
ABSTRACT. — We consider the Lagrangian
\[ L(y_1, \dot{y}_1, y_2, \dot{y}_2, q, \dot{q}) = \frac{1}{2} (\dot{y}_2^2 - \omega_2^2 y_2^2) + \frac{1}{2} (\dot{y}_1^2 - \omega_1^2 y_1^2) + \frac{1}{2} \dot{q}^2 + (1 + \delta(y_1, y_2))V(q), \]
where \( V \) is non-negative, periodic in \( q \) and such that \( V(0) = V'(0) = 0 \). We prove, using critical point theory, the existence of infinitely many solutions of the corresponding Euler–Lagrange equations which are asymptotic, as \( t \to \pm \infty \), to invariant tori in the center manifold of the origin, that is, to solutions of the form \( q(t) = 0, y_1(t) = R \cos(\omega_1 t + \phi_1), y_2(t) = R \cos(\omega_2 t + \phi_2) \).

KEY WORDS: Heteroclinic orbits; critical point theory; invariant tori; center manifold.


1. INTRODUCTION

The study of solutions asymptotic to invariant manifolds is important in order to understand the global dynamics of Hamiltonian systems. It is well known that they can indicate the existence of a complicated—even chaotic—behavior for the system under consideration.

Global variational methods have been employed by many authors to prove existence of solutions homoclinic or heteroclinic to hyperbolic stationary points (see [7, 9, 18, 21]), and to prove chaotic behavior for time-dependent systems (always having a hyperbolic stationary point); see [20] and [12]. The same kind of techniques have also been employed to study existence of solutions asymptotic to periodic orbits and to more general invariant manifolds having some kind of minimizing property (see [19] [8]).

More recently Patrick Bernard [1] [2] has considered a class of Hamiltonian systems having a saddle-center stationary point and has proved existence of solutions homoclinic to periodic orbits in the (global) center manifold. Motivated by such papers we have further investigated the situation in [10] [11]. All these papers consider autonomous Lagrangian systems of the form
\[ L(x, \dot{x}, q, \dot{q}) = \frac{1}{2} (x^2 - \omega^2 x^2) + \frac{1}{2} \dot{q}^2 + V(x, q), \]
where \( x \in \mathbb{R}, q \in \mathbb{T}, V(x, q) \geq V(x, 0) = 0 \) for all \( x, q \) (actually, Bernard considers more general Hamiltonian systems, and his results apply also to \( q \in \mathbb{T}^k \)). In this case the center manifold of the stationary point is 2-dimensional and is foliated by periodic solutions.

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In this paper we extend some of the above results to cover a class of Lagrangian systems having a 4-dimensional center manifold by considering Lagrangians of the form

$$
L(y_1, \dot{y}_1, y_2, \dot{y}_2, q, \dot{q}) = \frac{1}{2} (\dot{y}_1^2 - \omega_1^2 y_1^2) + \frac{1}{2} (\dot{y}_2^2 - \omega_2^2 y_2^2) + \frac{1}{2} \dot{q}^2 + (1 + \delta(y_1, y_2))V(q)
$$

with $y_1, y_2 \in \mathbb{R}$, $q \in \mathbb{T} = \mathbb{R}/[0, 2\pi]$. We look for solutions of the corresponding Euler–Lagrange equations

$$(1.1) \begin{cases}
\ddot{q} = (1 + \delta(y_1, y_2))V'(q), \\
\dot{y}_1 + \omega_1^2 y_1 = \frac{\partial}{\partial y_1} \delta(y_1, y_2)V(q), \\
\dot{y}_2 + \omega_2^2 y_2 = \frac{\partial}{\partial y_2} \delta(y_1, y_2)V(q).
\end{cases}$$

Note that the Lagrangian is autonomous and hence the total energy

$$
\frac{1}{2} (\dot{y}_1^2 + \omega_1^2 y_1^2) + \frac{1}{2} (\dot{y}_2^2 + \omega_2^2 y_2^2) + \frac{1}{2} \dot{q}^2 - (1 + \delta(y_1, y_2))V(q) = E
$$

is conserved along any solution.

Since we are assuming that $V$ has a strict global minimum at $q = 0$ and it is periodic in $q \in \mathbb{R}$, such a system admits, for all $R_1, R_2 \geq 0$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, the solutions

$$
\begin{cases}
q(t) \equiv 0, \\
y_1(t) = R_1 \cos(\omega_1 t + \varphi_1), \\
y_2(t) = R_2 \cos(\omega_2 t + \varphi_2),
\end{cases}
$$

having total energy $E = (\omega_1^2 R_1^2 + \omega_2^2 R_2^2)/2$. The set

$$
\mathcal{T}_{R_1, R_2} = \{(y_1 = R_1 \cos(\varphi_1), \ y_1 = -\omega_1 R_1 \sin(\varphi_1), \\
y_2 = R_2 \cos(\varphi_2), \ y_2 = -\omega_2 R_2 \sin(\varphi_2), \ q = 0, \ \dot{q} = 0) \mid (\varphi_1, \varphi_2) \in \mathbb{T}^2\}
$$

is an invariant torus for the system (1.1).

We look for solutions asymptotic to such tori, that is, solutions $y_1(t), y_2(t), q(t)$ such that

$$
\lim_{t \to +\infty} \text{dist}((y(t), \dot{y}(t), q(t), \dot{q}(t)), \mathcal{T}_{R_{1+}, R_{2+}}) = 0, \\
\lim_{t \to -\infty} \text{dist}((y(t), \dot{y}(t), q(t), \dot{q}(t)), \mathcal{T}_{R_{1-}, R_{2-}}) = 0.
$$

By energy conservation,

$$(1.2) \quad E = \omega_1^2 R_{1+}^2 + \omega_2^2 R_{2+}^2 = \omega_1^2 R_{1-}^2 + \omega_2^2 R_{2-}^2.$$

This kind of problem has been studied using perturbative methods by many authors mainly in the case of a 2-dimensional center manifold (see [16, 17, 15, 13], and the more
In these papers it is proved that there exist many solutions homoclinic to invariant tori of energy $E > 0$ and “not too small” (one does not expect to find solutions homoclinic to the stationary point $P_0 = (y_1 = 0, \dot{y}_1 = 0, y_2 = 0, \dot{y}_2 = 0, q = 0, \dot{q} = 0)$, which has one-dimensional stable and unstable manifold).

In this paper we obtain existence of infinitely many solutions, homoclinic to the invariant tori, under different kinds of assumptions on $\delta$ and $\omega_1$.

We remark that the condition that a solution is asymptotic to $T_{R_1, R_2}$ as $t \to \pm \infty$ can also be formulated by saying that $(y_1(t), y_2(t), q(t))$ is a solution of \((1.1)\) which satisfies, for some $f_{1\pm}, f_{2\pm} \in [0, 2\pi)$ and $k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{t \to \pm \infty} |y_1(t) - R_{1\pm} \cos(\omega_1 t + f_{1\pm})| = 0,$$

$$\lim_{t \to \pm \infty} |y_2(t) - R_{2\pm} \cos(\omega_2 t + f_{2\pm})| = 0,$$

$$\lim_{t \to \infty} q(t) = 0, \quad \lim_{t \to -\infty} q(t) = 2k\pi. \quad (1.3)$$

Throughout this paper, we will assume that $V$ and $\delta \in C^2(\mathbb{R})$ are such that

(V1) $V(q + 2\pi) = V(q)$ for all $q \in \mathbb{R}$;

(V2) $0 = V(0) < V(q)$ for all $q \in \mathbb{R} \setminus 2\pi \mathbb{Z}$;

(V3) $V''(0) = \mu > 0$;

(V4) $V'(q)q > 0$ for all $q \in [-\bar{\eta}, \bar{\eta}]$, $q \neq 0$;

($\delta_1$) $-1 < \delta \leq \delta(y_1, y_2) \leq \delta$ for all $(y_1, y_2) \in \mathbb{R}^2$;

($\delta_2$) $\|\nabla \delta(y_1, y_2)\| \leq C$ for all $(y_1, y_2) \in \mathbb{R}^2$ and for some positive constant $C$;

($\delta_3$) $|\langle \nabla \delta(y_1, y_2), (y_1, y_2) \rangle| \leq 2\alpha$ for all $(y_1, y_2) \in \mathbb{R}^2$ where $1 + \delta - \alpha > 0$.

We will often use the notation

$$V_\eta = \min\{V(s) : s \in [\eta, 2\pi - \eta]\} > 0, \quad \eta > 0. \quad (1.4)$$

**Remark 1.5.** Let us point out, for future reference, that \((V3)\) implies that there is an $\eta_0 \in (0, \bar{\eta}/2)$ such that

$$\mu/2 \leq V''(q) \leq 2\mu \quad \text{for all } |q| \leq \eta_0. \quad (1.6)$$

**Remark 1.7.** The above assumptions on $V$ and $\delta$ are satisfied, for example, if

$$V(q) = 1 - \cos q, \quad \delta(y_1, y_2) = \delta_{\infty} \arctan \lambda^2(y_1 + y_2) \quad (1.8)$$

provided $0 < \delta_{\infty} < 4/(2\pi + \lambda^2)$.

**Remark 1.9.** If $\delta(y_1, y_2) \equiv \delta_0$ is a constant, then under assumptions \((V1)-(V2)\) there is a solution $q_0(t)$ of $\ddot{q} = (1 + \delta_0)V'(q)$ homoclinic to 0 (see, for example \((1.1)-(1.3)\)) and hence $(R_1 \cos(\omega_1 t + f_1), R_2 \cos(\omega_2 t + f_2), q_0(t))$ is a solution of our problem for all $R_1, R_2 \geq 0$, $f_1, f_2 \in [0, 2\pi)$.

A first result is the existence of at least one solution asymptotic to an invariant torus $T_{R_1, R_2}$ as $t \to -\infty$ and to $T_{R_1, R_2}$ as $t \to +\infty$. 
THEOREM 1.10. Assume $V$ and $\delta$ satisfy (V1)–(V4) and (\delta4) $\tilde{\delta} + 2(\alpha - \delta) < 1$.

Then there is a solution of the system (1.1) satisfying (1.3) with $k = 1$ for some $R_{1\pm}, R_{2\pm} \in [0, +\infty)$ satisfying (1.2), and for some $f_{1\pm}, f_{2\pm} \in [0, 2\pi]$.

Note that we cannot prescribe, in the above theorem, the tori to which the solution we find is asymptotic, and not even its energy. We also observe that the above system should have a lot of solutions like the one we find.

In the next two theorems, we give, under different additional assumptions, a more precise result: the existence of infinitely many homoclinic solutions, that is, of solutions asymptotic as time goes to $-\infty$ and $+\infty$ ($\varphi_i$ in the statement of the theorem).

THEOREM 1.11. Assume $V$ and $\delta$ satisfy (V1)–(V4) (\delta1)–(\delta3) and (\delta4) $\tilde{\delta} - \delta + \alpha \leq \frac{\bar{\eta}(1 + \tilde{\delta} - \alpha)^{3/2}}{2\pi^2 + (1 + \tilde{\delta})^2} \frac{\sqrt{V_{\delta/2}}}{2}$

Assume moreover that (E) system (1.1) has no zero energy solutions satisfying $\lim_{t \to \pm \infty} q(t) = 0$ and $\lim_{t \to \mp \infty} \varphi(t) = 2\pi$.

Then for any $\varphi_1 \in (0, 2\pi)$ and (o1) any $\varphi_2 \in (0, 2\pi)$ if $\omega_2/\omega_1 \notin \mathbb{Q}$.

(\omega2) $\varphi_2 = j\varphi_1$ if $\omega_2/\omega_1 = j \in \mathbb{Q}$.

there exist $R_1$ and $R_2$ and a solution $(y_1(t), y_2(t), q(t))$ of (1.1), satisfying (1.2) with $k = 1, R_{1\pm} = R_1, R_{2\pm} = R_2, \omega_1^2 R_1^2 + \omega_2^2 R_2^2 > 0, \varphi_i = f_i - f_{i+} \mod 2\pi. Furthermore, q(t) \in [0, 2\pi]$ for all $t$.

REMARK 1.12. Assumption (\delta4) has already been used in (10) and it is satisfied if $V$ and $\delta$ are of the form (\delta3) with $\delta_{\infty} < 0.02$ and $\lambda = 1$. We also remark that (\delta4) implies (\delta4).

Assumption (E) is used in order to prove the existence of infinitely many solutions. If it is violated, we obtain a solution homoclinic to the stationary point $P_0 = (y_1 = 0, \dot{y}_1 = 0, y_2 = 0, \dot{y}_2 = 0, q = 0, \dot{q} = 0)$. As already remarked, this should not be the case in general (see (3)).

The next theorem shows the same result under a different set of assumptions. In order to state this result, let us introduce the notation

$$\bar{c} = 2\pi^2 + (1 + \tilde{\delta})V_{\infty}, \quad K = 1 + \tilde{\delta} - \alpha, \quad \tilde{K} = \max\{\bar{c}/K, \bar{c}\} > 1,$$

$$V = \bar{\eta} \sqrt{\frac{1 + \bar{\delta}(0, 0)}{2}} V_{\delta/2}, \quad C_{\varphi_i} = 2 + \left|\cos \varphi_i \right|$$
THEOREM 1.14. Assume $V$ and $\delta$ satisfy $(V1)$–$(V4)$, $(\delta1)$, $(\delta2)$, and assume that

$$(1.15) \quad C_{\psi_i} \frac{\|\nabla \delta\|_{\infty}}{\omega_i} \max\{1, \|\nabla \delta\|_{\infty}\} < \frac{V}{2\bar{K}} \quad \forall i = 1, 2.$$ 

Then there exist $R_1$ and $R_2$ and a solution $(y_1(t), y_2(t), q(t))$ of $(1.1)$ satisfying $(1.3)$ with $k = 1, R_{1\pm} = R_1, R_{2\pm} = R_2, \omega_1^2 R_1^2 + \omega_2^2 R_2^2 > 0, \bar{\psi}_i \equiv f_i_+ - f_i_- \mod 2\pi$. Furthermore, $q(t) \in [0, 2\pi]$ for all $t$.

REMARK 1.16. The above theorem states that one can find homoclinic solutions for all “phase shifts” uniformly far from 0 and $2\pi$ (so that $C_{\psi_i}$ is uniformly bounded) provided $\omega_i$ are large enough or $\delta$ is small enough.

Solutions of our problem will be found using variational methods as limits as $T \to +\infty$ of solutions of the following boundary value problem:

$$(PT) \quad \begin{cases}
\ddot{q} = (1 + \delta(y_1, y_2)) V'(q), \\
\dot{y}_1 + \omega_2^2 y_1 = \frac{\partial}{\partial y_1} \delta(y_1, y_2) V(q), \\
\dot{y}_2 + \omega_2^2 y_2 = \frac{\partial}{\partial y_2} \delta(y_1, y_2) V(q),
\end{cases}
$$

$$(2.1) \lim_{t \to -\infty} |y_i(t) - R_i - \cos(\omega_i t + f_i_+)| = 0,
$$

$$(2.2) \lim_{t \to +\infty} |y_i(t) - R_i + \cos(\omega_i t + f_i_-)| = 0
$$

for $i = 1, 2$ are satisfied by all solutions of

$$(2.3) \quad y_i(t) = \frac{1}{\omega_i} \int_{-\infty}^{t} \frac{\partial}{\partial y_i} \delta(y_1, y_2) V(q) \sin \omega_i (t - s) \, ds + R_i - \cos(\omega_i t + f_i_-)$$

such that $\int_{\mathbb{R}} V(q(t)) \, dt < +\infty$. Indeed, all solutions of $(2.2)$ can be expressed, for suitable $R_{i\pm}, f_{i\pm}, i = 1, 2$, as

$$(2.3) \quad y_i(t) = \frac{1}{\omega_i} \int_{-\infty}^{t} \frac{\partial}{\partial y_i} \delta(y_1, y_2) V(q) \sin \omega_i (t - s) \, ds + R_i - \cos(\omega_i t + f_i_-).$$
or

\[ y_i(t) = -\frac{1}{\omega_i} \int_0^\infty \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \sin \omega_i(t - s) \, ds + R_i t + f_i, \]

and passing to the limit as \( t \to \pm \infty \) in these expressions we obtain (2.1).

Now we give some preliminary estimates on the solutions of the boundary value problem (PT). In the following we will often use the notation

\[ \delta(y_1, y_2) = \int_0^T (\dot{y}_1^2 - \omega_i^2 \dot{y}_2^2) \, ds. \]

**LEMMA 2.6.** Assume that \( T \neq 2\pi N/\omega_i \) for all \( N \) and \( i = 1, 2 \) and that \( y_i \) is a solution of

\[ \begin{cases} \ddot{y}_i + \omega_i^2 y_i = \frac{\partial}{\partial y_i} \delta(y_1, y_2)V(q), \\ y_i(0) - y_i(T) = \dot{y}_i(0) - \dot{y}_i(T) = 0. \end{cases} \]

Then for \( i = 1, 2 \),

\[ \|y_i\|_\infty \leq \frac{1}{\omega_i} \left( 2 + \left| \cot \frac{\omega_i T}{2} \right| \right) \|\nabla \delta\|_\infty \int_0^T V(q(t)) \, dt, \]

\[ \|\dot{y}_i\|_\infty \leq \left( 2 + \left| \cot \frac{\omega_i T}{2} \right| \right) \|\nabla \delta\|_\infty \int_0^T V(q(t)) \, dt, \]

\[ |Q_i(y_i)| \leq \frac{1}{\omega_i} \left( 2 + \left| \cot \frac{\omega_i T}{2} \right| \right) \|\nabla \delta\|_\infty^2 \left( \int_0^T V(q(t)) \, dt \right)^2, \]

\[ \left| \frac{1}{2} Q_1(y_1) + \frac{1}{2} Q_2(y_2) \right| \leq \alpha \int_0^T V(q(t)) \, dt. \]

**PROOF.** An easy calculation shows that for all \( T \neq 2\pi N/\omega_i \) the solution of (2.7) is given by

\[ y_i(t) = \frac{1}{\omega_i} \int_0^t \frac{\partial}{\partial y_i} \delta(y_1, y_2)V(q) \sin \omega_i(t - s) \, ds 
\]

\[ + \left( \frac{H_i}{2} + \frac{L_i}{2} \cot \frac{\omega_i T}{2} \right) \cos \omega_i t + \left( \frac{L_i}{2} - \frac{H_i}{2} \cot \frac{\omega_i T}{2} \right) \sin \omega_i t, \]

where

\[ H_i = \frac{1}{\omega_i} \int_0^T \frac{\partial}{\partial y_i} \delta(y_1, y_2)V(q) \sin \omega_i(T - s) \, ds, \]

\[ L_i = \frac{1}{\omega_i} \int_0^T \frac{\partial}{\partial y_i} \delta(y_1, y_2)V(q) \cos \omega_i(T - s) \, ds. \]
We first observe that

$$|H_i| \leq \frac{1}{\omega_i} \|\nabla \delta\|_\infty \int_0^T V(q), \quad |L_i| \leq \frac{1}{\omega_i} \|\nabla \delta\|_\infty \int_0^T V(q).$$

Then we have the following estimates:

$$\|y_i\|_\infty \leq \frac{1}{\omega_i} \|\nabla \delta\|_\infty \int_0^T V(q) \leq \frac{1}{\omega_i} \|\nabla \delta\|_\infty \int_0^T V(q) \leq \frac{1}{\omega_i} \|\nabla \delta\|_\infty \int_0^T V(q).$$

Similarly we can estimate $\|\dot{y}_i\|_\infty$. Finally, an integration by parts shows that

$$Q_i(y_i) = \int_0^T -\frac{\partial}{\partial y_i} \delta(y_1, y_2) y_i(t) V(q(t)) dt \quad \text{for all } i = 1, 2,$$

so that

$$\left| \int_0^T (\dot{y}_i^2 - \omega_i^2 y_i^2) dt \right| \leq \|y_i\|_\infty \int_0^T \left| \frac{\partial}{\partial y_i} \delta(y_1, y_2) \right| V(q(t)) dt \leq \frac{1}{\omega_i} \left( 2 + \cot \frac{\omega_i T}{2} \right) \|\nabla \delta\|_\infty^2 \left( \int_0^T V(q(t)) dt \right)^2,$$

and

$$\left| \frac{1}{2} Q_1(y_1) + \frac{1}{2} Q_2(y_2) \right| = \left| \int_0^T -\frac{1}{2} \nabla \delta(y_1, y_2, (y_1, y_2)) V(q(t)) dt \right| \leq \alpha \int_0^T V(q(t)) dt. \quad \square$$

Let us remark that Lemma 2.25 requires the condition $T \neq 2\pi N/\omega_i$ for $i = 1, 2$. The following lemma shows that there exists a sequence $T_N \to +\infty$ such that $\omega_i T_N \neq 2\pi N$ for $i = 1, 2$.

**Lemma 2.8.** Fix $\varphi_1, \varphi_2 \in (0, 2\pi)$ and assume $\omega_2/\omega_1 \leq 1$ (otherwise exchange $\omega_1$ with $\omega_2$). Then, choosing $T_N = (2\pi N + \varphi_1)/\omega_1$, we have $\omega_1 T_N \bmod 2\pi = \varphi_1$ for all $N \in \mathbb{N}$.

Moreover, we can extract a subsequence $T_{N_k}$ such that

$$\omega_2 T_{N_k} \bmod 2\pi \to \varphi_2 \quad \text{as } k \to +\infty \text{ if } \omega_2/\omega_1 \notin \mathbb{Q},$$

$$\omega_2 T_{N_k} \bmod 2\pi = (\omega_2/\omega_1) \varphi_1 \quad \text{for all } k \text{ if } \omega_2/\omega_1 \in \mathbb{Q}.$$

**Proof:** If $\omega_2/\omega_1 \notin \mathbb{Q}$, then the set $\{\omega_2 T_N \bmod 2\pi\}_{N \in \mathbb{N}}$ is dense in $(0, 2\pi)$. Therefore, for any fixed $\varphi_2 \in (0, 2\pi)$, there is a sequence $\{N_k\}_{k \in \mathbb{N}}$ such that $\omega_2 T_{N_k} = 2\pi l_k + \varphi_2 k$ with $l_k \in \mathbb{N}$ and $\varphi_2 k \in (\varphi_2 - 1/k, \varphi_2 + 1/k)$, so that $\omega_2 T_{N_k} \bmod 2\pi \to \varphi_2$ as $k \to +\infty$. If $\omega_2/\omega_1 \in \mathbb{Q}$, then $\omega_2 T_N \bmod 2\pi$ is a rational multiple of $\varphi_2$. Therefore, for all $k \in \mathbb{N}$, $\omega_2 T_k \bmod 2\pi = (\omega_2/\omega_1) \varphi_2$. Thus, $\omega_2 T_{N_k} \bmod 2\pi = \omega_2/\omega_1 \varphi_2$ for all $k \in \mathbb{N}$. The result follows.
Let \( \omega_2/\omega_1 \in \mathbb{Q} \). If \( \omega_2/\omega_1 = 1 \) the statement is trivial. Otherwise we can assume \( \omega_2 = (n/m)\omega_1 \) with \( n, m \in \mathbb{N} \) coprime and \( m > n \). Then, setting \( N_k = km \) for all \( k \), we have \( \omega_2 T_{N_k} = (n/m)(2\pi km + \varphi_1) = 2\pi nk + (n/m)\varphi_1 \), that is, \( \omega_2 T_{N_k} \mod 2\pi = (n/m)\varphi_1 = (\omega_2/\omega_1)\varphi_1 \) for all \( k \).

From now on we will denote by \( T_N \) the subsequence as in Lemma 2.8 so that, for \( \varphi_1, \varphi_2 \) chosen according to \((\omega_1)–(\omega_2)\), we have

\[
\omega_1 T_N = 2\pi N + \varphi_1, \quad \omega_2 T_N = 2\pi l_N + \varphi_{2N} \quad \text{with} \quad \varphi_{2N} \to \varphi_2,
\]

and we will use the notation

\[
(2.9) \quad C_i^N = 2 + \left| \cot \frac{\omega_i T_N}{2} \right| \quad \text{for} \quad i = 1, 2;
\]

note that, for \( T_N \) as in Lemma 2.8 we have \( C_1^N = C_{\varphi_1} \) (see (1.13)) and \( C_2^N = C_{\varphi_{2N}} \to C_{\varphi_2} \) as \( N \to +\infty \).

The following lemma is a direct consequence of Lemma 2.6.

**Lemma 2.11.** Let \( T_N \) be as in Lemma 2.8 and let \( y_i^N \) be a solution of (2.7) for \( i = 1, 2 \). Then for \( i = 1, 2 \),

\[
\| y_i^N \|_\infty \leq \frac{C_i^N}{\omega_i} \| \nabla \delta \|_\infty \int_0^{T_N} V(q(t)) \, dt,
\]

\[
\| \dot{y}_i^N \|_\infty \leq \frac{C_i^N}{\omega_i} \| \nabla \delta \|_\infty \int_0^{T_N} V(q(t)) \, dt,
\]

\[
|Q_i(y_i^N)| \leq \frac{C_i^N}{\omega_i} \| \nabla \delta \|_\infty^2 \left( \int_0^{T_N} V(q(t)) \, dt \right)^2,
\]

where \( C_1^N = C_{\varphi_1} \) and \( C_2^N = C_{\varphi_{2N}} \to C_{\varphi_2} \) as \( N \to +\infty \).

**Remark 2.12.** If \( (y_{1N}, y_{2N}, q_N) \) is a solution of (PT) in the interval \([0, T_N]\), then \( y_{iN}(t), \ i = 1, 2 \), can be expressed, for suitable constants \( A_{iN} \) and \( \mu_{iN} \) (such that the periodic boundary conditions are satisfied), as

\[
(2.13) \quad y_{iN}(t) = \frac{1}{\omega_i} \int_0^t \frac{\partial \delta}{\partial y_i}(y_{1N}, y_{2N})V(q_N) \sin \omega_i(t - s) \, ds + A_{iN} \cos(\omega_i t + \mu_{iN}),
\]

or

\[
(2.14) \quad y_{iN}(t) = -\frac{1}{\omega_i} \int_0^{T_N} \frac{\partial \delta}{\partial y_i}(y_{1N}, y_{2N})V(q_N) \sin \omega_i(t - s) \, ds
\]

\[
+ A_{iN} \cos(\omega_i t + \mu_{iN} - \phi_{iN}),
\]

with \( \phi_{iN} \equiv \omega_i T_N \mod 2\pi \).
3. Variational Setting and Min-Max Procedure

From now on we assume that $T_N$ satisfies (2.9). Let

$$E_N = \{ x \in H^1_{\text{loc}}(\mathbb{R}) \mid x \text{ is } T_N\text{-periodic} \}$$

with scalar product $(u, v) = \int_0^{T_N} (\dot{u} \dot{v} + u v)$ and

$$\Gamma_N = \{ q \in H^1(0, T_N) \mid q(0) = 0, q(T_N) = 2\pi \}.$$ 

and let, for all $(y_1, y_2, q) \in E_N \times E_N \times \Gamma_N$,

$$f_N(y_1, y_2, q) = \int_0^{T_N} \left[ \frac{\dot{y}_1^2 - a_0^2 y_1^2}{2} + \frac{\dot{y}_2^2 - a_0^2 y_2^2}{2} + \frac{\dot{q}^2}{2} + (1 + \delta(y_1, y_2)) V(q) \right].$$

In the following we will omit the subscript $N$ when it is possible. It is straightforward to show that

**Lemma 3.1.** $f \in C^1(E \times E \times \Gamma; \mathbb{R})$. If $\nabla f(y_1, y_2, q) = 0$ then $(y_1, y_2, q)$ solves (PT).

We recall (see for example [10]), that, for fixed $N \in \mathbb{N}$, to the quadratic form $Q_1(y)$ on $E$ (see (2.5)), there is associated a splitting $E = E_1^- \oplus E_1^+$, and to $Q_2(y)$ a splitting $E = E_2^- \oplus E_2^+$. More precisely, for $i = 1, 2$ let

$$E_i^- = \left\{ y(s) = a_0^i + \sum_{k \mid 2\pi k < T_N} \left( a_k^i \cos \frac{2\pi k}{T_N} s + b_k^i \sin \frac{2\pi k}{T_N} s \right) \right\},$$

$$E_i^+ = \left\{ y(s) = \sum_{k \mid 2\pi k > T_N} \left( a_k^i \cos \frac{2\pi k}{T_N} s + b_k^i \sin \frac{2\pi k}{T_N} s \right) \right\}.$$

Then, for all $y \in E$ and $i = 1, 2$, $y = y_i^+ + y_i^-$, $y_i^+ \in E_i^+$, $y_i^- \in E_i^-$ and $\int_0^{T_N} y_i^+ y_i^- = 0$.

Note also that for suitable positive constants $\lambda_i^\pm(T_N)$ and $i = 1, 2$,

$$-Q_1(y) \geq \lambda_i^-(T_N) \| y \|^2 \quad \text{for all } y \in E_i^-,$$

$$Q_1(y) \geq \lambda_i^+(T_N) \| y \|^2 \quad \text{for all } y \in E_i^+.$$ 

**Proposition 3.3.** Assume $(y_{1n}, y_{2n}, q_n) \in E \times E \times \Gamma$ are such that

$$f(y_{1n}, y_{2n}, q_n) \to c, \quad \frac{\partial f}{\partial y_1}(y_{1n}, y_{2n}, q_n) \to 0, \quad i = 1, 2.$$

Then $(y_{1n}, y_{2n}, q_n)$ is bounded in $E \times E \times \Gamma$ and, up to a subsequence, $y_{1n} \to y_0$ in $E$ for $i = 1, 2$, $q_n \to q_0$ in $L^\infty$, $q_n \to q_0$ in $H^1$. Moreover, $(y_{10}, y_{20}, q_0)$ is a solution of (2.9) for $i = 1, 2$.

Furthermore, if $\nabla f(y_{1n}, y_{2n}, q_n) \to 0$, then, up to a subsequence, $(y_{1n}, y_{2n}, q_n) \to (y_{10}, y_{20}, q_0)$ and $f$ satisfies the PS condition.
and the sequences $y_i$ for
so that we obtain the boundedness of a subsequence, $q_n$ and since $q_n(0) = 0$, we see that $q_n$ is bounded in $H^1(0, T)$. We then deduce that, up to a subsequence, $q_n \to q_0$ in $L^2$, uniformly and weakly in $H^1$, and also $y_{in} \to y_{i0}$ in $L^2$, uniformly and weakly in $H^1$ for $i = 1, 2$. Since
\[
\int_0^T \dot{q}_n^2 = 2 f(y_{1n}, y_{2n}, q_n) - Q_1(y_{1n}) - Q_2(y_{2n}) - 2 \int_0^T (1 + \delta(y_{1n}, y_{2n})) V(q_n) \\
\leq 2(c + 1) + \max\{1, \omega_1^2\} \|y_{1n}\|_{H^1}^2 + \max\{1, \omega_2^2\} \|y_{2n}\|_{H^1}^2 \leq \text{const},
\]
and since $q_n(0) = 0$, we see that $q_n$ is bounded in $H^1(0, T)$. Then $q_n$ is bounded in $L^2$, uniformly and weakly in $H^1$, and also $y_{in} \to y_{i0}$ in $L^2$, uniformly and weakly in $H^1$ for $i = 1, 2$. Since
\[
\int_0^T |\dot{y}_{in} - \dot{y}_{i0}|^2 + \int_0^T |y_{in} - y_{i0}|^2 = \int_0^T \dot{y}_{in}(\dot{y}_{in} - \dot{y}_{i0}) \\
- \int_0^T \dot{y}_{i0}(\dot{y}_{in} - \dot{y}_{i0}) + \int_0^T |y_{in} - y_{i0}|^2,
\]
recalling that $\int_0^T |y_{in} - y_{i0}|^2 \to 0$ as well as (by weak convergence) $\int_0^T \dot{y}_{i0}(\dot{y}_{in} - \dot{y}_{i0}) \to 0$, to show that $y_{in} \to y_{i0}$ in $H^1$ it is enough to prove that
\[
\int_0^T \dot{y}_{in}(\dot{y}_{in} - \dot{y}_{i0}) \to 0.
\]
Since
\[
\int_0^T \dot{y}_{in}(\dot{y}_{in} - \dot{y}_{i0}) = \left(\frac{\partial f}{\partial y_i}(y_{in}, y_{2n}, q_n), y_{in} - y_{i0}\right) + \omega_i^2 \int_0^T y_{in}(y_{in} - y_{i0}) \\
- \int_0^T \frac{\partial}{\partial y_i} \delta(y_{1n}, y_{2n}) V(q_n)(y_{in} - y_{i0}),
\]
the result follows because $y_{in} - y_{i0}$ is bounded, $\frac{\partial f}{\partial y_i}(y_{in}, y_{2n}, q_n) \to 0$, $y_{in} \to y_{i0}$ in $L^2$ and the sequences $y_{in}$ and $\frac{\partial}{\partial y_i} \delta(y_{1n}, y_{2n}) V(q_n)$ are bounded in $L^\infty$. Then $y_{in} \to y_{i0}$ in $H^1$ for $i = 1, 2$.

Finally, if $\varphi$ is any test function, we have
\[
\left\langle \frac{\partial f}{\partial y_i}(y_{in}, y_{2n}, q_n), \varphi \right\rangle = \int_0^T (\dot{y}_{in} \dot{\varphi} - \omega_i^2 y_{in} \varphi) + \int_0^T \frac{\partial \delta}{\partial y_i}(y_{1n}, y_{2n}) V(q_n) \varphi \\
\to \int_0^T (\dot{y} \dot{\varphi} - \omega_i^2 y \varphi) + \int_0^T \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q_0) \varphi, \quad i = 1, 2,
\]
so that \( y_i \) is a weak solution of
\[
\ddot{y}_i + \omega_i^2 y_i = \frac{\partial \delta}{\partial y_i}(y_1, y_2) V(q_0), \quad i = 1, 2,
\]
and, by standard arguments, also a classical solution. Therefore \((y_{10}, y_{20}, q_0)\) is a solution of (2.7) for \(i = 1, 2\). If also \(\frac{\partial f}{\partial q}(y_{1n}, y_{2n}, q_n) \to 0\), using the same arguments, we have \(q_n \to q_0\) in \(H^1\) and \((y_{10}, y_{20}, q_0)\) is a solution of (PT). \(\square\)

We say that \(h \in H\) if
\begin{enumerate}
\item \(h : E^-_1 \times E^-_2 \to E \times E \times \Gamma\) is continuous,
\item there are \(R > 0\) and \(q_h \in \Gamma\) such that
\[
h(y_1, y_2) = (y_1, y_2, q_h) \quad \forall \| (y_1, y_2) \| \geq R.
\]
\end{enumerate}

Let us define
\begin{equation}
(3.4) \quad c(T_N) = \inf_{h \in H} \sup_{(y_1, y_2) \in E^-_1 \times E^-_2} f_N(h(y_1, y_2)).
\end{equation}

To estimate \(c(T_N)\) (see Lemma 3.6), we first prove the following inequality.

**Lemma 3.5.** For all \(q \in \Gamma\) we have
\[
\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_0^{2\pi} \sqrt{2V(s)} \, ds > 0.
\]

**Proof.** Let \(q \in \Gamma_\infty = \{q \in H^1_{\text{loc}}(\mathbb{R}) : q(-\infty) = 0, \quad q(+\infty) = 2\pi\}\) be such that
\[
\int_{\mathbb{R}} [\dot{q}^2/2 + V(q)] = \min_q \int_{\mathbb{R}} [\dot{q}^2/2 + V(q)].
\]

By energy conservation \(\dot{q}^2/2 - V(q) = 0\), so that for all \(q \in \Gamma\), we have
\[
\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_{\mathbb{R}} [\dot{q}^2/2 + V(q)] = \int_{\mathbb{R}} 2V(q).
\]

From our assumptions on \(V\) it follows that \(\dot{q}/2 > 0\) and using the change of variables \(s = q(t)\) we have
\[
\int_{\mathbb{R}} 2V(q) \, dt = \int_0^{2\pi} \frac{2V(s)}{\sqrt{2V(s)}} \, ds = \int_0^{2\pi} \sqrt{2V(s)} \, ds,
\]
so that
\[
\int_0^T [\dot{q}^2/2 + V(q)] \geq \int_0^{2\pi} \sqrt{2V(s)} \, ds > 0 \quad \forall q \in \Gamma. \quad \square
\]
LEMMA 3.6. Let \( T_N \) satisfy (2.9) and \( c(T_N) \) be defined as in (3.4). Then
\[
\int_0^{2\pi} \sqrt{2(1 + \delta)V(s)} \, ds =: \varepsilon \leq c(T_N) \leq \tilde{c} := 2\pi^2 + (1 + \delta)\|V\|_\infty \quad \text{for all } T_N.
\]

PROOF. Let \( \tilde{q}_T \) be such that
\[
\int_0^T [\tilde{q}_T^2/2 + (1 + \delta)V(\tilde{q}_T)] = \min_{q \in \Gamma} \int_0^T [\tilde{q}_T^2/2 + (1 + \delta)V(q)] = \tilde{c}(T).
\]
Then
\[
\tilde{c}(T) \leq \tilde{c}(1) \leq 2\pi^2 + (1 + \delta)\|V\|_\infty =: \tilde{c}.
\]

Letting \( \tilde{h}(y_1, y_2) = (y_1, y_2, \tilde{q}_T) \), we have
\[
c(T) = \inf_{h \in \mathcal{H}} \sup_{E_1^- \times E_2^-} f(h(y_1, y_2)) \leq \sup_{(y_1, y_2) \in E_1^- \times E_2^-} f(\tilde{h}(y_1, y_2))
\]
\[
\leq \int_0^T [\tilde{q}_T^2/2 + (1 + \delta)V(\tilde{q}_T)] = \tilde{c}(T) \leq \tilde{c}.
\]

On the other hand, for any \( h \in \mathcal{H} \), \( h(y_1, y_2) = (h_1(y_1, y_2), h_2(y_1, y_2), h_3(y_1, y_2)) \), consider the function \( \bar{h} : E_1^- \times E_2^- \rightarrow E_1^+ \times E_2^+ \) defined by \( \bar{h}(y_1, y_2) = (\pi_{E_1^-} h_1(y_1, y_2), \pi_{E_2^-} h_2(y_1, y_2)) \). Since \( \bar{h}|_{\partial B(0, R)} = \text{Id} \) for all \( R \) large enough, there is \( (\bar{y}_1, \bar{y}_2) \in E_1^+ \times E_2^+ \) such that \( \bar{h}((\bar{y}_1, \bar{y}_2) = (0, 0) \), i.e.
\[
h_1(\bar{y}_1, \bar{y}_2) \in E_1^+, \quad h_2(\bar{y}_1, \bar{y}_2) \in E_2^+.
\]
Then, letting \( q = h_3(\bar{y}_1, \bar{y}_2) \), we have, for all \( h \in \mathcal{H} \),
\[
\sup_{(y_1, y_2) \in E_1^- \times E_2^-} f(h(y_1, y_2)) \geq f(h(\bar{y}_1, \bar{y}_2)) \geq \int_0^T [\tilde{q}_T^2/2 + (1 + \delta)V(q)] \, dt
\]
\[
\geq \min_{q \in \Gamma} \int_0^T [\tilde{q}_T^2/2 + (1 + \delta)V(q)] \, dt = \varepsilon(T),
\]
and Lemma 3.5 yields
\[
\varepsilon(T) \geq \int_0^{2\pi} \sqrt{2(1 + \delta)V(s)} \, ds =: \varepsilon > 0. \quad \square
\]

PROPOSITION 3.7. Let \( T_N \) satisfy (2.9) and \( c(T_N) \) be defined as in (3.4). Then there is a critical point \( (y_{1N}, y_{2N}, q_N) \) for \( f_N \) at level \( c(T_N) \) that solves problem (PT). Moreover, \( q_N \) has the following properties:

(3.8) \( q_N(t) \in [0, 2\pi] \quad \forall t \in [0, T_N] \),
(3.9) \( q_N(0) = \bar{q}_N(T_N) \),
(3.10) \( \int_0^{T_N} V(q_N) \leq \frac{\bar{c}}{1 + \delta - \alpha} = \frac{\bar{c}}{K}, \quad \int_0^{T_N} \tilde{q}_N^2 \leq 2\bar{c} \).
PROOF: The existence of the critical point \((y_{1N}, y_{2N}, q_N)\) at level \(c(T_N)\) follows via the min-max principle, since \(f\) satisfies (PS) by Proposition 3.3 and by the estimates of Lemma 3.6.

To prove (3.8) let us introduce, for all \(T_N\),
\[
\begin{align*}
\Gamma^*_N &= \{ q \in \Gamma_N \mid q(s) \in [0, 2\pi] \forall s \in [0, T_N] \} \\
\mathcal{H}^* &= \{ h \in \mathcal{H} \mid h(y_1, y_2) \in E \times E \times \Gamma^*_N \forall (y_1, y_2) \in E^-_1 \times E^-_2 \}, \\
c^*(T_N) &= \inf_{h \in \mathcal{H}^*} \sup_{(y_1, y_2) \in E^-_1 \times E^-_2} f_N(h(y_1, y_2)).
\end{align*}
\]

It is easy to show that \(c^*(T_N) = c(T_N)\).

Indeed, \(\mathcal{H}^* \subseteq \mathcal{H}\) implies that \(c^*(T_N) \geq c(T_N)\). To prove the other inequality pick \(h \in \mathcal{H}\) and let \(h^* \in \mathcal{H}^*\) be defined as \(h^*(y_1, y_2) = h(y_1, y_2)^*, \) where \((y_1, y_2, q)^* = (y_1, y_2, q^*)\) and
\[
q^*(t) = \begin{cases} 
q(t) & \text{if } 0 \leq q(t) \leq 2\pi, \\
2\pi & \text{if } q(t) > 2\pi, \\
0 & \text{if } q(t) < 0.
\end{cases}
\]

Then, since
\[
(1 + \delta(y_1(t), y_2(t)))V(q^*(t)) \leq (1 + \delta(y_1(t), y_2(t)))V(q(t)) \quad \forall t \in [0, T],
\]
we immediately see that
\[
f_T(h^*(y_1, y_2)) \leq f_T(h(y_1, y_2)) \quad \forall (y_1, y_2) \in E^-_1 \times E^-_2, \forall h \in \mathcal{H}
\]
and
\[
c^*(T) \leq c(T)
\]
and also (3.8) follows.

Now let us show that (3.9) holds. Since \((y_{1N}, y_{2N}, q_N)\) is a solution of (PT), by energy conservation, we have
\[
\begin{align*}
\frac{\dot{y}_{1N}^2(0)}{2} + \frac{\omega_1^2 y_{1N}^2(0)}{2} + \frac{\dot{q}_N^2(0)}{2} &= \frac{\dot{y}_{2N}^2(T_N)}{2} + \frac{\omega_2^2 y_{2N}^2(T_N)}{2} + \frac{\dot{q}_N^2(T_N)}{2} \\
\frac{\dot{y}_{1N}(T_N)}{2} + \frac{\omega_1^2 y_{1N}^2(T_N)}{2} &= \frac{\dot{y}_{2N}(T_N)}{2} + \frac{\omega_2^2 y_{2N}^2(T_N)}{2} + \frac{\dot{q}_N(T_N)}{2} \\
\end{align*}
\]
Since \(q_N(0) = 0\), \(q_N(T_N) = 2\pi\), \(V(0) = V(2\pi) = 0\) and by periodicity of \(y_{1N}\) and \(y_{2N}\), we have
\[
\dot{q}_N(0) = \dot{q}_N(T_N).
\]

Since \(q_N(t) \in [0, 2\pi]\) for all \(t \in [0, T_N]\) we have \(\dot{q}_N(0) \geq 0, \dot{q}_N(T_N) \geq 0\) and so
\[
\dot{q}_N(0) = \dot{q}_N(T_N).
\]
Finally, using the last estimate of Lemma 2.6 we have
\[
c(T_N) = \frac{1}{2} Q_1(y_{1N}) + \frac{1}{2} Q_2(y_{2N}) + \int_0^{T_N} [\dot{q}_N^2/2 + (1 + \delta(y_{1N}, y_{2N})) V(q_N)]
\geq \int_0^{T_N} \dot{q}_N^2/2 + (1 + \delta - \alpha) V(q_N),
\]
and, by the estimate on \(c(T_N)\) in Lemma 3.6, (3.10) holds. □

4. PROOF OF THEOREM 1.10

We say that \(q(t)\) jumps from \(\eta\) to \(2\pi - \eta\) in an interval \([\alpha, \beta]\) if \(q(\alpha) = \eta, q(t) \in \eta, 2\pi - \eta\) for all \(t \in [\alpha, \beta]\), \(q(\beta) = 2\pi - \eta\). Note that if \(q(t)\) jumps in \([\alpha, \beta]\) from \(\eta\) to \(2\pi - \eta\), then defining
\[
\tilde{q}(t) = \begin{cases} 
q(t) & 0 \leq t \leq \alpha - 1, \\
\eta(t - \alpha + 1) & \alpha - 1 \leq t \leq \alpha, \\
q(t) & \alpha \leq t \leq \beta, \\
2\pi + \eta(t - \beta - 1) & \beta \leq t \leq \beta + 1, \\
2\pi & \beta + 1 \leq t \leq T,
\end{cases}
\]
and arguing as in Lemma 3.5, for any \(B > 0\) and \(\eta \leq \eta_0\) (\(\eta_0\) given by (1.6)), \(\eta\) sufficiently small, we have
\[
\int_\alpha^\beta [\dot{q}_N^2/2 + BV(q)] \geq \int_0^T [\dot{q}_N^2/2 + BV(\tilde{q})] - \eta^2 - BV(\eta) - BV(2\pi - \eta)
\geq \int_0^{2\pi} \sqrt{2BV(s)} ds - \eta^2 - 2B\mu\eta^2 > 0.
\]

LEMMA 4.2. Let \((y_1, y_2, q) \in E \times E \times \Gamma^* \) be a critical point for \(f_N\) at level \(c(T_N)\) as in Proposition 3.7 and assume (44) holds. Then there exists \(0 < \eta_1 \leq \eta_0\) (\(\eta_0\) given by (1.6)) such that for all \(0 < \eta \leq \eta_1, q(t)\) jumps only once from \(\eta\) to \(2\pi - \eta\). Moreover, if \([\alpha, \beta]\) is the interval where \(q(t)\) jumps from \(\eta\) to \(2\pi - \eta\), then \(|\beta - \alpha| \leq \tilde{c}/KV_\eta\) with \(V_\eta\) as in (1.4) and \(K\) as in (1.13).

PROOF. Arguing by contradiction, let us assume that \(q(t)\) jumps from \(\eta\) to \(2\pi - \eta\) in two intervals, \([\alpha_1, \beta_1]\) and \([\alpha_2, \beta_2]\) with \(\beta_1 < \alpha_2\). Without loss of generality we can assume that
\[
\int_0^{\beta_1} V(q) \leq \int_{\alpha_2}^T V(q).
\]
Define
\[
\tilde{q}(t) = \begin{cases} 
q(t) & t \in [0, \beta_1], \\
2\pi + \eta(t - \beta_1 - 1) & t \in [\beta_1, \beta_1 + 1], \\
2\pi & t \in [\beta_1 + 1, T],
\end{cases}
\]
and let $\tilde{h} : E_1^- \times E_2^- \to E_1 \times E_2 \times T$ be defined by $\tilde{h}(y_1^-, y_2^-) = (y_1^-, y_2^-, q)$ for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$. For all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$, using the last estimate of Lemma 2.6, we have

$$f(y_1, y_2, q) - f(y_1^-, y_2^-, q) \geq \frac{1}{2}Q(y_1) + \frac{1}{2}Q(y_2)$$

$$+ \int_0^T [\dot{q}/2 + (1 + \delta(y_1, y_2))V(q)] - \int_0^T [\dot{\bar{q}}/2 + (1 + \delta(y_1^-, y_2^-))V(\bar{q})]$$

$$\geq \int_0^T -\alpha V(q) + \int_{\beta_1}^{\beta_1} [\delta(y_1, y_2) - \delta(y_1^-, y_2^-)]V(q)$$

$$+ \int_{\beta_1}^{\beta_1} [\dot{\bar{q}}/2 + (1 + \delta(y_1^-, y_2^-))V(\bar{q})] - \int_{\beta_1}^{\beta_1+1}[\dot{q}/2 + (1 + \delta(y_1^-, y_2^-))V(q)]$$

$$\geq \int_0^{\bar{\beta}} (\bar{\delta} - \alpha - \bar{\delta})V(q) + \int_0^{\bar{\beta}} [\dot{\bar{q}}/2 + (1 + \bar{\delta} - \alpha) V(q)] - \eta^2/2 - (1 + \bar{\delta})\mu\eta^2.$$  

From assumption (4.4) it follows that $\bar{\delta} - \alpha - \bar{\delta} > -(1 + \bar{\delta} - \alpha)$; and thanks to (4.3) we have

$$\int_0^{\bar{\beta}} (\bar{\delta} - \alpha - \bar{\delta})V(q) > -(1 + \bar{\delta} - \alpha) \int_0^{\bar{\beta}} V(q) \geq -(1 + \bar{\delta} - \alpha) \int_{\alpha_2}^T V(q).$$

Therefore, using also (4.1) we have

$$f(y_1, y_2, q) - f(y_1^-, y_2^-, q) \geq \int_0^{\bar{\beta}} [\dot{\bar{q}}/2 + (1 + \bar{\delta} - \alpha) V(q)] - \eta^2 [1 + \mu(1 + \bar{\delta})]$$

$$\geq \int_0^{2\pi} \sqrt{2(1 + \bar{\delta} - \alpha) V(s)} ds$$

$$- \eta^2/2 - (1 + \bar{\delta} - \alpha)\mu\eta^2 - \eta^2 [1 + \mu(1 + \bar{\delta})].$$

Then, choosing $\eta_1$ sufficiently small, we have

$$f(y_1, y_2, q) - f(y_1^-, y_2^-, q) > 0, \quad \forall \eta \leq \eta_1, \forall (y_1^-, y_2^-) \in E_1^- \times E_2^-,$$

which is a contradiction, because $(y_1, y_2, q)$ is a critical point at level $c(T)$. Finally, by (3.10) we have

$$\frac{\bar{c}}{K} \geq \int_0^{T_N} V(q_N) \geq \int_\alpha^\beta V(q_N) \geq V_\eta |\beta - \alpha|.$$  

**Proof of Theorem 1.10.** Let $(y_{1N}, y_{2N}, q_N)$ be a critical point at level $c(T_N)$ which is a solution of (1.11), given by Proposition 3.7. Fix any $\eta \leq \eta_1$ and let $[\alpha_N, \beta_N]$ denote the unique (by Lemma 4.2) interval where $q_N$ jumps from $\eta$ to $2\pi - \eta$. Let $\tau_N \in [\alpha_N, \beta_N]$ be such that $q_N(\tau_N) = \pi$ and $q_N(t) \leq \pi$ for all $t \leq \tau_N$. Since $|\beta_N - \alpha_N| \leq \bar{c}/KV_\eta$ and $T_N \to +\infty$, we have either $\tau_N \to +\infty$ or $T_N - \tau_N \to +\infty$. Let us first analyze the case
where both \( \tau_N \to +\infty \) and \( T_N - \tau_N \to +\infty \). We define the function \( \tilde{q}_N \) in the interval \([-\tau_N, T_N - \tau_N]\) as

\[
\tilde{q}_N(t) = q_N(t + \tau_N) \quad \forall N \in \mathbb{N}.
\]

By definition \( \tilde{q}_N(t) \in [0, 2\pi] \) for all \( t \in [-\tau_N, T_N - \tau_N] \) and \( \tilde{q}_N(0) = \pi \) for all \( N \in \mathbb{N} \). Moreover, by Proposition 3.7 we have

\[
\int_{\tau_N}^{T_N - \tau_N} \tilde{q}_N(t)^2 \, dt = \int_0^{T_N} \tilde{q}_N(t)^2 \, dt \leq 2\bar{c}.
\]

Then, for any fixed \( a < b \in \mathbb{R} \), since both \( \tau_N \to +\infty \) and \( T_N - \tau_N \to +\infty \), we have \( \tilde{q}_N \in H^1(a, b) \) for all \( N \) large enough and

\[
\|\tilde{q}_N\|_{H^1(a, b)}^2 \leq 2\bar{c} + (b - a)4\pi^2.
\]

Therefore, up to a subsequence, \( \tilde{q}_N \rightharpoonup q \) in \( H^1(a, b) \), \( \tilde{q}_N \to q \) uniformly in \([a, b]\) and

\[
\int_a^b \tilde{q}_N^2 \leq \liminf_{N \to +\infty} \sup_{a < b} \int_a^b \tilde{q}_N^2 \leq 2\bar{c},
\]

\[
\int_a^b V(q) \leq \liminf_{N \to +\infty} \sup_{a < b} \int_a^b V(\tilde{q}_N) \leq \frac{\bar{c}}{1 + \delta - \alpha} = \frac{\bar{c}}{K}.
\]

Since \( V(q) = 0 \) only for \( q = 2k\pi, \, k \in \mathbb{Z} \), and since \( q_N \) jumps only once from \( \eta \) to \( 2\pi - \eta \) for all \( N \in \mathbb{N} \), we have

\[
\lim_{t \to -\infty} q(t) = 0, \quad \lim_{t \to +\infty} (q(t) - 2\pi) = 0, \quad \lim_{t \to \pm\infty} \dot{q}(t) = 0.
\]

Now let us analyze the case where only one of \( \tau_N \) and \( T_N - \tau_N \) diverges. We can assume that, up to a subsequence, \( T_N - \tau_N \) diverges and \( \tau_N < T_N/2 \). Define the function \( \tilde{q}_N \) in \([-T_N + \tau_N)/2, (T_N - \tau_N)/2] \) by

\[
\tilde{q}_N(t) = \begin{cases} q_N(t + T_N + \tau_N) - 2\pi & \text{if } t \in [-(T_N + \tau_N)/2, -\tau_N], \\ q_N(t + \tau_N) & \text{if } t \in [-\tau_N, (T_N - \tau_N)/2]. \end{cases}
\]

Then for all \( N \), \( \tilde{q}_N(0) = \pi \), \( \tilde{q}_N(t) \in [-2\pi + \eta, \pi] \) for all \( t \in [-(T_N + \tau_N)/2, 0] \), and \( \tilde{q}_N(t) \in [\eta, 2\pi] \) for all \( t \in (0, (T_N - \tau_N)/2] \). Then, arguing as in the first case, for \( a < b \) in \( \mathbb{R} \) we have, up to a subsequence, \( \tilde{q}_N \rightharpoonup q \) in \( H^1(a, b) \), \( \tilde{q}_N \to q \) uniformly in \([a, b]\), \( \|q\|_\infty \leq 2\pi \) and

\[
\int_a^b \tilde{q}_N^2 \leq 2\bar{c}, \quad \int_a^b V(q) \leq \frac{\bar{c}}{K}.
\]

Since \( V(q) = 0 \) only for \( q = 2k\pi \) and since for all \( N \in \mathbb{N} \), \( \tilde{q}_N \in [-2\pi + \eta, \pi] \) for all \( t \in [-(T_N + \tau_N)/2, 0] \), and \( \tilde{q}_N \in [\eta, 2\pi] \) for all \( t \in [0, (T_N - \tau_N)/2] \), we deduce also in this case that

\[
\lim_{t \to -\infty} q(t) = 0, \quad \lim_{t \to +\infty} (q(t) - 2\pi) = 0, \quad \lim_{t \to \pm\infty} \dot{q}(t) = 0.
\]
Now define, for $i = 1, 2$,
\[
\tilde{y}_i(t) = \begin{cases} 
    y_i(t + T_N + \tau_N) & \text{if } t \in [-(T_N + \tau_N)/2, -\tau_N], \\
    y_i(t + \tau_N) & \text{if } t \in [-\tau_N, (T_N - \tau_N)/2].
\end{cases}
\]

In view of Lemma 2.11 and (4.4) we know that $\tilde{y}_N(t)$ is bounded in $H^1(a, b)$, so that, up to a subsequence, $\tilde{y}_N \to \tilde{y}_i$ in $H^1(a, b)$ and $\tilde{y}_N \to \tilde{y}_i$ uniformly in $L^\infty(a, b)$ for $i = 1, 2$. We can now pass to the limit in the equations
\[
\begin{align*}
\ddot{q}_N &= (1 + \delta(\tilde{y}_1N, \tilde{y}_2N))V(\tilde{q}_N), \\
\ddot{\tilde{y}}_1N + \omega_1^2\tilde{y}_1N &= \frac{\partial}{\partial y_1}\delta(\tilde{y}_1N, \tilde{y}_2N)V(\tilde{q}_N), \\
\ddot{\tilde{y}}_2N + \omega_2^2\tilde{y}_2N &= \frac{\partial}{\partial y_2}\delta(\tilde{y}_1N, \tilde{y}_2N)V(\tilde{q}_N),
\end{align*}
\]

to deduce that $(y_1, y_2, q)$ is a solution of
\[
\begin{align*}
\ddot{q} &= (1 + \delta(y_1, y_2))V(q), \\
\ddot{y}_1 + \omega_1^2y_1 &= \frac{\partial}{\partial y_1}\delta(y_1, y_2)V(q), \\
\ddot{y}_2 + \omega_2^2y_2 &= \frac{\partial}{\partial y_2}\delta(y_1, y_2)V(q),
\end{align*}
\]
in the interval $[a, b]$ and hence also in $\mathbb{R}$.

Thus, as observed at the beginning of Section 2, conditions (1.3) are satisfied. Finally, by energy conservation, since $\dot{q}(\pm \infty) = 0$, also condition (1.2) holds.

5. Proof of Theorem 1.11

**Lemma 5.1.** Let $(y_1N, y_2N, q_N)$ be a critical point for $f_N$ at level $c(T_N)$ given by Proposition 5.7 and assume that (V4) holds. Then for all $0 < \eta \leq \eta_0$ (\(\eta_0\) given by (1.6)) there exist $0 < \tau_1 < \tau_2 < T_N$ such that
\[
0 \leq q_N(t) \leq \eta \quad \forall t \in [0, \tau_1], \\
q_N(t) \in [\eta, 2\pi - \eta] \quad \forall t \in [\tau_1, \tau_2], \\
2\pi - \eta \leq q_N(t) \leq 2\pi \quad \forall t \in [\tau_2, T_N].
\]

**Proof.** Let $\eta \leq \eta_0$, let $\tau_1 = \inf\{s \in [0, T] \mid q(s) > \eta\}$ and $\tau_2 = \sup\{s \in [0, T] \mid q(s) < 2\pi - \eta\}$. If the lemma does not hold, then there is $\tau' \in (\tau_1, T]$ such that $q(\tau') = \eta$ (or there is $\tau' \in (0, \tau_2)$ such that $q(\tau') = 2\pi - \eta$; we will only discuss the first case). Then $q(t)$ reaches a maximum at $\tau'' \in (\tau_1, \tau')$, hence $\dot{q}(\tau'') \leq 0$. But
\[
\ddot{q}(\tau'') = (1 + \delta(y_1(\tau'''), y_2(\tau''')))V(q(\tau'''))
\]
implies, by (V4) that $q(\tau'') \geq \tilde{\eta}$. Then there exists an interval where $q$ jumps from $\tilde{\eta}/2$ to $\tilde{\eta}$ and an interval where $q$ jumps from $\tilde{\eta}$ to $\tilde{\eta}/2$. In each of these intervals, say $[a, b]$,
\[
\frac{\tilde{\eta}}{2} = \int_a^b \dot{q} \leq \left(\int_a^b q^2 \right)^{1/2} \sqrt{b - a}
\]
so that for $V_{h/2}$ as in (1.4) we obtain

$$\int_a^b [\dot{q}^2 / 2 + (1 + \delta - \alpha) V(q)] \geq \frac{\tilde{\eta}^2}{8(b - a)} + (1 + \delta - \alpha) V_{h/2}(b - a)$$

$$\geq \tilde{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{h/2}.$$ 

Therefore

$$\int_{\tau_1}^{\tau_1'} [\dot{q}^2 / 2 + (1 + \delta(y_1, y_2) - \alpha) V(q)] \geq 2\tilde{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{h/2}.$$ 

Now we define a new function $\bar{q} \in \Gamma^\ast$ by setting

$$\bar{q}(t) = \begin{cases} 0, & 0 \leq t \leq \tau_1' - \tau_1, \\ q(t - \tau_1' + \tau_1), & \tau_1' - \tau_1 \leq t \leq \tau_1', \\ q(t), & \tau_1' \leq t \leq T. \end{cases}$$

We also introduce $\tilde{h}$ defined as $\tilde{h}(y_1^-, y_2^-) = (y_1^-, y_2^-)$ for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-$. Clearly $\tilde{h} \in \mathcal{H}^\ast$, so that, since $f_T(y_1, y_2, q) = \alpha(T)$,

$$0 \leq \sup_{(y_1^-, y_2^-) \in E_1^- \times E_2^-} f_T(\tilde{h}(y_1^-, y_2^-)) - f_T(y_1, y_2, q).$$

On the other hand, using (5.2) and arguing as in the proof of Lemma 4.2 (see also [10, Lemma 11]) we have, for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-,$

$$f_T(y_1, y_2, q) - f_T(y_1^-, y_2^-, \bar{q})$$

$$\geq \int_0^{\tau_1} [\dot{q}^2 / 2 + (1 + \delta - \alpha) V(q)] + \int_{\tau_1}^{\tau_1'} [\dot{q}^2 / 2 + (1 + \delta - \alpha) V(q)]$$

$$+ \int_{\tau_1'}^{T} (\delta - \delta - \alpha) V(q) - \int_{\tau_1 - \tau_1}^{\tau_1'} [\dot{q}^2 / 2 + (1 + \delta(y_1, y_2^-)) V(\bar{q})]$$

$$\geq -\tilde{\delta} - \tilde{\delta} + \alpha \int_0^{\tau_1'} V(q) + 2\tilde{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{h/2}.$$ 

Then, using the estimate (4.10) and by definition of $\tilde{\varepsilon}$ (see Lemma 5.6), we have, for all $(y_1^-, y_2^-) \in E_1^- \times E_2^-,$

$$f_T(y_1, y_2, q) - f_T(y_1^-, y_2^-, \bar{q})$$

$$\geq -\tilde{\delta} - \tilde{\delta} + \alpha \frac{\tilde{\varepsilon}}{1 + \delta - \alpha} + 2\tilde{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{h/2}$$

$$\geq -\frac{\tilde{\delta} - \tilde{\delta} + \alpha}{1 + \delta - \alpha} \{2\pi^2 + (1 + \delta)\|V\|_\infty\} + 2\tilde{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{h/2}.$$
Then, thanks to \eqref{5.4}, for all \((y_1^-, y_2^-) \in E_1^- \times E_2^-\) we have
\[
 f_T(y_1^-, y_2^-, q) - f_T(y_1, y_2, q) \leq -\bar{\eta} \sqrt{\frac{1 + \delta - \alpha}{2}} V_{\bar{h}/2},
\]
contradiction. \hfill \Box

**Lemma 5.3.** Let \((y_{1N}, y_{2N}, q_N)\) be a critical point for \(f_N\) at level \(c(T_N)\) given by Proposition 3.7 and assume \eqref{5.4} holds. For \(0 < \eta \leq \eta_0\), let \(\tau^1_N\) and \(\tau^2_N\) be given by Lemma 5.1. Then
\[
(5.4) \quad \tau^2_N - \tau^1_N \leq \frac{\tilde{c}}{(1 + \delta - \alpha) V_{\eta}},
\]
with \(V_{\eta}\) as in \eqref{1.4},
\[
w(t) \equiv \eta \frac{\sinh \sqrt{\bar{a}} t}{\sinh \sqrt{\bar{a}} \tau^1_N} \leq q_N(t) \leq \eta \frac{\sinh \sqrt{\bar{a}} t}{\sinh \sqrt{\bar{a}} \tau^1_N} \equiv z(t)
\]
for all \(t \in [0, \tau^1_N]\) and
\[
(5.5) \quad \tilde{w}(t) \equiv \eta \frac{\sinh \sqrt{\bar{a}} (T_N - t)}{\sinh \sqrt{\bar{a}} (T_N - \tau^2_N)} \leq 2\pi - q_N(t) \leq \eta \frac{\sinh \sqrt{\bar{a}} (T_N - t)}{\sinh \sqrt{\bar{a}} (T_N - \tau^1_N)} \equiv \tilde{z}(t)
\]
for all \(t \in [\tau^2_N, T_N]\), where \(\bar{a} = 2\mu(1 + \delta)\) and \(a = (\mu/2)(1 + \delta)\). Moreover, \(\tau^1_N \to +\infty\) and \(T_N - \tau^2_N \to +\infty\) as \(N \to +\infty\).

**Proof.** We give only a sketch of the proof, more details can be found in \cite{10} Lemmas 13–15 and Remark 14. Estimate \eqref{5.4} is an easy consequence of \eqref{3.10}. Thanks to Lemma 5.1 we can use a maximum principle argument to obtain exponential estimates on \(q_N\). Then, using the estimates on \(q_N\) and \eqref{3.9}, we conclude that both \(\tau^1_N\) and \(T_N - \tau^2_N\) diverge. \hfill \Box

In the following proposition we prove the first part of Theorem 1.11

**Proposition 5.6.** Let \((y_{1N}, y_{2N}, q_N)\) be as in Proposition 3.7 and assume \eqref{5.4} holds. Then for all \(N \in \mathbb{N}\) there is \(\tau_N \in [\tau^1_N, \tau^2_N]\) such that, up to a subsequence,
\[
q_N(\cdot - \tau_N) \to q, \quad y_{1N}(\cdot - \tau_N) \to y_1, \quad y_{2N}(\cdot - \tau_N) \to y_2,
\]
where \((y_1, y_2, q)\) is a solution of problem \eqref{1.1} satisfying \eqref{1.3}. Furthermore, \(q(t) \in [0, 2\pi]\) for all \(t\).

**Proof.** Fix \(\eta \leq \eta_0\). Then, by Lemmas 5.1 and 5.3 we can find \(\tau^1_N, \tau^2_N\) such that
\[
q_N(t) \in [0, \eta] \quad \forall t \in [0, \tau^1_N],
q_N(t) \in [\eta, 2\pi - \eta] \quad \forall t \in [\tau^1_N, \tau^2_N],
q_N(t) \in [2\pi - \eta, 2\pi] \quad \forall t \in [\tau^2_N, T_N],
\]
\[
|\tau^2_N - \tau^1_N| \leq \frac{\tilde{c}}{(1 + \delta - \alpha) V_{\eta}}.
\]
Let $\tau_N$ be the $\tau^1_N$ corresponding to $\eta_0$, and

$$\tilde{q}_N(t) = q_N(t + \tau_N), \quad t \in [-\tau_N, T_N - \tau_N],$$

so that $\tilde{q}_N(0) = \eta_0$ for all $N$. Also define

$$\tilde{y}_{1N}(t) = y_{1N}(t + \tau_N), \quad \tilde{y}_{2N}(t) = y_{2N}(t + \tau_N) \quad \forall t \in [-\tau_N, T_N - \tau_N].$$

Arguing as in the proof of Theorem 1.10, we find that for suitable constants $A_t$ for all $t$.

As in Proposition 5.6, we define

$$R_{i\pm} = R_i, \quad f_i - f_i = \varphi_i$$

for $i = 1, 2$. By (E) holds, then $\alpha_1^2 R_1^2 + \alpha_2^2 R_2^2 > 0$.

**PROOF.** Let $T_N$ satisfy (2.9), let $(y_{1N}, y_{2N}, q_N)$ be the solution of (P.T) given by Proposition 3.7 and let $(y_1, y_2, q)$ be the solution of (1.1) satisfying (1.3) obtained in Proposition 5.6 as the limit of $(y_{1N}, y_{2N}, q_N)$ for $N \to +\infty$.

Fix $\varepsilon > 0$; let $\eta \leq \eta_0$ such that

$$\frac{1}{\omega_i} \| \nabla \eta \| \frac{4\mu \eta^2}{\sqrt{2}} < \frac{\varepsilon}{4}, \quad \forall i = 1, 2,$$

and consider $\tau_N$ such that $q_N(\tau_N) = \eta$ and $q_N(t) \geq \eta$ for all $t \geq \tau_N$.

As in Proposition 5.6 we define

$$\tilde{q}_N(t) = q_N(t + \tau_N), \quad \tilde{y}_{iN}(t) = y_{iN}(t + \tau_N),$$

for all $t \in [-\tau_N, T_N - \tau_N], i = 1, 2$. By (2.13), (2.14), $\tilde{y}_{iN}$ has the following expression for suitable constants $A_{iN}$ and $\mu_{iN}$:

$$\tilde{y}_{iN}(t) = \frac{1}{\omega_i} \int_{-\tau_N}^t \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i(t - s) \, ds$$

$$+ A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN}),$$

or

$$\tilde{y}_{iN}(t) = -\frac{1}{\omega_i} \int_t^{T_N - \tau_N} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N}) V(\tilde{q}_N) \sin \omega_i(t - s) \, ds$$

$$+ A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_{iN} - \varphi_i N).$$
We claim that there exists \( N_1 \in \mathbb{N} \) such that for all \( N \geq N_1 \), all \( t \in [-\tau_N, T_N - \tau_N] \) and \( i = 1, 2 \) we have (with the notation \( \tilde{y}_N = (\tilde{y}_{1N}, \tilde{y}_{2N}) \), \( \alpha = (y_1, y_2) \))

\[
\left\{ \begin{array}{ll} \\
& 1 \left( \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_N)V(\tilde{q}_N) \sin \omega_i(t - s) \, ds - \left. \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y)V(q) \sin \omega_i(t - s) \, ds \right| < \varepsilon \\
& \end{array} \right.
\]

and

\[
\left\{ \begin{array}{ll} \\
& \left( \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_N)V(\tilde{q}_N) \sin \omega_i(t - s) \, ds - \left. \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y)V(q) \sin \omega_i(t - s) \, ds \right| < \varepsilon. \\
& \end{array} \right.
\]

We give a proof only of the first inequality, the other can be proved in the same way. Since \( \tilde{q}_N(-\tau_N) = 0 \) and \( \tilde{q}_N(T_N - \tau_N) = 2\pi \) we extend \( \tilde{q}_N \) by setting

\[
\tilde{q}_N(t) = \left\{ \begin{array}{ll} \\
& 0 \, \forall t \leq -\tau_N, \\
& 2\pi \, \forall t \geq T_N - \tau_N. \\
& \end{array} \right.
\]

and in view of (5.8) and (5.9) we extend \( \tilde{y}_N \) by setting

\[
\tilde{y}_N(t) = \left\{ \begin{array}{ll} \\
& A_{iN} \cos(\omega_0 t + \omega_0 \tau_N + \mu_i N) \quad \forall t \leq -\tau_N, \\
& A_{iN} \cos(\omega_0 t + \omega_0 \tau_N + \mu_i N - \varphi_i) \quad \forall t \geq T_N - \tau_N. \\
& \end{array} \right.
\]

With these extensions the claim follows if we prove that

\[
\int_{-\infty}^{a-1} V(\tilde{q}_N) + \int_{b+1}^{+\infty} V(q) \leq \frac{4\mu_2 \eta^2}{\sqrt{2}}, \quad \int_{-\infty}^{a-1} V(q) + \int_{b+1}^{+\infty} V(q) \leq \frac{4\mu_2 \eta^2}{\sqrt{2}}.
\]

Let us consider the case \( t > b+1 \) (the other cases being simpler). In view of the previous inequalities and by the choice of \( \eta \) we have, for all \( N \geq N_0 \),

\[
\int_{a-1}^{t} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \sin \omega_i(t - s) \, ds \\
\leq \frac{1}{\omega_i} \| \nabla \delta \|_\infty \int_{-\infty}^{a-1} \left( V(\tilde{q}_N) + V(q) \right) + \frac{1}{\omega_i} \| \nabla \delta \|_\infty \int_{b+1}^{t} \left( V(\tilde{q}_N) + V(q) \right) \\
+ \frac{1}{\omega_i} \int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \, ds \\
\leq \frac{1}{\omega_i} \| \nabla \delta \|_\infty \frac{8\mu_2 \eta^2}{\sqrt{2}} + \frac{1}{\omega_i} \int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \, ds \\
< \frac{\varepsilon}{2} + \frac{1}{\omega_i} \int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \, ds.
\]
Since $\tilde{q}_N \to q$ and $\tilde{y}_i \to y_i$ in $L^\infty(a-1,b+1)$, using the dominated convergence theorem we find that
\[
\int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \, ds \to 0 \quad \text{as } N \to \infty.
\]
Thus, there exists $N_1 \geq N_0$ such that for all $N \geq N_1$ we have
\[
\frac{1}{\omega_i} \int_{a-1}^{b+1} \left| \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) - \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \right| \, ds < \frac{\varepsilon}{2}, \quad i = 1, 2,
\]
and the claim is proved.

Using (2.3) and (5.8) we deduce for all $t \in [-2\pi/\omega_i, 4\pi/\omega_i]$ the estimate
\[
|R_{i-} \cos(\omega_i t + f_{i-}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_i N)|
\]
\[
= \left| y_i(t) - \frac{1}{\omega_i} \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \sin(\omega_i t - s) \, ds - \tilde{y}_i(t) \right|
\]
\[
+ \frac{1}{\omega_i} \int_{-\tau_N}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) \sin(\omega_i t - s) \, ds
\]
\[
\leq \frac{1}{\omega_i} \left| \int_{-\infty}^{t} \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q) \sin(\omega_i t - s) \, ds 
\right|
\]
\[
- \int_{-\tau_N}^{t} \frac{\partial \delta}{\partial y_i}(\tilde{y}_{1N}, \tilde{y}_{2N})V(\tilde{q}_N) \sin(\omega_i t - s) \, ds
\]
\[
+ |y_i(t) - \tilde{y}_i(t)| < \varepsilon + |y_i(t) - \tilde{y}_i(t)|.
\]
Since $\tilde{y}_i \to y_i$ in $L^\infty_{loc}$, there exists $N_2 \geq N_1$ such that for all $N \geq N_2$ we have
\[
|R_{i-} \cos(\omega_i t + f_{i-}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_i N)| < 2\varepsilon
\]
\[
\forall t \in [-2\pi/\omega_i, 4\pi/\omega_i],
\]
and, arguing in the same way, we also have
\[
|R_{i+} \cos(\omega_i t + f_{i+}) - A_{iN} \cos(\omega_i t + \omega_i \tau_N + \mu_i N - \varphi_i N)| < 2\varepsilon
\]
\[
\forall t \in [-2\pi/\omega_i, 4\pi/\omega_i].
\]
Rewriting (5.10) for $t = s - \varphi_i N/\omega_i$ we have, for all $N \geq N_2$,
\[
|R_{i-} \cos(\omega_i s + f_{i-} - \varphi_i N) - A_{iN} \cos(\omega_i s + \omega_i \tau_N + \mu_i N - \varphi_i N)| < 2\varepsilon \quad \forall s \in [0, 2\pi/\omega_i].
\]
Putting together this estimate and (5.11) we obtain, for all $N \geq N_2$,
\[
|R_{i-} \cos(\omega_i s + f_{i-} - \varphi_i N) - R_{i+} \cos(\omega_i s + f_{i+})| < 4\varepsilon \quad \forall s \in [0, 2\pi/\omega_i].
\]
Therefore, recalling that $\varphi_i N \equiv \varphi_i$ for all $N$, (5.12) becomes
\[
|R_{i-} \cos(\omega_i s + f_{i-} - \varphi_i) - R_{i+} \cos(\omega_i s + f_{i+})| < 4\varepsilon \quad \forall s \in [0, 2\pi/\omega_1] \forall N \geq N_2;
\]
since ε was arbitrarily chosen, this immediately implies that

$$R_{1-} = R_{1+} \quad \text{and} \quad f_{1+} = f_{1-} - \varphi_1 \mod 2\pi.$$  

Moreover, since \(\varphi_{2N} \to \varphi_2\), there exists \(N_3 \geq N_2\) such that for all \(N \geq N_3\) we have

$$|R_{2-} \cos(\omega_2s + f_{2-} - \varphi_2) - R_{2+} \cos(\omega_2s + f_{2+})|$$

$$\leq |R_{2-} \cos(\omega_2s + f_{2-} - \varphi_2) - R_{2-} \cos(\omega_2s + f_{2-} - \varphi_{2N})|$$

$$+ |R_{2-} \cos(\omega_2s + f_{2-} - \varphi_{2N}) - R_{2+} \cos(\omega_2s + f_{2+})|$$

$$< \epsilon + 4\epsilon \quad \forall s \in [0, 2\pi/\omega_1] \forall N \geq N_3;$$

since \(\epsilon\) was arbitrarily chosen, this implies that

$$R_{2-} = R_{2+} \quad \text{and} \quad f_{2+} = f_{2-} - \varphi_2 \mod 2\pi.$$

6. PROOF OF THEOREM 1.14

In this section we will use the notation already introduced (see (1.13), (2.10)) and we will consider a sequence \(T_N\) satisfying (2.9).

Let us define

$$\tilde{\Gamma}_N = \{ q \in \Gamma_N : \int_0^{T_N} V(q) \leq \tilde{c} \} K;$$

we recall that for all \(N\), \(c(T_N) \leq \tilde{c} := 2\pi^2 + (1 + \tilde{\delta})\|V\|\).  

REMARK 6.1. By Lemma 2.11, if, for \(i = 1, 2\), \(y_i\) is a solution of

$$\ddot{y}_i + \omega_i^2 y_i = \frac{\partial \delta}{\partial y_i}(y_1, y_2)V(q),$$

$$y_i(0) - y_i(T_N) = \dot{y}_i(0) - \dot{y}_i(T_N) = 0,$$

with \(q \in \tilde{\Gamma}_N\), then

$$\|y_i\|_\infty \leq C_N \omega_i \|\nabla \delta\|_\infty K, \quad i = 1, 2,$$

$$|Q_i(y_i)| \leq C_N \omega_i \|\nabla \delta\|_\infty^2 K^2, \quad i = 1, 2.$$  

Also Proposition 3.7 implies that any critical point \((y_{1N}, y_{2N}, q_N)\) of the functional \(f_{T_N}\) at level \(c(T_N)\) is such that \(q_N \in \tilde{\Gamma}_N\), and thus estimates (6.2) hold for \(y_{1N}\) and \(y_{2N}\).

LEMMA 6.3. Let \((y_{1N}, y_{2N}, q_N)\) be a critical point at level \(c(T_N)\) as in Proposition 3.7.

Then

$$\left| c(T_N) - \int_0^{T_N} \left[ \dot{y}_{2N}^2/2 + (1 + \delta(0, 0))V(q_N) \right] \right| \leq \frac{3}{2} \|\nabla \delta\|_\infty^2 K^2 \left( \frac{C_N}{\omega_1} + \frac{C_N}{\omega_2} \right).$$
PROOF. In view of Remark 6.1 the estimates (6.2) hold and we have
\[
\gamma_0(T_N) = \int_0^{T_N} \left[ \bar{q}^2_0/2 + (1 + \delta(0,0))V(q_0) \right] dt
\]
\[
= \frac{1}{2} Q_1(y_{1N}) + \frac{1}{2} Q_2(y_{2N}) + \int_0^{T_N} [\delta(y_{1N}, y_{2N}) - \delta(0,0)]V(q_0) dt
\]
\[
\leq \frac{1}{2} |Q_1(y_{1N})| + \frac{1}{2} |Q_2(y_{2N})| + \|\nabla \bar{q}\|_\infty (\|y_{1N}\|_\infty + \|y_{2N}\|_\infty) \int_0^{T_N} V(q_0) dt
\]
\[
\leq \frac{1}{2} \|\nabla \bar{q}\|_\infty^2 \bar{K}^2 \left( \frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right)\|\nabla \bar{q}\|_\infty \bar{K} \left( \frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right)
\]
\[
= \frac{3}{2} \|\nabla \bar{q}\|_\infty^2 \bar{K}^2 \left( \frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right). \quad \Box
\]

Let us define \(\gamma_0(T_N)\) and \(q_0N\) such that (6.4)
\[
\gamma_0(T_N) = \min_{q \in F} \int_0^{T_N} \left[ \bar{q}^2/2 + (1 + \delta(0,0))V(q) \right] dt = \int_0^{T_N} \left[ \bar{q}^2_0/2 + (1 + \delta(0,0))V(q_0) \right] dt.
\]

Arguing as in Lemma 3.6 it is easy to show that
\[
\gamma_0(T_N) \leq 2\pi^2 + (1 + \delta(0,0))\|V\|_\infty \leq \tilde{c}.
\]

LEMMA 6.5. If \((y_1, y_2) \in E \times E\) is such that
\[
\|y_i\|_\infty \leq V \frac{1 + \delta(0,0)}{\|\nabla \bar{q}\|_\infty \gamma_0(T_N)}, \quad i = 1, 2,
\]

then
\[
\int_0^{T_N} \left[ \bar{q}^2_0/2 + (1 + \delta(y_1, y_2))V(q_0) \right] dt \leq \gamma_0(T_N) + 2\gamma;
\]

moreover, if also \(|Q_1(y_i)| \leq 2\gamma, i = 1, 2\), then
\[
|f(y_1, y_2, q_0)| \leq \gamma_0(T_N) + 4\gamma.
\]

PROOF. We have
\[
\int_0^{T_N} \left[ \bar{q}^2_0/2 + (1 + \delta(y_1, y_2))V(q_0) \right] dt
\]
\[
= \gamma_0(T_N) + \int_0^{T_N} [\delta(y_1, y_2) - \delta(0,0)]V(q_0) dt
\]
\[
\leq \gamma_0(T_N) + \|\nabla \bar{q}\|_\infty (\|y_1\|_\infty + \|y_2\|_\infty) \int_0^{T_N} V(q_0) dt.
\]

Then, by definition of \(\gamma_0(T_N)\) and using the assumptions we obtain
\[
\int_0^{T_N} \left[ \bar{q}^2_0/2 + (1 + \delta(y_1, y_2))V(q_0) \right] dt \leq \gamma_0(T_N) + 2\gamma \frac{1 + \delta(0,0)}{\gamma_0(T_N)} \gamma_0(T_N)
\]
\[
= \gamma_0(T_N) + 2\gamma.
\]
and the first inequality is proved. Next

\[ |f(y_1, y_2, q_{0N})| \leq \frac{1}{2}|Q_1(y_1)| + \frac{1}{2}|Q_2(y_2)| + \int_0^{T_N} \frac{q_{0N}^2}{2} + (1 + \delta(y_1, y_2))V(q_{0N}) \]

\[ \leq 2V + \gamma_0(T_N) + 2V = \gamma_0(T_N) + 4V, \]

and the second inequality is also proved. \( \square \)

**Lemma 6.6.** There exists \( \chi > 0 \) such that

\[ \max \left\{ \left\| \frac{\partial f}{\partial y_1}(y_1, y_2, q) \right\|_\infty, \left\| \frac{\partial f}{\partial y_2}(y_1, y_2, q) \right\|_\infty \right\} \geq \chi \]

for all \( (y_1, y_2, q) \in E \times E \times \bar{F} \) satisfying

\[ |f(y_1, y_2, q)| \leq C_1 = 3\bar{c} \]

and at least one of the following four inequalities:

\[ (6.7) \]

(6.7)

\[ \|y_1\|_\infty \geq \frac{2C^n_1}{\omega_1} \|\nabla \delta\|_\infty \bar{K}, \quad |Q_1(y_1)| \geq \frac{2C^n_1}{\omega_1} (\|\nabla \delta\|_\infty \bar{K})^2, \]

\[ \|y_2\|_\infty \geq \frac{2C^n_2}{\omega_2} \|\nabla \delta\|_\infty \bar{K}, \quad |Q_2(y_2)| \geq \frac{2C^n_2}{\omega_2} (\|\nabla \delta\|_\infty \bar{K})^2. \]

**Proof.** By contradiction, assume that there exists \( (y_{1n}, y_{2n}, q_n) \in E \times E \times \bar{F} \) satisfying

\[ |f(y_{1n}, y_{2n}, q_n)| \leq C_1, \quad \frac{\partial f}{\partial y_i}(y_{1n}, y_{2n}, q_n) \to 0 \quad \text{for all } i = 1, 2, \]

and at least one of the inequalities in \((6.7)\). By Proposition 3.3, up to a subsequence, \( y_{1n} \to y_1 \) in \( E \), \( q_n \to q \) in \( L^\infty \), \( (y_1, y_2, q) \) is a solution of \((2.7)\) and satisfies at least one of the inequalities in \((6.7)\). Moreover, \( q \in \bar{F}_N \), since \( V(q_n) \to V(q) \) almost everywhere and by the Fatou lemma

\[ \int_0^T V(q) \leq \lim \inf_{n \to +\infty} \int_0^T V(q_n) \leq \bar{K}. \]

Thus we get a contradiction with Remark 6.1. \( \square \)

**Lemma 6.8.** Assume

\[ \frac{C_{2i}}{\omega_i} \|\nabla \delta\|_\infty \bar{K} \max\{1, \|\nabla \delta\|_\infty \bar{K}\} \leq \frac{V}{2\bar{K}}, \quad i = 1, 2. \]

Then for \( N \) large enough,

\[ \gamma_0(T_N) - 3V \leq c(T_N) \leq \gamma_0(T_N) + 8V. \]
PROOF. Let \((y_{1N}, y_{2N}, q_{N})\) be a critical point at level \(c(T_N)\). By Lemma 6.3 and by definition of \(\gamma_0(T_N)\) we have
\[
c(T_N) \geq \int_0^{T_N} \left[ q_N/2 + (1 + \delta(0,0))V(q_N) \right] - \frac{3}{2} \left( \frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right) \| \nabla \delta \|^2 \| K \|^2
\]
\[
\geq \gamma_0(T_N) - \frac{3}{2} \left( \frac{C_1^N}{\omega_1} + \frac{C_2^N}{\omega_2} \right) \| \nabla \delta \|^2 \| K \|^2.
\]
Since \(C_1^N = C_{\psi_1}\) and \(C_2^N \to C_{\psi_2}\), for \(N\) large we have
\[
c(T_N) \geq \gamma_0(T_N) - 3\mathcal{V}.
\]

In order to prove the other inequality we will construct an admissible path \(h = (h_1, h_2, q_{0N}) \in \mathcal{H}\) along which the value of the functional \(f\) is less than \(\gamma_0(T_N) + 3\mathcal{V}\). To show the existence of such functions \(h_1, h_2\) we will deform, using a suitable pseudo-gradient vector field, the identity map \(Id: E_1^- \times E_2^- \rightarrow E_1^- \times E_2^-\).

Let \(\psi: \mathbb{R} \rightarrow [0, 1]\) and \(\psi: [0, +\infty) \rightarrow [0, 1]\) be defined by
\[
\psi(s) = \begin{cases} 0, & s \leq 0, \\ \frac{2s}{\gamma_0(T_N) + 4\mathcal{V}}, & 0 \leq s \leq (\gamma_0(T_N) + 4\mathcal{V})/2, \\ 1, & (\gamma_0(T_N) + 4\mathcal{V})/2 \leq s \leq C_1 := 3\bar{\epsilon}, \\ C_1 + 1 - s, & C_1 \leq s \leq C_1 + 1, \\ 0, & s \geq C_1 + 1. \\
\end{cases}
\]
and define the vector field \(v: E \times E \rightarrow E \times E\) by
\[
v_i(y_1, y_2) = -\left[ \psi \left( \frac{\omega_1 \| y_1 \|_{\infty}}{C_1^0 \| \nabla \delta \|_{\infty} \| K \|^2} \right) + \psi \left( \frac{\omega_2 \| y_2 \|_{\infty}}{C_2^0 \| \nabla \delta \|_{\infty} \| K \|^2} \right) + \psi \left( \frac{\omega_1 |Q_1(y_1)|}{C_1^0 \| \nabla \delta \|_{\infty} \| K \|^2} \right) \right]
+ \psi \left( \frac{\omega_2 |Q_2(y_2)|}{C_2^0 \| \nabla \delta \|_{\infty} \| K \|^2} \right) \frac{\partial f}{\partial y_i}(y_1, y_2, q_{0N}) \frac{\partial y_i}{\partial y_i}(y_1, y_2, q_{0N})
+ \psi \left( \frac{\omega_2 |Q_2(y_2)|}{C_2^0 \| \nabla \delta \|_{\infty} \| K \|^2} \right) \frac{\partial f}{\partial y_i}(y_1, y_2, q_{0N})
\]

Since \(v\) is a bounded locally Lipschitz function of \((y_1, y_2)\), the Cauchy problem
\[
\begin{align*}
d\eta \\
\eta(0, y_1, y_2) &= (y_1, y_2),
\end{align*}
\]
has a unique solution for every \((y_1, y_2) \in E \times E\), defined on \([0, +\infty)\).

We claim that, setting \(\tau_3 = (C_1 - \gamma_0(T_N))(1 + \chi)/\chi^2\) (\(\chi\) given by Lemma 6.6), we have
\[
f(\eta_1(\tau_3, y_1, y_2), \eta_2(\tau_3, y_1, y_2), q_{0N}) \leq \gamma_0(T_N) + 3\mathcal{V}
\]
for all \((y_1, y_2)\) such that \(f(y_1, y_2, q_{0N}) \leq C_1\). First of all,

\[
\frac{df}{ds} (\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) = \left \{ \frac{df}{d\eta_1} (\eta_1, \eta_2, q_{0N}), \frac{d\eta_1}{ds}(s, y_1, y_2) \right \} + \left \{ \frac{df}{d\eta_2} (\eta_1, \eta_2, q_{0N}), \frac{d\eta_2}{ds}(s, y_1, y_2) \right \}
\]

and hence \(f(\eta_1(s, y_1, y_2), q_{0N})\) is a nonincreasing function of \(s\) and the claim follows for all \((y_1, y_2)\) such that \(f(y_1, y_2, q_{0N}) \leq y_{0N} + 8V\).

Take now any \((y_1, y_2) \in E \times E\) such that

\[
y_{0}(T_{x}) + 8V < f(y_1, y_2, q_{0N}) \leq C_1.
\]

Assume, by contradiction, that

\[
f(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) > y_{0}(T_{x}) + 8V, \quad \forall s \in [0, \tau_3].
\]

Fix \(s \in [0, \tau_3]\). If \(\|\eta_i(s, y_1, y_2)\|_{\infty} \geq (2C_i^N/\omega_i)\|\nabla \delta\|_{\infty} \tilde{K}\) for \(i = 1, 2\) then by Lemma 6.6 we have

\[
\max \left \{ \left \| \frac{df}{d\eta_1} (\eta_1, \eta_2, q_{0N}) \right \|_{\infty}, \left \| \frac{df}{d\eta_2} (\eta_1, \eta_2, q_{0N}) \right \|_{\infty} \right \} \geq \chi
\]

and, by definition of \(\psi\),

\[
\psi \left (\frac{\omega_i\|\eta_i(s, y_1, y_2)\|_{\infty}}{C_i^N\|\nabla \delta\|_{\infty} \tilde{K}} \right ) = 1.
\]

Otherwise \(\|\eta_i(s, y_1, y_2)\|_{\infty} < (2C_i^N/\omega_i)\|\nabla \delta\|_{\infty} \tilde{K}\) for \(i = 1, 2\). Using the assumption and the definition of \(\tilde{K}\), we obtain, for \(i = 1, 2\) and \(N\) sufficiently large,

\[
\|\eta_i\|_{\infty} < \frac{2C_i^N}{\omega_i}\|\nabla \delta\|_{\infty}^2 \tilde{K}^2 \frac{1}{\|\nabla \delta\|_{\infty}^2 \tilde{K}} < \frac{\|\nabla \delta\|_{\infty} \tilde{K}}{\|\nabla \delta\|_{\infty} \tilde{K}} < \frac{1}{\gamma_{0}(T_{x})}\|\nabla \delta\|_{\infty} (1 + \delta(0, 0)),
\]

so that the first conclusion of Lemma 6.5 holds for \((\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2))\). Then if

\[
|Q_1(\eta_1(s, y_1, y_2))| < \frac{2C_i^N}{\omega_i}\|\nabla \delta\|_{\infty}^2 \tilde{K}^2
\]

(the same argument applies if \(|Q_2(\eta_2(s, y_1, y_2))| < (2C_i^N/\omega_2)\|\nabla \delta\|_{\infty}^2 \tilde{K}^2\), for \(N\)
sufficiently large we have
\[ \frac{1}{2} Q_2(\eta_2(s, y_1, y_2)) = f(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) \]
\[ - \frac{1}{2} Q_1(\eta_1(s, y_1, y_2)) - \int_0^T \left[ \dot{q}_{0N}^2/2 + (1 + \delta(\eta_1, \eta_2)) V(q_{0N}) \right] \]
\[ > \gamma_0(T_N) + 8V - \frac{C^N_1}{\omega_1} \| \nabla \delta \|_\infty^2 \bar{K}^2 \]
\[ = 6V - \frac{C^N_1}{\omega_1} \| \nabla \delta \|_\infty^2 \bar{K}^2 \geq 6V - \frac{\nu}{2\bar{K}} \geq 5V \geq \frac{C^N_2}{\omega_2} \| \nabla \delta \|_\infty^2 \bar{K}^2. \]
that is,
\[ Q_2(\eta_2(s, y_1, y_2)) \geq 10 \frac{C^N_2}{\omega_2} \| \nabla \delta \|_\infty^2 \bar{K}^2, \]
so that by Lemma 6.6 we have
\[ \max \left\{ \left\| \frac{\partial f}{\partial y_1}(\eta_1, \eta_2, q_{0N}) \right\|_\infty, \left\| \frac{\partial f}{\partial y_2}(\eta_1, \eta_2, q_{0N}) \right\|_\infty \right\} \geq \chi \]
and, by definition of \( \psi \),
\[ \psi \left( \frac{\omega_1 |Q_1(\eta_1(s, y_1, y_2))|}{C^N_1 \| \nabla \delta \|_\infty^2 \bar{K}^2} \right) = 1. \]
Therefore we always have
\[ \max \left\{ \left\| \frac{\partial f}{\partial y_1}(\eta_1, \eta_2, q_{0N}) \right\|_\infty, \left\| \frac{\partial f}{\partial y_2}(\eta_1, \eta_2, q_{0N}) \right\|_\infty \right\} \geq \chi \]
and
\[ \psi \left( \frac{\omega_1 |Q_1(\eta_1(s, y_1, y_2))|}{C^N_1 \| \nabla \delta \|_\infty^2 \bar{K}^2} \right) + \psi \left( \frac{\omega_2 |Q_2(\eta_2(s, y_1, y_2))|}{C^N_2 \| \nabla \delta \|_\infty^2 \bar{K}^2} \right) \geq 1. \]
We also have, for all \( s \in [0, \tau_3] \),
\[ \psi(f(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N})) = 1. \]
Then
\[ C_1 - \gamma_0(T_N) - 8V > f(y_1, y_2, q_{0N}) - f(\eta_1(\tau_3, y_1, y_2), \eta_2(\tau_3, y_1, y_2), q_{0N}) \]
\[ = - \int_0^{\tau_3} \frac{df}{ds}(\eta_1(s, y_1, y_2), \eta_2(s, y_1, y_2), q_{0N}) \frac{\dot{q}_{0N}}{ds} \]
\[ \geq \frac{\chi^2}{1 + \chi} \tau_3 = C_1 - \gamma_0(T_N), \]
a contradiction which proves the claim.
We now define

\[ h : E_1^- \times E_2^- \to E \times E \times \bar{\Gamma}, \]
\[(y_1, y_2) \mapsto h(y_1, y_2) = (\eta_1(\tau_3, y_1, y_2), \eta_2(\tau_3, y_1, y_2), q_0N).\]

There exists \(L\) large such that

\[ f(y_1, y_2, q_0N) < 0 \quad \forall \| (y_1, y_2) \| \geq L, \]

and hence, by definition of \((\eta_1, \eta_2)\),

\[ h(y_1, y_2) = (y_1, y_2, q_0N) \quad \forall \| (y_1, y_2) \| \geq L, \]

which shows that \(h \in H\). Finally, since \(f(y_1, y_2, q_0N) \leq C_1\) for all \((y_1, y_2) \in E_1^- \times E_2^-\), we have

\[ f(h(y_1, y_2)) \leq \gamma_0(T_N) + 8V \quad \forall (y_1, y_2) \in E_1^- \times E_2^- , \]

and

\[ c(T_N) \leq \gamma_0(T_N) + 8V. \]

**Lemma 6.11.** Let \((y_{1N}, y_{2N}, q_N) \in E \times E \times \bar{\Gamma}\) be a critical point at level \(c(T_N)\) and assume

\[ \frac{C_{\bar{\delta_i}}}{\omega_i} \| \nabla \delta \| \bar{K} \max \{ 1, \| \nabla \delta \| \bar{K} \} < \frac{\gamma}{2\bar{K}}, \quad i = 1, 2. \]

Then, for \(N\) large, there exists \(\eta_1 \leq \eta_0\) such that for all \(0 < \eta \leq \eta_1\), there exist \(0 < \tau_1^N < \tau_2^N < T_N\) such that

\[ 0 \leq q_N(t) \leq \eta \quad \forall t \in [0, \tau_1^N], \]
\[ q_N(t) \in [\eta, 2\pi - \eta] \quad \forall t \in [\tau_1^N, \tau_2^N], \]
\[ 2\pi - \eta \leq q_N(t) \leq 2\pi \quad \forall t \in [\tau_2^N, T_N], \]

**Proof.** In the proof we will omit the superscripts and subscripts \(N\) for brevity. Let \(\eta \leq \eta_1\), let \(\tau_1 = \inf \{ s \in [0, T] \mid q(s) > \eta \}\) and \(\tau_2 = \sup \{ s \in [0, T] \mid q(s) < 2\pi - \eta \}\). If the lemma does not hold, then arguing as in Lemma 5.1, we deduce that there is \(\tau_1' \in (\tau_1, T]\) such that \(q(\tau_1') = \eta\) and

\[ \int_{\tau_1}^{\tau_1'} \left[ \dot{q}^2/2 + (1 + \delta(0, 0))V(q) \right] \geq 2\bar{\delta} \sqrt{\frac{1 + \delta(0, 0)}{2}} V_{\delta/2}. \]

Now we define a new function \(\bar{q} \in \bar{\Gamma}\) by setting

\[ \bar{q}(t) = \begin{cases} 0, & 0 \leq t \leq \tau_1' - 1, \\ \eta t - \eta \tau_1' + \eta, & \tau_1' - 1 \leq t \leq \tau_1', \\ q(t), & \tau_1' \leq t \leq T. \end{cases} \]
In view of Remark 6.1, we have, for $N$ sufficiently large,

$$\frac{1}{2}Q_1(y) + \frac{1}{2}Q_2(y) \geq -\frac{1}{2} \left( \frac{C_N^1}{\omega_1} + \frac{C_N^2}{\omega_2} \right) \| \nabla \delta \|_\infty^2 \hat{K}^2 > -\mathcal{V},$$

and

$$\int_0^T [\delta(y_1, y_2) - \delta(0, 0)] V(q) \geq -\| \nabla \delta \|_\infty (\| y_1 \|_\infty + \| y_2 \|_\infty) \int_0^T V(q)$$

$$\geq - \left( \frac{C_N^1}{\omega_1} + \frac{C_N^2}{\omega_2} \right) \| \nabla \delta \|_\infty^2 \hat{K}^2 > -2\mathcal{V}.$$

Then, by the previous two estimates, we have

$$c(T) = \frac{1}{2}Q_1(y) + \frac{1}{2}Q_2(y) + \int_0^T \left[ \dot{q}^2 / 2 + (1 + \delta(y_1, y_2)) V(q) \right]$$

$$> -\mathcal{V} + \int_0^T \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right] + \int_0^T [\delta(y_1, y_2) - \delta(0, 0)] V(q)$$

$$> \int_0^T \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right] - 3\mathcal{V}.$$

By (6.12) and by definition of $\mathcal{V}$ (see (1.13)), we have

$$\int_0^T \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right]$$

$$\geq 2\eta \sqrt{1 + \delta(0, 0) / 2} V_{\hat{q}/2} + \int_0^T \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right]$$

$$= 24\mathcal{V} + \int_{\tau_1}^T \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right] - \int_{\tau_1}^{\tau_2} \left[ \dot{q}^2 / 2 + (1 + \delta(0, 0)) V(q) \right]$$

$$\geq 24\mathcal{V} + \gamma_0(T) - \eta^2 / 2 - (1 + \delta(0, 0)) V(\eta)$$

$$\geq 24\mathcal{V} + \gamma_0(T) - \eta^2 / 2 + \mu(1 + \delta(0, 0)).$$

Then, putting together (6.13) and (6.14) and using Lemma 6.8, we obtain

$$c(T) > -3\mathcal{V} + 24\mathcal{V} + \gamma_0(T) - \eta^2 / 2 + \mu(1 + \delta(0, 0))$$

$$\geq -3\mathcal{V} + 24\mathcal{V} + c(T) - 8\mathcal{V} - \eta^2 / 2 + \mu(1 + \delta(0, 0))$$

$$= 13\mathcal{V} - \eta^2 / 2 + \mu(1 + \delta(0, 0)) + c(T)$$

$$> \eta \sqrt{\frac{1 + \delta(0, 0)}{2}} V_{\hat{q}/2} - \eta^2 / 2 + \mu(1 + \delta(0, 0)) + c(T).$$

Therefore, since $\eta \leq \eta_1$, choosing $\eta_1$ small enough, we get the contradiction $c(T) > c(T)$. \hfill \Box

**Proof of Theorem 1.14** Let $(y_{1N}, y_{2N}, q_N) \in E \times E \times E$ be a critical point at level $c(T_N)$. For $0 < \eta \leq \eta_1$, let $r_1^N$ and $r_2^N$ be given by Lemma 6.1. Then, we can repeat the same arguments of Lemma 5.3, Proposition 5.6, and Proposition 5.7, and the theorem is proved. \hfill \Box
REFERENCES


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