A weighted version of Journé's Lemma

Donald Krug and Alberto Torchinsky

In this paper we discuss a weighted version of Journé's covering lemma, a substitute for the Whitney decomposition of an arbitrary open set in $\mathbb{R}^2$ where squares are replaced by rectangles. We then apply this result to obtain a sharp version of the atomic decomposition of the weighted Hardy spaces $H^p_w(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ and a description of their duals when $p$ is close to 1.

A nonnegative locally integrable function $w(x, y)$ on $\mathbb{R}^2$ is called a weight. A weight $w$ is said to satisfy Muckenhoupt's $A_p(\mathbb{R} \times \mathbb{R})$ condition on rectangles, or plainly the $A_p$ condition, $1 < p < \infty$, provided that

$$\sup_R \left( \frac{1}{|R|} \int_R w(x, y) \, dx \, dy \right) \left( \frac{1}{|R|} \int_R w(x, y)^{-1/(p-1)} \, dx \, dy \right)^{p-1} \leq c,$$

where $R$ runs over all rectangles with sides parallel to the coordinate axes. When $p = 1$ this condition reduces to

$$\frac{1}{|R|} \int_R w(x, y) \, dx \, dy \leq c \inf_{(x, y) \in R} \, w(x, y), \quad \text{all } R.$$

We say that $w$ satisfies the $A_{\infty}(\mathbb{R} \times \mathbb{R})$ condition if it satisfies the $A_p$ condition for some $p < \infty$. The constant $c$ that appears on the right-hand side in the inequalities above is called the $A_p$ constant of $w$, and a property is said to be independent in $A_p$ provided it depends on $c$, and
not on the particular weight \( w \) in \( A_p \) involved. By the Lebesgue differentiation theorem it readily follows that if \( w \) satisfies the \( A_p \) condition, then \( w(x, \cdot) \) satisfies Muckenhoupt's \( A_p(\mathbb{R}) \) condition, uniformly for a.e. \( x \), with constant \( \leq c \), the \( A_p \) constant for \( w \); similarly for \( w(\cdot, y) \).

The same holds for \( A_{\infty} \): an \( A_{\infty} \) weight \( w \) is uniformly in \( A_{\infty}(\mathbb{R}) \) for a.e. \( x \), or \( y \), fixed. By well-known properties of \( A_{\infty} \) weights, if \( w(x, \cdot) \) is an \( A_{\infty}(\mathbb{R}) \) weight uniformly in \( x \), then the following holds: given \( x \in \mathbb{R} \) and \( 0 < \varepsilon < 1 \), there exists \( \eta > 0 \), such that if \( A \subseteq I \) and

\[
\frac{w(x, A)}{w(x, I)} > \eta, \quad \text{then} \quad \frac{w(x', A)}{w(x', I)} > \varepsilon \quad \text{for a.e. } x' \in \mathbb{R}.
\]

It is clear that we may always choose \( \eta \geq 1/2 \) above, and we do so.

Under the assumption that \( w \) is uniformly \( A_{\infty} \) for a variable fixed and uniformly doubling for the other variable fixed, the weighted strong maximal operator \( M_{S,w}f(x, y) \) given by

\[
M_{S,w}f(x, y) = \sup_{(x,y) \in \mathbb{R}} \frac{1}{w(R)} \int_R \int_R |f(u, v)| w(u, v) \, du \, dv,
\]

is known to be bounded in \( L^2_w(\mathbb{R}^2) \), say, cf. [JT] and [F1].

Given a bounded open set \( \Omega \subseteq \mathbb{R}^2 \), \( x \in \mathbb{R} \) and \( t > 0 \), following [J], let

\[ E_{x, t} = \{ y \in \mathbb{R} : [x-t, x+t] \times \{y\} \subseteq \Omega \} \]

Each \( E_{x, t} \) is open, because \( \Omega \) is open, and, for each \( x \), \( E_{x, t} \) is decreasing in \( t \).

Let \( E_{x, t} = \bigcup_k J^k_{x, t} \) denote the decomposition of \( E_{x, t} \) into open interval components, and let \( t(k, x) \) be the infimum over those \( \tau \geq t \) such that

\[
 w(x, J^k_{x, t} \cap E_{x, \tau}) \leq \eta w(x, J^k_{x, t}) ,
\]

where \( 1/2 \leq \eta < 1 \) corresponds to the value \( \varepsilon = 1/2 \) above.

**Proposition 1.** Given a bounded open set \( \Omega \), let

\[
\hat{\Omega} = \bigcup_{x,t,k} (x-t(k,x), x+t(k,x)) \times J^k_{x, t} ,
\]

and assume that the weight \( w(x, y) \) is uniformly \( A_{\infty}(\mathbb{R}) \) for a variable fixed, and uniformly doubling for the other variable fixed. Then \( w(\hat{\Omega}) \leq cw(\Omega) \), where \( c \) is independent of \( \Omega \).
A weighted version of Journé’s Lemma

\[ \text{PROOF. As it is readily seen by the containment relation between the sets involved, we have} \]

\[ (3) \, w\left(\{(x - s, x + s) \times J_{x,t}^k \cap \Omega\} \right) \geq w\left(\{(x - s, x + s) \times (J_{x,t}^k \cap E_{x,s})\} \right). \]

Now, if \( s < t(k, x) \), from (2) and (1) it follows that

\[ (4) \, w(x', J_{x,t}^k \cap E_{x,s}) > \frac{1}{2} w(x', J_{x,t}^k), \quad \text{a.e.} \, \, x' \in \mathbb{R}. \]

Thus, integrating (4) over \( (x - s, x + s) \), combining the resulting expression with (3), and setting \( R = (x - s, x + s) \times J_{x,t}^k \), we obtain

\[ (5) \, \int \int_R \chi_{\Omega}(x, y) w(x, y) \, dx \, dy > \frac{1}{2} \int \int_R w(x, y) \, dx \, dy. \]

Now, if \((x', y') \in \hat{\Omega}\), there exist \( x, t, k \) such that \( x' \in (x - t(k, x), x + t(k, x)) \), and also \( s < t(k, x) \) so that \((x', y') \in (x - s, x + s) \times J_{x,t}^k = R\). Whence, by (5),

\[ \hat{\Omega} \subseteq \{ M_{S,w}(\chi_{\Omega}) > \frac{1}{2} \}, \]

and by the continuity of \( M_{S,w} \) in \( L_w^2(\mathbb{R}^2) \),

\[ w(\hat{\Omega}) \leq cw(\Omega), \]

with \( c \) independent of \( \Omega \).

**Proposition 2.** Suppose \( \Omega \) and \( w \) are as in Proposition 1, and that \( \phi \) is a nondecreasing function with \( \phi(0) = 0 \). Then

\[ \int_0^{+\infty} \int_{\mathbb{R}} \sum_k \int_{J_{x,t}^k} \phi\left(\frac{t}{t(k, x)}\right) w(x, y) \, dy \, dx \, \frac{dt}{t} \leq c w(\Omega) \int_0^1 \phi(s) \, \frac{ds}{s}, \]

where \( c \) is a constant independent of \( \Omega \) and \( \phi \).

**PROOF.** From (2) it readily follows that

\[ w(x, J_{x,t}^k) \leq \frac{1}{1 - \eta} w(x, J_{x,t}^k \cap E_{x,t(k,x)}). \]
Thus, save for the factor $1/(1 - \eta)$, the left-hand side of the above expression does not exceed
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{\infty} \sum_k \chi_{J_{x,t}^k \setminus E_{x,t}(k, x)} (y) \phi\left( \frac{t}{t(k, x)} \right) w(x, y) \frac{dt}{t} dy dx
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} B(x, y) w(x, y) dx dy,
\]
say. We want to show that
\[
B(x, y) \leq c \chi_{\Omega}(x, y) \int_0^1 \phi(s) \frac{ds}{s}.
\]

Clearly if $(x, y) \notin \Omega$, then $B(x, y) = 0$. Also, if $(x, y) \in \Omega$, at most one summand in the above sum does not vanish, the one corresponding to the index $k$, say. Thus,
\[
B(x, y) = \chi_{\Omega}(x, y) \int_0^{\infty} \chi_{J_{x,t}^k \setminus E_{x,t}(k, x)} (y) \phi\left( \frac{t}{t(k, x)} \right) \frac{dt}{t}.
\]

Let $T(x, y) = \sup \{s : [x - s, x + s] \times \{y\} \subseteq \Omega\}$. Since $J_{x,t}^k$ is an interval component of $E_{x,t}$, from this definition it readily follows that $t \leq T(x, y)$. We may also assume that $T(x, y) \leq t(k, x)$, for if $t(k, x) < T(x, y)$, then it follows that $y \in E_{x,t(k,x)}$, and the integrand above vanishes. Whence
\[
B(x, y) \leq \chi_{\Omega}(x, y) \int_0^{T(x,y)} \chi_{J_{x,t}^k \setminus E_{x,t}(k, x)} (y) \phi\left( \frac{t}{t(k, x)} \right) \frac{dt}{t}
\]
\[
\leq \chi_{\Omega}(x, y) \int_0^{T(x,y)} \phi\left( \frac{t}{T(x,y)} \right) \frac{dt}{t}
\]
\[
= \chi_{\Omega}(x, y) \int_0^1 \phi(s) \frac{ds}{s}.
\]

Replacing this in the expression above gives the desired estimate.

Now we pass to discuss the discrete version of Journé’s covering lemma. For $\Omega$ as before, let $M_2(\Omega)$ denote the collection of those rectangles (dyadic) $R = I \times J$ so that $I, J$ are dyadic and $J$ is maximal with respect to the inclusion property in $\Omega$. 
Given arbitrary intervals \( I, J \), not necessarily dyadic, let

\[
J^{I'} = \{ y \in J : I \times \{ y \} \subseteq \Omega \}.
\]

If by \( rI \) we denote the interval concentric with \( I \) with sidelength \( r \) times that of \( I \), we define \( \tilde{I} \) as follows: it is the smallest interval \( I' \) concentric with \( I \) such that

\[
w(x, J^I) \leq \frac{1}{2} w(x, J) \quad \text{for a.e. } x \in \mathbb{R}.
\]

**Proposition 3.** Suppose the open set \( \Omega \), weight \( w \) and the function \( \phi \) are as in Proposition 2. Then

\[
\sum_{R \in M_2(\Omega)} w(R) \phi\left(\frac{|I|}{|I|}\right) \leq c \left( \int_0^1 \phi(8s) \frac{ds}{s} \right) w(\Omega).
\]

**Proof.** Let \( \mathcal{I}_n \) denote the collection of those dyadic intervals \( I \) such that \( R = I \times j \in M_2(\Omega) \) for some dyadic interval \( J \), and \( |I| = 2^n \), \( n = 0, \pm 1, \pm 2, \ldots \). Then, since \( J^I \supseteq J \) for \( R = I \times J \in M_2(\Omega) \), the sum we want to estimate does not exceed

\[
\sum_n \sum_{I \in \mathcal{I}_n} \int_I \sum_{J' \supseteq J} \int_{J'} \phi\left(\frac{|I|}{|J'|}\right) w(x, y) \, dx \, dy
\]

\[
\leq \sum_n \int_{2^{n-3}} \sum_{I \in \mathcal{I}_n} \int_I \sum_{J' \supseteq J} \int_{J'} w(x, y) \, dx \, dy \phi\left(\frac{|I|}{|J'|}\right) \frac{dt}{t}.
\]

Fix now \( n \), and \( I \in \mathcal{I}_n \). Let \( S = \{ x \in I : [x - t, x + t] \subseteq I \} \), and note that for \( t \in (2^{n-3}, 2^{n-2}) \), since \( |I| = 2^n \), \( 2S \supseteq I \). Thus by the uniform doubling property of \( w(\cdot, y) \), the above expression does not exceed

\[
c \sum_n \int_{2^{n-3}} \sum_{I \in \mathcal{I}_n} \int_I \sum_{J' \supseteq J} \int_{J'} \phi\left(\frac{|I|}{|J'|}\right) w(x, y) \, dy \, dx \, \frac{dt}{t}.
\]

Furthermore, since \( t \geq 2^{n-3} = |(1/8)I| \), and since \( x \in S \), it readily follows that \( y \in J_{x,t}^k \), one of the components of \( E_{x,t} \), and the above expression is dominated by

\[
c \sum_n \int_{2^{n-3}} \sum_{I \in \mathcal{I}_n} \int_I \sum_{J_{x,t}^k} \phi\left(\frac{|I|}{|J_{x,t}^k|}\right) w(x, y) \, dy \, dx \, \frac{dt}{t}.
\]
Since in the above expression \(|I| \leq 8t\), and since \([x - t, x + t] \subseteq I\) and consequently \(J = J^I \subseteq J_{x,t}^k\), we see from the definitions of \(t(k, x)\) and \(|\tilde{I}|\) (recall that \(1/2 \leq \eta < 1\)) that these quantities are essentially the same. Moreover, since in the definition of \(t(k, x)\) the right-hand side is larger, so must be the left-hand side, and consequently \(t(k, x) \leq |\tilde{I}|\). Thus we may continue our estimation by

\[
c \sum_n \int_{2n+3}^{2n+2} \sum_{I \in \mathcal{I}_n} \int_S \sum_k \int_{J_{x,t}^k} \phi\left(\frac{8t}{t(k, x)}\right) w(x, y) \, dy \, dx \, \frac{dt}{t}
\leq c \sum_n \int_{2n+3}^{2n+2} \int_{\mathbb{R}} \sum_k \int_{J_{x,t}^k} \phi\left(\frac{8t}{t(k, x)}\right) w(x, y) \, dy \, dx \, \frac{dt}{t}
\leq c \int_0^{+\infty} \int_{\mathbb{R}} \sum_k \int_{J_{x,t}^k} \phi\left(\frac{8t}{t(k, x)}\right) w(x, y) \, dy \, dx \, \frac{dt}{t}.
\]

Then the proof proceeds exactly as that of Proposition 2.

**Proposition 4.** Under the conditions of Proposition 3, we have

\[
w\left( \bigcup_{R \in \mathcal{S}_3(\Omega)} \tilde{I} \times J \right) \leq c w(\Omega), \quad c \text{ independent of } \Omega.
\]

Because the proof is similar to that of Proposition 1 it is omitted.

As a first application of the weighted version of Journé's lemma we discuss the atomic decomposition of the weighted Hardy spaces \(H^p_w(\mathbb{R}_+^2 \times \mathbb{R}_+^2), 0 < p \leq 1\).

Given a smooth function \(\psi\) supported in \((-1, 1)\) with nonvanishing integral, put

\[
\psi_{s,t}(x, y) = \frac{1}{s} \psi\left(\frac{x}{s}\right) \frac{1}{t} \psi\left(\frac{y}{t}\right), \quad s, t > 0,
\]

and for a distribution \(f\) in \(\mathbb{R}^2\), let

\[
f^*(x, y) = \sup_{\varepsilon_1, \varepsilon_2 > 0} |f * \psi_{\varepsilon_1, \varepsilon_2}(x, y)|.
\]

Then \(H^p_w(\mathbb{R}_+^2 \times \mathbb{R}_+^2)\) consists of those distributions \(f\) such that \(f^* \in L^p_w(\mathbb{R}^2)\), and we set \(\|f\|_{H^p_w} = \|f^*\|_{L^p_w}\). We would like to discuss
the so-called atomic decomposition of elements of these spaces when \( w \in A_r(\mathbb{R} \times \mathbb{R}), 1 \leq r \leq 2 \).

A function \( a(x, y) \) is called a \((p, w)\)-atom, if

a) the set where \( a(x, y) \neq 0 \) is contained in a set \( \Omega \), with
\[
\|a\|_{L^p_\omega} \leq w(\Omega)^{1/2-1/p} < +\infty,
\]

b) \( a = \sum a_R \), where the subatoms \( a_R \) have the following properties:

i) if \( a_R(x, y) \neq 0 \), then \( (x, y) \in \hat{R} = 3I \times 3J \), and \( \hat{R} \subseteq \Omega \),

ii) \( R = I \times J \) is a dyadic rectangle, and no rectangle is repeated,

iii) for all integers \( \alpha \leq [r/p - 1] \),
\[
\int_I x^{\alpha} a_R(x, y) \, dx = \int_J y^{\alpha} a_R(x, y) \, dy = 0,
\]

iv) \( \left( \sum \|a_R\|_{L^p_\omega}^2 \right)^{1/2} \leq w(\Omega)^{1/p - 1/2} \).

The atomic decomposition states that \( f \in H^p_\omega(\mathbb{R} \times \mathbb{R}) \) if and only if \( f = \sum \lambda_i a_i \), where the \( a_i \)'s are \((p, w)\)-atoms, the sum is taken in the sense of distributions and in the norm sense, and \( \sum \lambda_i^p \leq c \|f\|^p_{H^p_\omega} \).

That \( f \in H^p_\omega \) can be decomposed into such sum is very similar to the unweighted case considered by R. Fefferman in [F2], and the proof is not discussed here.

Thus, we propose to prove the following result

**Proposition 5.** Suppose that \( w \in A_r \) and that \( a \) is a \((p, w)\)-atom. Then \( \|a\|_{H^p_\omega} \leq c \), where \( c \) is independent of \( a \) and independent in \( A_r \).

**Proof.** Given \( R = I \times J \subseteq \Omega \), let \( \hat{I} \) now denote the interval which is the largest between \( \hat{I} \) from Journě's lemma and \( 2I \); and similarly for \( J \). Let \( \hat{R} = (\hat{I} \times J) \cup (I \times \hat{J}) = \hat{I} \times \hat{J} \). If
\[
\hat{\Omega} = \bigcup_{R \subseteq \Omega} \hat{R},
\]
then by Proposition 4 above, \( w(\hat{\Omega}) \leq c w(\Omega) \), where \( c \) is independent of \( \Omega \) and \( w \).
In order to estimate \( \|a\|_{L^p_0} = \|a^*\|_{L^p_0} \), we break up the integral that gives the \( L^p_0 \) norm into \( \hat{\Omega} \) and \( \mathbb{R}^2 \setminus \hat{\Omega} \). The contribution over \( \hat{\Omega} \) is readily handled: indeed, if \( M_S \) denotes the strong maximal function, then since \( w \in A_\delta(\mathbb{R} \times \mathbb{R}) \), and \( a^*(x, y) \leq c M_S a(x, y) \), by Proposition 4 it follows that

\[
\int_{\hat{\Omega}} a^*(x, y)^p \, w(x, y) \, dx \, dy \\
\leq c \int_{\hat{\Omega}} M_S a(x, y)^p \, w(x, y)^{p/2} \, w(x, y)^{1-p/2} \, dx \, dy \\
\leq c \left( \int_{\hat{\Omega}} M_S a(x, y)^2 \, w(x, y) \, dx \, dy \right)^{p/2} w(\hat{\Omega})^{1-p/2} \\
\leq c \|a\|_{L^p_0}^p w(\hat{\Omega})^{1-p/2} \\
\leq c w(\Omega)^{p(1/2-1/p)} w(\Omega)^{1-p/2} \\
\leq c.
\]

Next, if \( a = \sum_n a_R \), we consider each subatom \( a_R \) separately; by translation if necessary we may assume that \( a_R \) is centered at the origin, and if \( R = I \times J \), we estimate the larger expression

\[
\int_{\mathbb{R} \setminus J} \int_{\mathbb{R} \setminus I} a_R^*(x, y)^p \, w(x, y) \, dx \, dy.
\]

For this purpose we show that the following two estimates hold:

\[
\int_{\mathbb{R} \setminus J} \int_{\mathbb{R} \setminus I} a_R^*(x, y)^p \, w(x, y) \, dx \, dy \leq c \left( \frac{|R|}{|I|} \right)^p,
\]

and

\[
\int_{\mathbb{R} \setminus I} \int_{\mathbb{R} \setminus J} a_R^*(x, y)^p \, w(x, y) \, dx \, dy \leq c \left( \frac{|J|}{|I|} \right)^p.
\]

We do (6) first. Let \( p_N(\psi, \cdot) \) denote the Taylor expansion of degree \( N \) of \( \psi \). By the moment condition on \( a_R \) it readily follows that

\[
|a_R * \psi_{\varepsilon_1, \varepsilon_2}(x, y)| \\
\leq \frac{1}{\varepsilon_1 \varepsilon_2} \int_R \left| \psi \left( \frac{x-u}{\varepsilon_1} \right) - p_N \left( \psi, \frac{u}{\varepsilon_1} \right) \right| \\
\cdot \left| \psi \left( \frac{y-v}{\varepsilon_2} \right) - p_N \left( \psi, \frac{v}{\varepsilon_2} \right) \right| \, |a_R(u, v)| \, du \, dv \\
\leq \frac{c}{\varepsilon_1 \varepsilon_2} \int_R \left( \frac{|u|}{\varepsilon_1} \right)^{N+1} \left( \frac{|v|}{\varepsilon_2} \right)^{N+1} \, |a_R(u, v)| \, du \, dv.
\]
Notice that if \( x \notin 2I \) and \( u \in I \), then \(|x|/2 \leq |x - u| \leq 2|x|\), so that if \( \varepsilon_1 \leq |x|/2 \), then \( \psi_{\varepsilon_1}(x - u) = 0 \). We may thus assume that \( \varepsilon_1 \geq |x|/2 \), and likewise that \( \varepsilon_2 \geq |x_2|/2 \). Therefore, since \(|u v| \leq |\hat{R}|\), the above expression does not exceed

\[
\frac{c|\hat{R}|^{N+1}}{(|x| |y|)^{N+2}} \int_R |a_R(u, v)| w(u, v)^{1/2} w(u, v)^{-1/2} du dv 
\leq \frac{c|\hat{R}|^{N+1}}{(|x| |y|)^{N+2}} \|a_R\|_{L^2_w} \left( \int_R w(u, v)^{-1/2} du dv \right)^{1/2}.
\]

Now, by the bound on \( a_R \), and since \( w \in A_2(\mathbb{R} \times \mathbb{R}) \), this expression does not exceed

\[
c \frac{|R|^{N+1}}{(|x| |y|)^{N+2}} \frac{1}{w(\Omega)^{1/p-1/2}} \frac{|\hat{R}|}{w(\hat{R})^{1/2}}.
\]

Thus

\[
\int_{\mathbb{R}\setminus I} \int_{\mathbb{R}\setminus J} a_R^*(x, y)^p w(x, y) \, dx \, dy 
\leq c \frac{|\hat{R}|^{(N+2)p}}{w(\Omega)^{1-p/2}} \frac{1}{w(\hat{R})^{p/2}} \int_{\mathbb{R}\setminus I} \int_{\mathbb{R}\setminus J} \frac{w(x, y)}{|x|^{(N+2)p}} \, dx \, dy.
\]

In order to estimate the integral in (8) note that if \( w \in A_r(\mathbb{R} \times \mathbb{R}) \), then by the choice of \( N, N(p + 2) \geq r \); the argument proceeds now using well-known estimates in the case of the line, cf. [T, Proposition IX, 4.5 (iv)], and the fact that the restrictions of \( w \) are uniformly in \( A_r(\mathbb{R}) \) for each variable fixed. Indeed, the expression in question does not exceed

\[
c \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{w(x, y)}{(|x| + |\hat{I}|)^{(N+2)p}} \, dx \, dy 
\leq \frac{1}{\hat{I}^{(N+2)p}} \int_{\mathbb{R}} \frac{1}{(|y| + |\hat{J}|)^{(N+2)p}} \int_{\mathbb{R}} \frac{w(x, y)}{(|x| + |\hat{I}|)^{(N+2)p}} \, dx \, dy
\]

\[
\leq \frac{1}{\hat{I}^{(N+2)p}} \int_{\mathbb{R}} \frac{1}{(|y| + |\hat{J}|)^{(N+2)p}} \int_I w(x, y) \, dx \, dy 
= c \frac{1}{\hat{I}^{(N+2)p}} \int_I \int_{\mathbb{R}} \frac{w(x, y)}{|J|^{(N+2)p}} \, dx \, dy
\]
$$\leq c \frac{1}{|I|^{(N+2)p}} \frac{1}{|J|^{(N+2)p}} \int_I \int_J w(x, y) \, dx \, dy$$

$$= c \frac{w(\hat{R})}{(|I| |J|)^{(N+2)p}}.$$ 

Thus, replacing this estimate in the right-hand side of (8), and by Proposition 4, we obtain that the left-hand side there does not exceed

$$c \frac{|R|^{(N+2)p}}{w(\Omega)^{1-p/2} w(R)^{p/2} (|I| |J|)^{(N+2)p}} \frac{w(\hat{R})}{w(\Omega)} \leq c \left( \frac{|R|}{|\hat{R}|} \right)^{(N+1)p} \left( \frac{|R|}{|I| |J|} \right)^p \left( \frac{w(R)}{w(\Omega)} \right)^{1-p/2} \leq c \left( \frac{|R|}{|\hat{R}|} \right)^p,$$

which, of course, gives (6).

We show now estimate (7). By the moment condition on \( a_R \) we get

$$|a_R \ast \psi_{\varepsilon_1, \varepsilon_2}(x, y)| \leq \frac{1}{\varepsilon_1 \varepsilon_2} \left| \int_I \int_J \left( \psi \left( \frac{x - u}{\varepsilon_1} \right) - \psi \left( \frac{y - v}{\varepsilon_2} \right) \right) a_R(u, v) \, du \, dv \right|$$

$$\leq \frac{c}{\varepsilon_1} \int_I \left( \frac{|u|}{\varepsilon_1} \right)^{N+1} \ |M^2 a_R(u, y)| \, du,$$

where \( M^2 \) denotes the Hardy maximal operator in the second variable only. Thus

$$a^*_R(x, y) \leq c \frac{I_{\mathbb{R}^N}^{N+1}}{|x|^{(N+2)p}} \int_I M^2 a_R(u, y) \, du,$$

and consequently,

$$\int_{\mathbb{R}^N} \int_J a^*_R(x, y) w(x, y) \, dx \, dy$$

$$\leq c |I|^{(N+1)p} \int_{\mathbb{R}^N} \frac{1}{|x|^{(N+2)p}} \left( \int_I M^2 a_R(u, y) \, du \right)^p \, w(x, y) \, dy \, dx$$

$$\leq c |I|^{(N+2)p} \int_I \left( \frac{1}{|I|} \int_J M^2 a_R(u, y) \, du \right)^p \int_{\mathbb{R}^N} \frac{w(x, y)}{|x|^{(N+2)p}} \, dx \, dy$$

$$\leq c |I|^{(N+2)p} \int_J M^1(M^2 a_R)(x, y)^p \int_{\mathbb{R}^N} \frac{w(x, y)}{|x|^{(N+2)p}} \, dx \, dy,$$
where $M^1$ denotes the Hardy maximal operator in the first variable only.

As before, by the usual $A_v(\mathbb{R})$ properties it follows that for $y$-a.e.

$$\int_{\mathbb{R} \setminus I} \frac{w(x, y)}{|x|^{(N+2)p}} \, dx \leq \frac{c}{|I|^{(N+2)p}} \int_I w(x, y) \, dx,$$

and consequently,

$$\int_{\mathbb{R} \setminus I} \int_I a^w_R(x, y)^p \, w(x, y) \, dx \, dy \leq c \left( \frac{|I|}{|I|} \right)^{(N+2)p} \int_I \int_J M^1(M^2 a_R)(x, y)^p \, w(x, y) \, dy \, dx.$$

Note that the above integral looks similar to the first expression we estimated, and, in fact, since $w \in A_2(\mathbb{R} \times \mathbb{R})$, it does not exceed

$$c \|a_R\|_{L^p_w}^p w(\tilde{R})^{1-p/2} \leq c \|w(\Omega)^{p/2-1}w(\tilde{R})^{1-p/2} \leq c,$$

which completes the proof of (7).

We would like now to improve on these estimates; following R. Fefferman, put

$$b_R(x, y) = \frac{w(R)^{1/2 - 1/p}}{\|a_R\|_{L^p_w}} a_R(x, y),$$

and observe that $b_R(x, y)$ is an atom supported on $R$, and that the above estimate applied to $b_R$ gives

$$\int_{\mathbb{R} \setminus I} \int_{\mathbb{R}} b^*_R(x, y)^p w(x, y) \, dy \, dx \leq c \left( \frac{|I|}{|I|} \right)^p + c \left( \frac{|R|}{|I|} \right)^p \leq c \left( \frac{|I|}{|I|} \right)^p.$$

Thus, replacing $b_R$ by its expression in terms of $a_R$, it readily follows that

$$\int_{\mathbb{R} \setminus I} \int_{\mathbb{R}} a^*_R(x, y)^p w(x, y) \, dx \, dy \leq c \|a_R\|_{L^p_w}^p w(R)^{1-p/2} \left( \frac{|I|}{|I|} \right)^p.$$
This is all we need, as we are now ready to sum over the collection of all the maximal dyadic rectangles $R$ contained in $\Omega$. In fact, by Hölder’s inequality and the properties of atoms, it follows that

$$
\sum_R \int_{\mathbb{R}} \int_{\mathbb{R}} a_R^*(x, y)^p \, w(x, y) \, dx \, dy
\leq c \sum_R \|a_R\|_{L^p_{\omega}}^p w(R)^{1-p/2} \left(\frac{|I|}{|\Omega|}\right)^p
\leq c \left(\sum_R \|a_R\|_{L^p_{\omega}}^p\right)^{p/2} \left(\sum_R w(R)^{(1-p/2)(2/p)'} \left(\frac{|I|}{|\Omega|}\right)^{p(2/p)' - 1/2}\right)^{1/p'}
\leq c w(\Omega)^{p/2 - 1} \left(\sum_R w(R) \left(\frac{|I|}{|\Omega|}\right)^{(2-p)/2}\right)^{1-p/2}
$$

We now invoke Journé’s lemma with $\phi(s) = s^{(2-p)/2}$, and note that the above expression is then dominated by

$$
c w(\Omega)^{p/2 - 1} w(\Omega)^{1-p/2} \leq c,
$$
and the proof is complete.

To complete the results discussed here we consider a description of the duals to the Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $r/2 < p \leq 1$, when $w \in A_r(\mathbb{R} \times \mathbb{R})$; by known properties of weights the case $H^1_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ when $w \in A_2(\mathbb{R} \times \mathbb{R})$ is included.

Given a real-valued function $b$ on $\mathbb{R}^2$, and a weight $v \in A_r(\mathbb{R} \times \mathbb{R})$, consider the following expression: if $\Omega$ is a bounded open set in $\mathbb{R}^2$, and $R$ runs over the collection of the maximal dyadic rectangles contained in $\Omega$, then set

$$
\|b\|_{y,v} = \sup_{b_R} \left(\frac{1}{w(\Omega)^v} \sum_R \|b - b_R\|^2_{L^2(v)}\right)^{1/2},
$$

where $b_R$ runs over the family of functions of the form

$$
b_R(x, y) = c_1 b_1(y) + c_2 b_2(x),
\text{ supp } b_1 \subseteq J, \text{ supp } b_2 \subseteq I, \quad R = I \times J.
$$

We then have
Proposition 6. $H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$, the dual of the Hardy space $H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $r/2 < p \leq 1$ can be identified with $B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$, the collection of those square integrable functions $b$ such that $\|b\|_{2/p-1,1/w} < +\infty$.

**Proof.** We begin by showing that each $b \in B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$ induces a bounded linear functional on $H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, with norm less than or equal to $c \|b\|_{2/p-1,1/w}$.

Suppose, then, that $a = \sum_R a_R$ is a $(p,w)$-atom in $H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, and let $b \in B_{2/p-1,1/w}(\mathbb{R}\times\mathbb{R})$. Then, by the properties of atoms, a judicious choice of the $b_R$’s, and Cauchy’s inequality,

$$\left| \iint_{\mathbb{R}^2} a(x,y)b(x,y) \, dx \, dy \right| \leq \sum_R \left| \iint_{\mathbb{R}^2} a_R(x,y)b(x,y) \, dx \, dy \right| \leq \sum_R \iint_{\mathbb{R}^2} |a_R(x,y)| |b(x,y) - b_R(x,y)| w(x,y)^{1/2} w(x,y)^{-1/2} \, dx \, dy \leq \left( \sum_R \|a_R\|_{L^2_w}^2 \right)^{1/2} \left( \sum_R \|b - b_R\|_{L^2_{1/w}}^2 \right)^{1/2} \leq w(\Omega)^{(1-2/p)/2} \left( \sum_R \|b - b_R\|_{L^2_{1/w}}^2 \right)^{1/2} \leq \|b\|_{2/p-1,1/w}.$$ 

Next, if $f \in H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, then it admits an atomic decomposition $f = \sum_j \lambda_j a_j$, where the $a_j$’s are $(p,w)$-atoms and $\|f\|_{H_w^p} \sim (\sum_j |\lambda_j|^p)^{1/p}$. Thus,

$$\left| \iint_{\mathbb{R}^2} f(x,y)b(x,y) \, dx \, dy \right| \leq \sum_j |\lambda_j| \left| \iint_{\mathbb{R}^2} a_j(x,y)b(x,y) \, dx \, dy \right| \leq \left( \sum_j |\lambda_j|^p \right)^{1/p} \|b\|_{2/p-1,1/w},$$

and the assertion follows.

Conversely, suppose that $L \in H_w^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)^*$. Then on a dense subset there, consisting of smooth functions, $L$ can be represented by
\( b(x, y) \) in the form

\[
L(f) = \int_{\mathbb{R}^2} f(x, y) b(x, y) \, dx \, dy .
\]

Let now \( \Omega \) be a bounded open set in \( \mathbb{R}^2 \), and suppose that \( \Omega = \bigcup_R R \), where the \( R \)'s are the maximal dyadic rectangles contained in \( \Omega \). Now, given a function \( g \in L^2_w(\mathbb{R}^2) \) and \( R = I \times J \), set

\[
g_R(x, y) = \frac{1}{|I|} \int_I g(u, y) \, du + \frac{1}{|J|} \int_J g(x, v) \, dv
\]

\[- \frac{1}{|R|} \int \int_R g(u, v) \, du \, dv .\]

Then

\[
\int_I (g(x, y) - g_R(x, y)) \, dx = \int_J (g(x, y) - g_R(x, y)) \, dy = 0 ,
\]

and

\[
\| g - g_R \|_{L^2_w(R)} \leq c \| g \|_{L^2_w(\Omega)} .
\]

The first assertion is readily verified, and to see the second we consider the first term in \( g_R \), the others being handled analogously. Note that since \( w(\cdot, y) \in A_2(\mathbb{R}) \) uniformly in \( y \),

\[
\int_J \int_I \left( \frac{1}{|I|} \int_I g(u, y) \, du \right)^2 w(x, y) \, dx \, dy
\]

\[
\leq \int_J \int_I \left( \frac{1}{|I|} \int_I g(u, y)^2 w(u, y) \, du \right) \left( \frac{1}{|I|} \int_I \frac{1}{w(u, y)} \, du \right) w(x, y) \, dx \, dy
\]

\[
= \int_J \int_I g(u, y)^2 w(u, y) \left( \frac{1}{|I|} \int_I \frac{1}{w(u, y)} \, du \right) \left( \frac{1}{|I|} \int_I w(x, y) \, dx \right) \, du \, dy .
\]

Now, since \( w(\cdot, y) \in A_2(\mathbb{R}) \), uniformly in \( y \), the above expression involving the inner integrals does not exceed the \( A_2 \) constant of \( w \), and the whole expression is less than or equal to \( c \| g \|_{L^2_w} \), as claimed. The other terms are dealt with in a similar fashion.

Suppose now that \( \Omega \) is a bounded open subset in \( \mathbb{R}^2 \), and that \( f \in L^2(\Omega) \) is such that

\[
\left( \sum_R \| f \|_{L^2_w(R)}^2 \right)^{1/2} = 1 .
\]
Then, by the above remark, there is a constant $c$ such that
\begin{equation}
\alpha(x, y) = c \frac{1}{w(\Omega)^{1/p-1/2}} \sum_R (f(x, y) - f_R(x, y)) \chi_R(x, y),
\end{equation}
is a $(p, w)$-atom of norm 1, and consequently,
\begin{align*}
\|L\| &\geq \|L(a)\| \\
&= c \frac{1}{w(\Omega)^{1/p-1/2}} \sum_R \iint_R (f(x, y) - f_R(x, y)) b(x, y) \, dx \, dy \\
&= c \frac{1}{w(\Omega)^{1/p-1/2}} \iint_R f(x, y) (b(x, y) - b_R(x, y)) \, dx \, dy.
\end{align*}
Since this estimate holds for all such $f$'s, by duality it readily follows that
\begin{equation}
c \left( \frac{1}{w(\Omega)^{2/p-1}} \sum_R \|b - b_R\|_{L^1_{w(R)}}^2 \right)^{1/2} \leq \|L\|,
\end{equation}
which is precisely what we wanted to show.

References.


*Recibido: 4 de febrero de 1.993*

Donald Krug  
Department of Mathematics and Computer Science  
Northern Kentucky University  
Highland Heights KY 41009, U.S.A.

and

Alberto Torchinsky  
Department of Mathematics  
Indiana University  
Bloomington IN 47405, U.S.A.