Partial Differential Equations — A semilinear problem with a $W^{1,1}_0$ solution, by Lucio Boccardo, Gisella Croce and Luigi Orsina.

Abstract. — We study a degenerate elliptic equation, proving the existence of a $W^{1,1}_0$ distributional solution.

Key words: Elliptic equations, $W^{1,1}$ solutions, Degenerate equations.

MSC 1991 Classification: 35J61, 35J70, 35J75.

In the study of elliptic problems, it is quite standard to find solutions belonging either to $BV(\Omega)$ or to $W^{1,s}(\Omega)$, with $s > 1$. In this paper we prove the existence of a $W^{1,1}_0$ distributional solution for the following boundary value problem:

$$
\begin{cases}
-\text{div} \left( \frac{a(x)\nabla u}{(1 + b(x)|u|)^\alpha} \right) + u = f & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1)

Here $\Omega$ is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $a(x)$, $b(x)$ are measurable functions such that

$$0 < \alpha \leq a(x) \leq \beta, \quad 0 \leq b(x) \leq B,$$

(2)

with $\alpha, \beta \in \mathbb{R}^+$, $B \in \mathbb{R}$ and

$$f(x) \text{ belongs to } L^2(\Omega).$$

(3)

We are going to prove that problem (1) has a distributional solution $u$ belonging to the non-reflexive Sobolev space $W^{1,1}_0(\Omega)$.

Problems like (1) have been extensively studied in the past. In [4], existence and regularity results were obtained for

$$
\begin{cases}
-\text{div} \left( \frac{a(x)\nabla u}{(1 + |u|)^\eta} \right) = f & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(4)

where $0 < \theta \leq 1$ and $f$ belongs to $L^m(\Omega)$ for some $m \geq 1$. A whole range of existence results was proved, yielding solutions belonging to some Sobolev space $W^{1,q}_0(\Omega)$, with $q = q(2,m) \leq 2$ or entropy solutions. In the case where $\theta > 1$ a non-existence result for constant sources has been proved in [1].
As pointed out in [2], existence of solutions can be recovered for any value of \( \theta > 0 \), by adding a lower order term of order zero. If we consider the problem

\[
\begin{aligned}
    &(-\text{div} \left( \frac{a(x)\nabla u}{(1 + |u|)^2} \right)) + u = f & \text{in } \Omega, \\
    &u = 0 & \text{on } \partial \Omega,
\end{aligned}
\]

(5)

with \( f \) in \( L^m(\Omega) \), then the following results can be proved (see [2] and [5]):

i) if \( 2 < m < 4 \), then there exists a distributional solution in \( W^{1,\frac{2m}{m+2}}_0(\Omega) \cap L^m(\Omega) \);

ii) if \( 1 \leq m \leq 2 \), then there exists an entropy solution in \( L^m(\Omega) \) whose gradient belongs to the Marcinkiewicz space \( M^{m/2}(\Omega) \).

In this paper we deal with the borderline case \( m = 2 \), improving the above results as follows.

**Theorem 1.** Assume (2) and (3). Then there exists a distributional solution \( u \in W^{1,1}_0(\Omega) \cap L^2(\Omega) \) to problem (1), in the sense that

\[
\int_{\Omega} a(x) \nabla u \cdot \nabla \varphi + \int_{\Omega} u \varphi = \int_{\Omega} f \varphi,
\]

for all \( \varphi \in W^{1,\infty}_0(\Omega) \).

**Remark 2.** If the operator is nonlinear with respect to the gradient, existence of distributional solutions are studied in [3].

**Proof of Theorem 1.**

Step 1. We begin by approximating our boundary value problem (1) and we consider a sequence \( \{f_n\} \) of \( L^\infty(\Omega) \) functions such that \( f_n \) strongly converges to \( f \) in \( L^2(\Omega) \), and \( |f_n| \leq |f| \) for every \( n \in \mathbb{N} \). The same technique of [2] assures the existence of a solution \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) of

\[
\begin{aligned}
    &(-\text{div} \left( \frac{a(x)\nabla u_n}{(1 + b(x)|u_n|)^2} \right)) + u_n = f_n & \text{in } \Omega, \\
    &u_n = 0 & \text{on } \partial \Omega.
\end{aligned}
\]

(6)

Indeed, let \( M_n = \|f_n\|_{L^\infty(\Omega)} + 1 \), and consider the problem

\[
\begin{aligned}
    &(-\text{div} \left( \frac{a(x)\nabla w}{(1 + b(x)|T_{M_n}(w)|)^2} \right)) + w = f_n & \text{in } \Omega, \\
    &w = 0 & \text{on } \partial \Omega,
\end{aligned}
\]

(7)

where \( T_k(s) = \max(-k, \min(s, k)) \) for \( k \geq 0 \) and \( s \) in \( \mathbb{R} \). The existence of a weak solution \( w \) in \( H^1_0(\Omega) \) of (7) follows from Schauder’s theorem. Choosing
As a test function we obtain, dropping the nonnegative first term,
\[
\int_{\Omega} |w|(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \leq \int_{\Omega} \|f_n\|_{L^\infty(\Omega)}(|w| - \|f_n\|_{L^\infty(\Omega)})_+.
\]
Thus,
\[
\int_{\Omega} (|w| - \|f_n\|_{L^\infty(\Omega)})(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \leq 0,
\]
so that \(|w| \leq \|f_n\|_{L^\infty(\Omega)} < M_n\). Therefore, \(T_{M_n}(w) = w\), and \(w\) is a bounded weak solution of (6).

**Step 2.** We prove some *a priori* estimates on the sequence \(\{u_n\}\). Let \(k \geq 0\), \(i > 0\), and let \(\psi_{i,k}(s)\) be the function defined by
\[
\psi_{i,k}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ i(s - k) & \text{if } k < s \leq k + \frac{1}{i}, \\ 1 & \text{if } s > k + \frac{1}{i}, \\ \psi_{i,k}(s) = -\psi_{i,k}(-s) & \text{if } s < 0. \end{cases}
\]
Note that
\[
\lim_{i \to +\infty} \psi_{i,k}(s) = \begin{cases} 1 & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ -1 & \text{if } s < -k. \end{cases}
\]
We choose \(|u_n|\psi_{i,k}(u_n)\) as a test function in (6), and we obtain
\[
\int_{\Omega} \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} |\psi_{i,k}(u_n)| + \int_{\Omega} \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} \psi_{i,k}'(u_n)|u_n| + \int_{\Omega} u_n|u_n|\psi_{i,k}(u_n)
\]
\[
= \int_{\Omega} f_n|u_n|\psi_{i,k}(u_n).
\]
Since \(\psi_{i,k}'(s) \geq 0\), we can drop the second term; using (2), and the assumption \(|f_n| \leq |f|\), we have
\[
\int_{\{\|u_n\| \geq k\}} \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} |\psi_{i,k}(u_n)| + \int_{\{\|u_n\| \geq k\}} u_n|u_n|\psi_{i,k}(u_n) \leq \int_{\{\|u_n\| \geq k\}} |f| |u_n| |\psi_{i,k}(u_n)|.
\]
Letting \(i\) tend to infinity, we thus obtain, by Fatou’s lemma (on the left hand side) and by Lebesgue’s theorem (on the right hand side, recall that \(u_n\) belongs to \(L^\infty(\Omega)\)),
\[
n\int_{\{\|u_n\| \geq k\}} \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} + \int_{\{\|u_n\| \geq k\}} |u_n|^2 \leq \int_{\{\|u_n\| \geq k\}} |f| |u_n|.
\]

Dropping the nonnegative first term in (8) and using Hölder’s inequality on the right hand side, we obtain

\[
\int_{\{|u_n| \geq k\}} |u_n|^2 \leq \left[ \int_{\{|u_n| \geq k\}} |f|^2 \right]^{1/2} \left[ \int_{\{|u_n| \geq k\}} |u_n|^2 \right]^{1/2}.
\]

Simplifying equal terms we thus have

(9) \[
\int_{\{|u_n| \geq k\}} |u_n|^2 \leq \int_{\{|u_n| \geq k\}} |f|^2.
\]

For \( k = 0 \), (9) gives

(10) \[
\int_{\Omega} |u_n|^2 \leq \int_{\Omega} |f|^2,
\]

so that \( \{u_n\} \) is bounded in \( L^2(\Omega) \). This fact implies in particular that

(11) \[
\lim_{k \to +\infty} \text{meas}(\{|u_n| \geq k\}) = 0, \quad \text{uniformly with respect to } n.
\]

From (8), written for \( k = 0 \), dropping the nonnegative second term and using that \( b(x) \leq B \), we have

\[
x \int_{\Omega} \frac{\nabla u_n}{(1 + B|u_n|)^2} \leq \int_{\Omega} |f| |u_n|.
\]

Hölder’s inequality on the right hand side then gives

\[
x \int_{\Omega} \frac{\nabla u_n}{(1 + B|u_n|)^2} \leq \left[ \int_{\Omega} |f|^2 \right]^{1/2} \left[ \int_{\Omega} |u_n|^2 \right]^{1/2},
\]

so that, by (10), we infer that

(12) \[
x \int_{\Omega} \frac{\nabla u_n}{(1 + B|u_n|)^2} \leq \int_{\Omega} |f|^2.
\]

Step 3. We prove that, up to subsequences, the sequence \( \{u_n\} \) strongly converges in \( L^2(\Omega) \) to some function \( u \).

From (12) we deduce that \( v_n = \log(1 + B|u_n|) \text{ sgn}(u_n) \) is bounded in \( H^1_0(\Omega) \). Therefore, up to subsequences, it converges to some function \( v \) weakly in \( H^1_0(\Omega) \), strongly in \( L^2(\Omega) \), and almost everywhere in \( \Omega \). If we define \( u = e^{v_n - 1} \text{ sgn}(v) \), then \( u_n \) converges almost everywhere to \( u \) in \( \Omega \). Let now \( E \) be a measurable subset of \( \Omega \); then
\[
\int_E |u_n|^2 \leq \int_{E \cap \{|u_n| \geq k\}} |u_n|^2 + \int_{E \cap \{|u_n| < k\}} |u_n|^2 \\
\leq \int_{\{|u_n| \geq k\}} |f|^2 + k^2 \text{meas}(E),
\]

where we have used (9) in the last passage. Thanks to (11), we may choose \(k\) large enough so that the first integral is small, uniformly with respect to \(n\); once \(k\) is chosen, we may choose the measure of \(E\) small enough such that the second term is small. Thus, the sequence \(\{u_n^2\}\) is equiintegrable and so, by Vitali’s theorem, \(u_n\) strongly converges to \(u\) in \(L^2(\Omega)\).

**Step 4.** We prove that, up to subsequences, the sequence \(\{u_n\}\) weakly converges to \(u\) in \(W^{1,1}_0(\Omega)\).

Let again \(E\) be a measurable subset of \(\Omega\), and let \(i\) be in \(\{1, \ldots, N\}\). Then

\[
\int_E |\partial_i u_n| \leq \int_E |\nabla u_n| = \int_E \frac{|\nabla u_n|}{1 + B|u_n|} (1 + B|u_n|) \\
\leq \left[ \int_E \frac{|\nabla u_n|^2}{(1 + B|u_n|)^2} \right]^{1/2} \left[ \int_E (1 + B|u_n|)^2 \right]^{1/2} \\
\leq \left[ \frac{1}{\Omega} \int_\Omega |f|^2 \right]^{1/2} \left[ 2 \text{meas}(E) + 2B^2 \int_E |u_n|^2 \right]^{1/2},
\]

where we have used (12) in the last passage. Since the sequence \(\{u_n\}\) is compact in \(L^2(\Omega)\), we have that the sequence \(\{\partial_i u_n\}\) is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists \(Y_i\) in \(L^1(\Omega)\) such that \(\partial_i u_n\) weakly converges to \(Y_i\) in \(L^1(\Omega)\). Since \(\partial_i u_n\) is the distributional derivative of \(u_n\), we have, for every \(n\) in \(\mathbb{N}\),

\[
\int_\Omega \partial_i u_n \varphi = -\int_\Omega u_n \partial_i \varphi, \quad \forall \varphi \in C^\infty_0(\Omega).
\]

We now pass to the limit in the above identities, using that \(\partial_i u_n\) weakly converges to \(Y_i\) in \(L^1(\Omega)\), and that \(u_n\) strongly converges to \(u\) in \(L^2(\Omega)\); we obtain

\[
\int_\Omega Y_i \varphi = -\int_\Omega u \partial_i \varphi, \quad \forall \varphi \in C^\infty_0(\Omega),
\]

which implies that \(Y_i = \partial_i u\), and this result is true for every \(i\). Since \(Y_i\) belongs to \(L^1(\Omega)\) for every \(i\), \(u\) belongs to \(W^{1,1}_0(\Omega)\), as desired.

Note now that, since \(s \mapsto \log(1 + Bs)\) is Lipschitz continuous on \(\mathbb{R}^+\), and \(u\) belongs to \(W^{1,1}_0(\Omega)\), by the chain rule we have

\[
\nabla \log(1 + B|u|) \text{sgn}(u) = \frac{\nabla u}{1 + B|u|}, \quad \text{almost everywhere in } \Omega.
\]
Hence, from the weak convergence of \( v_n \) to \( v \) in \( H_0^1(\Omega) \) we deduce that

\[
\lim_{n \to +\infty} \frac{\nabla u_n}{1 + B|u_n|} = \frac{\nabla u}{1 + B|u|}, \quad \text{weakly in } (L^2(\Omega))^N.
\]

**Step 5.** We now pass to the limit in the approximate problems (6).
Both the lower order term and the right hand side give no problems, due to the strong convergence of \( u_n \) to \( u \), and of \( f_n \) to \( f \), in \( L^2(\Omega) \).

For the operator term we can write, if \( \phi \) belongs to \( W^{1,0}(\Omega) \),

\[
\int_{\Omega} \frac{a(x)\nabla u_n \cdot \nabla \phi}{(1 + b(x)|u_n|)^2} = \int_{\Omega} \frac{\nabla u_n}{1 + B|u_n|} \cdot \nabla \phi \frac{1 + B|u_n|}{(1 + b(x)|u_n|)^2}.
\]

In the last integral, the first term is fixed in \( L^\infty(\Omega) \), the second is weakly convergent in \( (L^2(\Omega))^N \) by (13), the third is fixed in \( (L^\infty(\Omega))^N \), and the fourth is strongly convergent in \( L^2(\Omega) \), since is bounded from above by \( 1 + B|u_n| \), which is compact in \( L^2(\Omega) \). Therefore, we can pass to the limit to have that

\[
\lim_{n \to +\infty} \int_{\Omega} \frac{a(x)\nabla u_n \cdot \nabla \phi}{(1 + b(x)|u_n|)^2} = \int_{\Omega} \frac{\nabla u \cdot \nabla \phi}{(1 + b(x)|u|)^2},
\]

as desired.

**Remark 3.** Note that if \( b(x) \geq b > 0 \) in \( \Omega \), then we can choose test functions \( \phi \) in \( H_0^1(\Omega) \). Indeed,

\[
0 \leq \frac{1 + B|u_n|}{(1 + b(x)|u_n|)^2} \leq \frac{1 + B|u_n|}{(1 + b|u_n|)^2} \leq C(B, b),
\]

for some nonnegative constant \( C(B, b) \), so that we can rewrite (14) as

\[
\int_{\Omega} \frac{a(x)\nabla u_n \cdot \nabla \phi}{(1 + b(x)|u_n|)^2} = \int_{\Omega} \frac{\nabla u_n}{1 + B|u_n|} \cdot \nabla \phi \frac{1 + B|u_n|}{(1 + b(x)|u_n|)^2},
\]

with the first term fixed in \( L^\infty(\Omega) \), the second weakly convergent in \( (L^2(\Omega))^N \), and the third strongly convergent in the same space by Lebesgue’s theorem.

**References**


A SEMILINEAR PROBLEM WITH A $W^{1,1}_0$ SOLUTION


Received 28 April 2011,
and in revised form 19 September 2011.

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