for any \( z \in \mathbb{T}^2 \) we have \( T(Q)z = m_0 z + \sum_{m \in \mathbb{N}} k_m m(q_1 + \cdots + q_n) + c \), where \( m_0 \in \mathbb{N} \), \( m \in \mathbb{Z} \), and \( c \in \mathbb{T}^2 \) do not depend on \( z \) and \( Q \). Recall that \( s(Q) = q_1 + \cdots + q_n \) for \( Q = [q_1, \ldots, q_n] \) and set \( s_1 = s \circ F \), i.e., \( s_1(Q) = (s \circ F)(Q) = s(T(Q)Q) = T(Q)q_1 + \cdots + T(Q)q_n \). Using the explicit formula for \( T(Q)z \) for
Here we construct configuration spaces of the universal Teichmüller family of tori and describe their holomorphic self-maps.

The Teichmüller space \( T(1, 1) \) of tori with one marked point is isomorphic to the upper half plane \( \mathbb{H}^+ \). The group \( H = \mathbb{Z} \times \mathbb{Z} \) acts discontinuously and freely in the space \( V = T(1, 1) \times \mathbb{C} = \mathbb{H}^+ \times \mathbb{C} \) by weighted translations \((\tau, z) \mapsto (\tau, z + l + m\tau), (l, m) \in H\). Let \( V(1, 1) = V/H \); the map \( \psi : V \to V(1, 1) \) is a covering, and the holomorphic projection \( \pi : V(1, 1) \to \mathbb{H}^+ = T(1, 1) \) is called the universal Teichmüller family of tori with one marked point (see [4] Sec. 4.11)]. All fibres \( \pi^{-1}(\tau) \) are tori; each of them carries a natural group structure, marked points are neutral elements and they form a holomorphic section of \( \pi \).

**Definition 6.1.** Let \( C_\pi^0(V(1, 1)) \) be the complex subspace of the configuration space \( C^0(V(1, 1)) \) consisting of all \( \mathbf{Q} = [q_1, \ldots, q_n] \) in \( C^0(V(1, 1)) \) such that \( \pi(q_1) = \cdots = \pi(q_n) \). Define the holomorphic projection \( \rho : C_\pi^0(V(1, 1)) \to T(1, 1) \) by \( \rho(\mathbf{Q}) = \pi(q_1) = \cdots = \pi(q_n), \mathbf{Q} = [q_1, \ldots, q_n] \in C_\pi^0(V(1, 1)) \); the triple \([\rho, C_\pi^0(V(1, 1)), T(1, 1)]\), or simply \( C_\pi^0(V(1, 1)) \to T(1, 1) \), is called the fibred configuration space of the universal Teichmüller family \( \pi : V(1, 1) \to T(1, 1) \)(cf. M. Engber [5]). A fibred morphism of fibred configuration spaces is a holomorphic map \( F : C_\pi^0(V(1, 1)) \to C_\pi^0(V(1, 1)) \) which respects the projection \( \rho \), that is, \( \rho \circ F = \rho \). One can easily check that \( C_\pi^0(V(1, 1)) \) is a connected complex manifold.

**Definition 6.2.** Let \( g : C_\pi^0(V(1, 1)) \to V(1, 1) \) be a fibred morphism. Any point \( \mathbf{Q} \in C_\pi^0(V(1, 1)) \) belongs to a certain fibre \( \rho^{-1}(\tau) \), which is the configuration space \( C^0(\pi^{-1}(\tau)) \) of the torus \( T_\tau^2 = \pi^{-1}(\tau) \); so \( \mathbf{Q} \) may be viewed as an n-point subset of \( T^2_\tau \). Since \( g \) is a fibred morphism, \( g(\mathbf{Q}) \) is a point of the same torus \( T_\tau^2 \); thus, \( \mathbf{Q} + g(\mathbf{Q}) \) and \( -\mathbf{Q} + g(\mathbf{Q}) \) are well-defined n-point subsets of \( T^2_\tau \), or which is the same, points of \( C^0(T^2_\tau) \subset C_\pi^0(V(1, 1)) \). This provides us with two fibred maps \( G_\pm = \pm \text{Id} + g : C_\pi^0(V(1, 1)) \to C_\pi^0(V(1, 1)) \) defined by \( \mathbf{Q} \mapsto \pm \mathbf{Q} + g(\mathbf{Q}) \). It can be easily shown that the fibred maps \( G_\pm \) are holomorphic.
One can prove statements analogous to Theorem 1.2 for the case of fibred morphisms. For instance, we sketch the proof of the following theorem.

**Theorem 6.3.** Let \( n > 4 \) and \( F : C^n_\pi(V(1, 1)) \to C^n_\pi(V(1, 1)) \) be a fibred non-abelian morphism. There exists a fibred morphism \( g : C^n_\pi(V(1, 1)) \to V(1, 1) \) such that \( F \) is either \( \text{Id} + g \) or \( -\text{Id} + g \).

**Sketch of Proof.** According to Theorem 1.2, for any \( \tau \in T(1, 1) \) there exists a unique holomorphic map \( T_\tau : \rho^{-1}(\tau) \to \text{Aut} \pi^{-1}(\tau) \) such that \( F(Q) = T_\tau(Q)Q \) for any \( Q \in \rho^{-1}(\tau) \subset C^n_\pi(V(1, 1)) \). There is no complex multiplication on a generic torus. Thus, for any generic \( \tau \in T(1, 1) \) and any \( Q \in \rho^{-1}(\tau) \), there exists \( c_\tau(Q) \in \pi^{-1}(\tau) \) such that the automorphism \( T_\tau(Q) \) maps a point \( z \in \pi^{-1}(\tau) \) either to \( z + c_\tau(Q) \) or to \( -z + c_\tau(Q) \).

Since the representation of \( T_\tau(Q) \) is unique, \( F \) is continuous and the fibred configuration spaces are irreducible, it can be easily seen that for all \( Q \in C^n_\pi(V(1, 1)) \) only one of the above-mentioned possibilities takes place; moreover, there exists a fibred morphism \( g : C^n_\pi(V(1, 1)) \to V(1, 1) \) such that \( c_\tau(Q) = g(Q) \).

**Remark 6.4.** For an automorphism \( F \) the above statement holds true for \( n = 3, 4 \).

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**References**


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