Conservation of the noetherianity by perfect transcendental field extensions

Magdalena Fernández Lebrón and Luis Narváez Macarro

Abstract

Let \( k \) be a perfect field of characteristic \( p > 0 \), \( k(t)_{\text{per}} \) the perfect closure of \( k(t) \) and \( A \) a \( k \)-algebra. We characterize whether the ring

\[
A \otimes_k k(t)_{\text{per}} = \bigcup_{m \geq 0} (A \otimes_k k(t^{1/p^m}))
\]

is noetherian or not. As a consequence, we prove that the ring \( A \otimes_k k(t)_{\text{per}} \) is noetherian when \( A \) is the ring of formal power series in \( n \) indeterminates over \( k \).

Introduction

Motivated by the generalization of the results in [7] (for the case of a perfect base field \( k \) of characteristic \( p > 0 \)) in this paper we study the conservation of noetherianity by the base field extension \( k \to k(t)_{\text{per}} \), where \( k(t)_{\text{per}} \) is the perfect closure of \( k(t) \). Since this extension is not finitely generated, the conservation of noetherianity is not clear \textit{a priori} for \( k \)-algebras which are not finitely generated.

Our main result states that \( k(t)_{\text{per}} \otimes_k A \) is noetherian if and only if \( A \) is noetherian and for every prime ideal \( p \subset A \) the field \( \bigcap_{m \geq 0} Qt(A/p)^{1/p^m} \) is algebraic over \( k \) (see theorem 3.6). In particular, we are able to apply this result to the case where \( A \) is the ring of formal power series in \( n \) indeterminates over \( k \).

We are indebted to J. M. Giral for giving us the proof of proposition 2.5 and for other helpful comments.

\textit{2000 Mathematics Subject Classification}: 13E05, 13B35, 13A35.

\textit{Keywords}: Perfect field, power series ring, noetherian ring, perfect closure, complete local ring.
1. Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If $B$ is a ring, we shall denote by $\dim(B)$ its Krull dimension and by $\Omega(B)$ the set of its maximal ideals. We shall use the letters $K, L, k$ to denote fields and $\mathbb{F}_p$ to denote the finite field of $p$ elements, for $p$ a prime number. If $p \in \text{Spec}(B)$, we shall denote by $ht(p)$ the height of $p$. Remember that a ring $B$ is said to be equicodimensional if all its maximal ideals have the same height. Also, $B$ is said to be biequicodimensional if all its saturated chains of prime ideals have the same length.

If $B$ is an integral domain, we shall denote by $\mathbb{Q}_t(B)$ its quotient field.

For any $\mathbb{F}_p$-algebra $B$, we denote $B^\sharp = \bigcap_{m \geq 0} B^p^m$.

We shall first study the contraction-extension process for prime ideals relative to the ring extension $K[t] \subset K[t^\frac{1}{p}]$, $K$ being a field of characteristic $p > 0$.

Let us recall the following well known result (cf. for example [4, th. 10.8]):

**Proposition 1.1** Let $K$ be a field of characteristic $p > 0$. Let $g(X)$ be a monic polynomial of $K[X]$. Then, the polynomial $f(X) = g(X^p)$ is irreducible in $K[X]$ if and only if $g(X)$ is irreducible in $K[X]$ and not all its coefficients are in $K^p$.

From the above result, we deduce the following corollary.

**Corollary 1.2** Let $K$ be a field of characteristic $p > 0$. Let $P$ be a non zero prime ideal in $K[t^\frac{1}{p}]$ and let $F(t) \in K[t]$ be the monic irreducible generator of the contraction $P^c = P \cap K[t]$. Then the following conditions hold:

1. If $F(t) = a_0 + a_1 t + \cdots + t^d \in K^p[t]$, then $P = (a_0 + a_1 t^\frac{1}{p} + \cdots + t^\frac{d}{p})$.

2. The equality $P = P^c K[t^\frac{1}{p}]$ holds if and only if $F(t) \notin K^p[t]$.

**Proof:**

1. Consider the polynomial $G(\tau) = a_0 + a_1 \tau + \cdots + \tau^d \in K[\tau](\tau = t^\frac{1}{p})$ and the ring homomorphism $\mu : K[\tau] \rightarrow K[t]$ defined by

$$\mu\left(\sum a_i \tau^i\right) = \sum a_i^p t^i.$$ 

From the identity $\mu(G) = F$ we deduce that $G(\tau)$ is irreducible. Since $G(t^\frac{1}{p})^p = F(t) \in P$, we deduce that $G(t^\frac{1}{p}) \in P$ and then $P = (G(t^\frac{1}{p})).$
2. The equality $P = P^e K[t^\frac{1}{e}]$ means that $F(t) = F(t^p) \in K[\tau]$ generates the ideal $P$, but that is equivalent to saying that $F(t^p)$ is irreducible in $K[\tau]$. To conclude, we apply proposition 1.1. ■

For each $k$-algebra $A$, we define $A(t) := k(t) \otimes_k A$. We also consider the field extension

$$k(\infty) = \bigcup_{m \geq 1} k(t^\frac{m}{p^m}).$$

If $k$ is perfect, $k(\infty)$ coincides with the perfect closure of $k(t)$, $k(t)_{\text{per}}$.

For the sake of brevity, we will write $t_m = t^\frac{1}{p^m}$. We also define

$$A_{(m)} := A(t_m) := A \otimes_k k(t_m) = A(t) \otimes_{k(t)} k(t_m), \quad A_{[m]} := A[t_m]$$

and

$$A_{(\infty)} := A \otimes_k k(\infty) = \bigcup_{m \geq 0} A_{(m)}, \quad A_{[\infty]} := \bigcup_{m \geq 0} A[t_m].$$

Each $A_{(m)}$ (resp. $A_{[m]}$) is a free module over $A(t)$ (resp. over $A[t]$) of rank $p^m$ (because $(t_m)^{p^m} - t = 0$).

For each prime ideal $P$ of $A(\infty)$ we denote $P_{[\infty]} := P \cap A_{[\infty]}$, $P_{[m]} := P \cap A_{[m]} \in \text{Spec}(A_{[m]})$ and $P_{(m)} := P \cap A_{(m)} \in \text{Spec}(A_{(m)})$.

In a similar way, if $Q$ is a prime ideal of $A_{[\infty]}$ we denote $Q_{[m]} := Q \cap A_{[m]} \in \text{Spec}(A_{[m]})$. We have:

- $P = \bigcup_{m \geq 0} P_{(m)}, \quad P_{[\infty]} = \bigcup_{m \geq 0} P_{[m]}$, \quad (resp. $Q = \bigcup_{m \geq 0} Q_{[m]}$).

- $P_{(n)} \cap A_{(m)} = P_{(m)}$ and $P_{[n]} \cap A_{[m]} = P_{[m]}$ for all $n \geq m$ (resp. $Q_{[n]} \cap A_{[m]} = Q_{[m]}$ for all $n \geq m$).

The following properties are straightforward:

1. The $k$-algebras $A_{[m]}$ (respectively $A_{(m)}$) are isomorphic to each other.

2. If $S_m = k[t_m] - \{0\}$, then $A_{(m)} = S_m^{-1} A_{[m]}$.

3. Since $(S_m)^{p^m} \subset S_0 \subset S_m$, we have $A_{(m)} = S_0^{-1} A_{[m]}$ for $m \geq 0$. Consequently $A_{(\infty)} = S_0^{-1} A_{[\infty]}$.

4. If $A$ is a domain (integrally closed), then $A_{[m]}$ and $A_{(m)}$ are domains (integrally closed) for all $m \geq 0$ or $m = \infty$.

5. If $A$ is a noetherian $k$-algebra, then $A_{[m]}$ and $A_{(m)}$ are noetherian rings, for every $m \geq 0$. 


6. If \( A = k[X] = k[X_1, \ldots, X_n] \), then \( A_\infty \) is not noetherian (the ideal generated by the \( t_m, m \geq 0 \), is not finitely generated).

7. If \( I \subset A \) is an ideal, then \( (A/I)_\infty = A_\infty / A_\infty I \).

8. If \( T \subset A \) is a multiplicative subset, then \( (T^{-1}A)_\infty = T^{-1}A_\infty \).

9. If \( A = k[X] \), then \( A_\infty = k_\infty [X] \), hence \( A_\infty \) is noetherian. Moreover, \( A_\infty \) is noetherian for every finitely generated \( k \)-algebra \( A \).

The main goal of this paper is to characterize whether the ring \( A_\infty \) is noetherian (see theorem 3.6 and corollary 3.8).

**Proposition 1.3** With the above notations, the following properties hold:

1. The extensions \( A_{[m-1]} \subset A_{[m]} \) and \( A_{(m-1)} \subset A_{(m)} \) are finite and free, and therefore integral and faithfully flat.

2. The corresponding extensions to their quotient fields are purely inseparable.

**Proof:** Straightforward. ■

**Corollary 1.4** \( A_{[\infty]} \) (resp. \( A_{\infty} \)) is integral and faithfully flat over each \( A_{[m]} \) (resp. over each \( A_{(m)} \)).

From the properties above, we obtain the following lemmas:

**Lemma 1.5** Let \( P' \subset P \) be prime ideals of \( A_{(\infty)} \) (resp. of \( A_{[\infty]} \)). The following conditions are equivalent:

1. \( P' \not\subset P \)
2. There exists an \( m \geq 0 \) such that \( P'_{(m)} \not\subset P_{(m)} \) (resp. \( P'_{[m]} \not\subset P_{[m]} \)).
3. For every \( m \geq 0 \), \( P'_{(m)} \not\subset P_{(m)} \) (resp. \( P'_{[m]} \not\subset P_{[m]} \)).

**Lemma 1.6** Let \( P \) prime ideal of \( A_{(\infty)} \) (resp. of \( A_{[\infty]} \)). The following conditions are equivalent:

1. \( P \) is maximal.
2. \( P_{(m)} \) (resp. \( P_{[m]} \)) is maximal for some \( m \geq 0 \).
3. \( P_{(m)} \) (resp. \( P_{[m]} \)) is maximal for every \( m \geq 0 \).
Corollary 1.7 With the notations above, for every prime ideal $P$ of $A_{(\infty)}$ we have $ht(P) = ht(P_{(m)}) = ht(P_{[m]})$ for all $m \geq 0$. Moreover, $dim(A_{(\infty)}) = dim(A_{(m)})$.

Proof: Since flat ring extensions satisfy the “going down” property, corollary 1.4 implies that $ht(P \cap A_{(m)}) \leq ht(P)$. By corollary 1.4 again, $A_{(\infty)}$ is integral over $A_{(m)}$, then $ht(P) \leq ht(P \cap A_{(m)})$.

The equality $ht(P_{(m)}) = ht(P_{[m]})$ comes from the fact that $A_{(m)}$ is a localization of $A_{[m]}$.

The last relation is a standard consequence of the “going up” property.

Remark 1.8 Corollary 1.7 remains true if we replace $A_{(m)} \subset A_{(\infty)}$ by $A_{[m]} \subset A_{[\infty]}$.

Corollary 1.9 With the notations above, for every $Q \in \text{Spec}(A_{(m)})$ there is a unique $\tilde{Q} \in \text{Spec}(A_{(m+1)})$ such that $\tilde{Q}^c = Q$. Moreover, the ideal $\tilde{Q}$ is given by $\tilde{Q} = \{y \in A_{(m+1)} \mid y^p \in Q\}$.

Proof: This is an easy consequence of the fact that $(A_{(m+1)})^p \subset A_{(m)}$.

Corollary 1.10 Let us assume that $A$ is noetherian and for every maximal ideal $m$ of $A$, the residue field $A/m$ is algebraic over $k$. Then for every $m \geq 0$ we have:

1. $dim(A_{[\infty]}) = dim(A_{[m]}) = dim(A[t]) = n + 1$.

2. $dim(A_{(\infty)}) = dim(A_{(m)}) = dim(A(t)) = n$.

Proof: The first relation comes from remark 1.8 and the noetherianity hypothesis. The second relation comes from corollary 1.7 and [7, proposition 1.4].

The following result is a consequence of [7, theorem 1.6], lemma 1.6 and corollary 1.10.

Corollary 1.11 Let $A$ be a noetherian, biequidimensional, universally catenarian $k$-algebra of Krull dimension $n$, such that for any maximal ideal $m$ of $A$, the residue field $A/m$ is algebraic over $k$. Then every maximal ideal of $A_{(\infty)}$ has height $n$. 
2. The biggest perfect subfield of a formal function field

Throughout this section, $k$ will be a perfect field of characteristic $p > 0$, $A = k[[X]]$, $p \subset A$ a prime ideal, $R = A/p$ and $K = Qt(R)$.

The aim of this section is to prove that the biggest perfect subfield of $K$, $K^\# = \bigcap_{e \geq 0} K^{p^e}$, is an algebraic extension of the field of constants, $k$. This result is proved in proposition 2.5 and it is one of the ingredients in the proof of corollary 3.8.

Proposition 2.1 Under the above hypothesis, it follows that $k = R^\#$.

Proof: Let $m$ be the maximal ideal of $R$. It suffices to prove that $R^\# \subseteq k$.

If $f \in R^\#$, then for every $e > 0$ there exists an $f_e \in R$ such that $f = f_e^{p^e}$.

- Suppose at first that $f$ is not a unit, then $f_e$ is not a unit for any $e > 0$, and $f_e \in m$ for every $e > 0$. Thus, $f \in m^{p^e}$ for every $e > 0$ and by Krull's intersection theorem,

$$f \in \bigcap_{e \geq 0} m^{p^e} = \bigcap_{r \geq 0} m^r = (0).$$

- If $f$ is unit, then $f = f_0 + \tilde{f}$, with $f_0 \in k \subset R^\#$ and $\tilde{f} \in R^\#$ and $f_0$ is unit. By the above case $f = 0$, hence $f \in k$. $\blacksquare$

Proposition 2.2 If $p = (0)$, that is $R = k[[X]]$ and $K = k((X))$, then $k = K^\#$. 

Proof: This is a consequence of prop. 2.1 and the fact that $R$ is a unique factorization domain. $\blacksquare$

In order to treat the general case, let us look at some general lemmas.

Lemma 2.3 (cf. [3, Chap. 5, §15, ex. 8]) If $L$ is a separable algebraic extension of a field $K$ of characteristic $p > 0$, then $L^\#$ is an algebraic extension of $K^\#$.

Proof: If $x \in L^\#$, then $x = y_e^{p^e}$ with $y_e \in L$ for all $e \geq 0$. Since $y_e$ is separable over $K$, $K(y_e) = K(y_e^{p^e}) = K(x)$, it follows that $y_e = x^{p^{-e}} \in K(x)$ and then $x \in K^{p^e} (x^{p^e})$. Therefore

$$[K^{p^e} (x) : K^{p^e}] = [K^{p^e} (x^{p^e}) : K^{p^e}] = [K(x) : K].$$

Thus $x$ satisfies the same minimal polynomial over $K^{p^e}$ and over $K$ for all $e \geq 0$, and the coefficients of this minimal polynomial must be in $K^\#$. So $x$ is algebraic over $K^\#$. $\blacksquare$
Lemma 2.4 Every algebraic extension of a perfect field is perfect.

Proof: This is obvious because this is true for the finite algebraic extensions. ■

Proposition 2.5 Let $k$ be a perfect field of characteristic $p > 0$, $A = k[[X]] = k[[X_1, \ldots, X_n]]$, $p \subset A$ a prime ideal, $R = A/p$ and $K = \text{Qt}(R)$. Then $K^\sharp$ is an algebraic extension of $k$.

Proof: Let $r = \text{dim}(A/p) \leq n$. By the normalization lemma for power series rings (cf. [1, 24.5 and 23.7])\(^2\), there is a new system of formal coordinates $Y_1, \ldots, Y_n$ of $A$, such that

- $p \cap k[[Y_1, \ldots, Y_r]] = \{0\}$,
- $k[[Y_1, \ldots, Y_r]] \hookrightarrow \frac{A}{p} = R$ is a finite extension, and
- $k((Y_1, \ldots, Y_r)) \hookrightarrow K$ is a separable finite extension.

The proposition is then a consequence of proposition 2.2 and lemma 2.3.\(^3\) ■

Remark 2.6 Actually, under the hypothesis of proposition 2.5, J.M. Giral and the authors have proved that the following stronger properties hold:

1. If $R$ is integrally closed in $K$, then $K^\sharp = k$.
2. In the general case, $K^\sharp$ is a finite extension of $k$.

3. Noetherianity of $A \otimes_k k(t)_{\text{per}}$

Throughout this section, $k$ will be a perfect field of characteristic $p > 0$, keeping the notations of section 1.

Proposition 3.1 Let $K$ be a field extension of $k$ and suppose that $K^\sharp$ is algebraic over $k$. For every prime ideal $\mathcal{P} \in \text{Spec}(K_{[\infty]})$ such that $\mathcal{P} \cap k[t] = 0$ there exists an $m_0 \geq 0$ such that $\mathcal{P}_m$ is the extended ideal of $\mathcal{P}_{m_0}$ for all $m \geq m_0$.

\(^1\)Due to J. M. Giral.

\(^2\)The proof of the normalization lemma for power series rings in [1] uses generic linear changes of coordinates and needs the field $k$ to be infinite. This proof can be adapted for an arbitrary perfect coefficient field (infinite or not) by using non linear changes of the form $Y_i = X_i + F_i(X_{i+1}^p, \ldots, X_n^p)$, where the $F_i$ are polynomials with coefficients in $\mathbb{F}_p$.

\(^3\)In particular, if $k$ is algebraically closed, we would have $K^\sharp = k$. 
Proof: The extension $k[t] \subset K[t]$ is integral and then $\mathcal{P} \cap K[t] = 0$.

We can suppose $\mathcal{P} \neq (0)$. From Remark 1.8, we have $\text{ht}(\mathcal{P}[i]) = \text{ht}(\mathcal{P}) = 1$ for every $i \geq 0$. Let $F_i(t_i) \in K[t_i]$ be the monic irreducible generator of $\mathcal{P}[i]$. From Corollary 1.2, for each $i \geq 0$ there are two possibilities:

1. $F_i \in K^p[t_i]$, then $F_{i+1}(t_{i+1}) = F_i(t_i)^{1/p}$.

2. $F_i \notin K^p[t_i]$, then $\mathcal{P}_{[i+1]}^{(p)} = (\mathcal{P}_{[i]}^{(p)})^c$ and $F_{i+1}(t_{i+1}) = F_i(t_i) = F_i(t_i^{p-1})$.

Since $\mathcal{P} \cap K^2[t] = (0)$,

$$F_0(t_0) \notin \bigcap_{m \geq 0} K^{p^m}[t_0] = \bigcap_{m \geq 0} K^{p^m}[t_0]$$

and there exists an $m_0 \geq 0$ such that $F_0(t_0) \in K^{p^{m_0}}[t_0]$ and $F_0(t_0) \notin K^{p^{m_0+1}}[t_0]$.

From (1) we have $F_i(t_i) = F_0(t_0)^{1/p^i} \in K^{p^{m_0-i}}[t_i]$ for $i = 0, \ldots, m_0 - 1$ and $F_{m_0}(t_{m_0}) \notin K^p[t_{m_0}]$. Hence, applying (2) repeatedly we find $F_{i+m_0}(t_{i+m_0}) = F_{m_0}(t_{m_0}) = F_{m_0}(t_{j+m_0})$ and $\mathcal{P}_{[j+m_0]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for all $j \geq 1$.

Corollary 3.2 Under the same hypothesis of proposition 3.1, $\mathcal{P}$ is the extended ideal of some $\mathcal{P}_{[m_0]}$.

Proof: This is a consequence of prop. 3.1 and the equality $\mathcal{P} = \bigcup_{m \geq 0} \mathcal{P}_{[m]}$.

Let $B$ be a free algebra over a ring $A$ and $S \subset A$ a multiplicative subset. We denote by $I \mapsto I^S, J \mapsto J^C$ (resp. $I \mapsto I^e, J \mapsto J^c$) the extension-contraction process between the rings $A$ or $S^{-1}A$ (resp. $A$ or $B$) and the rings $B$ or $S^{-1}B$ (resp. $S^{-1}A$ or $S^{-1}B$).

Proposition 3.3 With the notations above, let $\mathcal{P}_1$ be a prime ideal in $B$ such that $\mathcal{P}_1 \cap S = \emptyset$. Let $\mathcal{P}_0 = \mathcal{P}_1^C, \mathcal{P}_1^c = \mathcal{P}_1^C$ and $\mathcal{P}_0 = \mathcal{P}_1^c$. If $\mathcal{P}_1 = \mathcal{P}_0^e$, then $\mathcal{P}_1 = \mathcal{P}_0^e$.

Proof: Let $\{\epsilon_i\}$ be a $A$-basis of $B$. Since $\mathcal{P}_1 \cap S = \emptyset$, it is clear that $\mathcal{P}_1^e = \mathcal{P}_1, \mathcal{P}_1^c = \mathcal{P}_0$ and $\mathcal{P}_0 = \mathcal{P}_1^c$. If $\mathcal{P}_1 = \mathcal{P}_0^e$, we have

$$\mathcal{P}_1 = \mathcal{P}_1^{ce} = \mathcal{P}_1 = (\mathcal{P}_0^e)^c = (\mathcal{P}_0^e)^c = (\mathcal{P}_0^e)^{ce} = \bigcup_{s \in S} (\mathcal{P}_0^e : s)_B \supset \mathcal{P}_0^e.$$ 

To prove the other inclusion, take an $s \in S$ and let $f = \sum a_i \epsilon_i$ be an element of $(\mathcal{P}_0^e : s)_B$ with $a_i \in A$. Then, $sf = \sum (sa_i) \epsilon_i \in \mathcal{P}_0^e$ and from the equality $\mathcal{P}_0^e = \{ \sum b_i \epsilon_i \mid b_i \in \mathcal{P}_0 \}$ we deduce that $sa_i \in \mathcal{P}_0$ and $a_i \in (\mathcal{P}_0^e : s)_A = \mathcal{P}_0$. Therefore $f \in \mathcal{P}_0^e$.
**Proposition 3.4** Let $R$ be an integral $k$-algebra, $K = \text{Qt}(R)$, and suppose that $K^\natural$ is algebraic over $k$. Then any prime ideal $\mathcal{P} \in \text{Spec}(R[\infty])$ with $\mathcal{P} \cap k[t] = 0$ and $\mathcal{P} \cap R = 0$ is the extended ideal of some $\mathcal{P}_{[m_0]}$, $m_0 \geq 0$.

**Proof:** Let us write $T = R \setminus \{0\}$. We have $K = T^{-1}R$ and $K_{[m]} = T^{-1}R_{[m]}$ for all $m \geq 0$ or $m = \infty$. We define $\mathcal{P} = T^{-1}\mathcal{P}$. We easily deduce that $\mathcal{P}_{[m]} = T^{-1}\mathcal{P}_{[m]}$ for all $m \geq 0$.

From proposition 3.1, there exists an $m_0 \geq 0$ such that $\mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for every $m \geq m_0$. Then, proposition 3.3 tells us that $\mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for every $m \geq m_0$, so $\mathcal{P} = \bigcup \mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$.


**Proposition 3.5** Let $K$ be a field extension of $k$ and suppose that $K^\natural$ is not algebraic over $k$. Then $K(\infty)$ is not noetherian.

**Proof:** Let $s \in K^\natural$ be a transcendental element over $k$.

For each $m \geq 0$, let $s_m = s^{mp} \in K$ and $\alpha_m = t_m - s_m$. Let $P$ be the ideal in $K(\infty)$ generated by the $\alpha_m$, $m \geq 0$. We have $\alpha_m = \alpha_{m+1}^p$ and $P_{(m)} = K_{(m)}\alpha_m$ for all $m \geq 0$.

Suppose that $P$ is finitely generated. Then, there exists an $m_0 \geq 0$ such that $P_{[m]}$ is the extended ideal of $P_{[m_0]}$ for every $m \geq m_0$. Then, proposition 3.3 tells us that $P_{[m]}$ is the extended ideal of $P_{[m_0]}$ for every $m \geq m_0$, so $\mathcal{P} = \bigcup \mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$.

**Theorem 3.6** Let $k$ be a perfect field of characteristic $p > 0$ and let $A$ be a $k$-algebra. The following properties are equivalent:

(a) The ring $A$ is noetherian and for any $\mathfrak{p} \in \text{Spec}(A)$, the field $\text{Qt}(A/\mathfrak{p})^\natural$ is algebraic over $k$.

(b) The ring $A(\infty)$ is noetherian.
Proof: Let first prove \((a) \Rightarrow (b)\). By Cohen’s theorem (cf. [6, (3.4)]), it is enough to prove that any \(P \in \text{Spec}(A_{(\infty)}) - \{(0)\}\) is finitely generated.

From corollaries 1.7 and 1.10, we have

\[
ht(P_{[m]}) = ht(P_{(m)}) = ht(P_{[\infty]}) = ht(P) = r \leq n.
\]

Consider the prime ideal of \(A\):

\[
p := A \cap P = A \cap P_{[\infty]} = A \cap P_{[m]} = A \cap P_{(m)}.
\]

There are two possibilities (cf. [5, prop. 5.5.3]):

(i) \(ht(p) = r = ht(P_{[m]})\) and \(P_{[m]} = p[t_m]\), for every \(m \geq 0\).

(ii) \(ht(p) = r - 1 = ht(P_{[m]}) - 1\), \(p[t_m] \not\subseteq P_{[m]}\) and \(A/p \not\subseteq A[t_m]/P_{[m]}\) is algebraic generated by \(t_m \mod P_{[m]}\), for every \(m \geq 0\).

In case (i), \(P_{[\infty]}\) and \(P\) are the extended ideals of \(p\) and they are finitely generated.

Suppose we are in case (ii). We denote \(R = A/p, K = Qt(R)\). Then:

\[
R_{[m]} = A_{[m]}/p[t_m], \quad R_{[\infty]} = A_{[\infty]}/A_{[\infty]}p = A_{[\infty]}/\bigcup_{m \geq 0} p[t_m].
\]

Define

\[
\mathcal{P} := R_{[\infty]}P_{[\infty]} = P_{[\infty]}/\bigcup_{m \geq 0} p[t_m] \in \text{Spec}(R_{[\infty]}).
\]

We have \(\mathcal{P}_{[m]} = \mathcal{P} \cap R_{[m]} = P_{[m]}/p[t_m], \mathcal{P} \cap R = \mathcal{P} \cap k[t] = 0\) and

\[
ht(P_{[m]}) = ht\left( (P_{[m]}/p[t_m]) \right) = 1, \quad ht(\mathcal{P}) = ht\left( P_{[\infty]}/\bigcup_{m \geq 0} p[t_m] \right) = 1.
\]

We conclude by applying proposition 3.4: there exists an \(m_0 \geq 0\) such that \(\mathcal{P}\) is the extended ideal of \(P_{[m_0]}\). Then, \(P_{[\infty]}\) is the extended ideal of \(P_{[m_0]}\) and \(P = A_{(\infty)}P_{[\infty]} = A_{(\infty)}P_{[m_0]}\) is finitely generated.

Let us prove now \((b) \Rightarrow (a)\). Since \(A_{(\infty)}\) is faithfully flat over \(A\), we deduce that \(A\) is noetherian.

Let \(p \in \text{Spec}(A)\) and let \(R = A/p, K = Qt(R)\). Noetherianity of \(A_{(\infty)}\) implies, first, noetherianity of \(R_{(\infty)}\), and second, noetherianity of \(K_{(\infty)}\). To conclude we apply proposition 3.5. ■
**Corollary 3.7** Let \( k \) be a perfect field of characteristic \( p > 0 \) and let \( A \) be a noetherian \( k \)-algebra. The following properties are equivalent:

(a) The ring \( A(\infty) \) is noetherian.

(b) The ring \( (A_m)(\infty) \) is noetherian for any maximal ideal \( m \in \Omega(A) \).

**Proof:** For (a) \( \Rightarrow \) (b) we use the fact that \((A_m)(\infty) = A_m \otimes_A A(\infty)\).

For (b) \( \Rightarrow \) (a), let \( p \subset A \) be a prime ideal and let \( m \) be a maximal ideal containing \( p \). From hypothesis (b), the ring \((A_m)(\infty)\) is noetherian. Then, from theorem 3.6 we deduce that the field \( Qt(A/p)^t = Qt(A_m/A_mp)^t\) is algebraic over \( k \). From theorem 3.6 again we obtain (a). ■

**Corollary 3.8** Let \( k \) be a perfect field of characteristic \( p > 0 \), \( k' \) an algebraic extension of \( k \) and \( A = k'[X_1, \ldots, X_n] \). Then, the ring \( A(\infty) = k(t)_{per} \otimes_k A \) is noetherian.

**Proof:** It is a consequence of lemma 2.4, proposition 2.5 and theorem 3.6. ■

**Corollary 3.9** Let \( k \) be a perfect field of characteristic \( p > 0 \). If \((B, m)\) is a local noetherian \( k \)-algebra such that \( B/m \) is algebraic over \( k \), then \( B(\infty) = k(t)_{per} \otimes_k B \) is noetherian. In particular, the field \( Qt(B/p)^t \) is algebraic over \( k \) for every prime ideal \( p \subset B \).

**Proof:** Let \( k' = B/m \). By Cohen structure theorem (cf. [5, Chap. 0, th. 19.8.8]), the completion \( \widehat{B} \) of \( B \) is a quotient of a power-series ring \( A \) with coefficients in \( k' \). Since \( \widehat{B}(\infty) \) is also a quotient of \( A(\infty) \), we deduce from corollary 3.8 that \( \widehat{B}(\infty) \) is noetherian. Since \( \widehat{B} \) is faithfully flat over \( B \), the ring \( \widehat{B}(\infty) \) is also faithfully flat over \( B(\infty) \). So, \( B(\infty) \) is noetherian.

The last assertion is a consequence of theorem 3.6. ■

**Corollary 3.10** Let \( k \) be a perfect field of characteristic \( p > 0 \). For any noetherian \( k \)-algebra \( A \) such that the residue field \( A/m \) of every maximal ideal \( m \in \Omega(A) \) is algebraic over \( k \), the ring \( A(\infty) \) is noetherian. Furthermore, if \( A \) is regular and equicodimensional then \( A(\infty) \) is also regular and equicodimensional of the same dimension as \( A \).

**Proof:** The first part is a consequence of corollaries 3.7 and 3.9. For the last part, we use corollary 1.11, the fact that all \( A_{(m)}, m \geq 0 \) are regular and of the same (global homological = Krull) dimension ([7, th. 1.6] and [2]). ■
References


Recibido: 4 de abril de 2002

Magdalena Fernández Lebrón
Departamento de Álgebra
Facultad de Matemáticas, Universidad de Sevilla
P.O. Box 1160, 41080 Sevilla, Spain
lebron@algebra.us.es

Luis Narváez Macarro
Departamento de Álgebra
Facultad de Matemáticas, Universidad de Sevilla
P.O. Box 1160, 41080 Sevilla, Spain
narvaez@algebra.us.es

Dedicated to Prof. José L. Vicente on his sixtieth birthday. Both authors are partially supported by DGESIC, PB97-0723, BFM2001-3207 and FEDER.