**Abstract.** We prove analogue statements of the spherical maximal theorem of E. M. Stein, for the lattice points $\mathbb{Z}^n$. We decompose the discrete spherical “measures” as an integral of Gaussian kernels $s_{t,e}(x) = e^{2\pi i |x|^2(t+i\varepsilon)}$. By using Minkowski’s integral inequality it is enough to prove $L^p$-bounds for the corresponding convolution operators. The proof is then based on $L^2$-estimates by analysing the Fourier transforms $\hat{s}_{t,e}(\xi)$, which can be handled by making use of the “circle” method for exponential sums. As a corollary one obtains some regularity of the distribution of lattice points on small spherical caps.

1. Introduction.

Let us denote by $\sigma_\lambda$ the characteristic function of the sphere of radius $\lambda^{1/2}$ in $\mathbb{Z}^n$, i.e.

$$\sigma_\lambda = \chi_{\{x \in \mathbb{Z}^n : |x|^2 = \lambda\}}$$

and

$$S_\lambda = \sum_{x \in \mathbb{Z}^n} \sigma_\lambda(x).$$

Let $\Lambda$ be a fixed positive number and define the spherical maximal operator as

$$M_\Lambda f(x) = \sup_{\Lambda \leq \Lambda < 2\Lambda} \left| \left( \frac{\sigma_\lambda}{S_\lambda} * f \right)(x) \right|.$$
It is proved that

**Theorem 1.** If \( n \geq 5, \ p > n/(n - 2), \ f \in L^p(\mathbb{Z}^n), \) then

\[
\|M_{\Lambda} f\|_{L^p(\mathbb{Z}^n)} \leq c_{n,p} \|f\|_{L^p(\mathbb{Z}^n)},
\]

where the constant \( c_{n,p} \) is independent of \( \Lambda. \)

We generalize estimate (2) to the case of the \( k \)-spheres, which are defined by

\[
\sigma_{\lambda} = \chi_{\{x \in \mathbb{Z}^n : |x|^2 = \lambda\}}
\]

and it is proved.

**Theorem 2.** Let \( k \geq 2, \ K = 2^{k-1}, \) then for \( n > 4Kk, \ p > n/(n - 4Kk) \) we have

\[
\|M_{\Lambda,k} f\|_{p} \leq c_{n,k,p} \|f\|_{p},
\]

where the constant \( c_{n,k,p} \) is independent of \( \Lambda. \)

It is well-known, that for \( n \geq 5, \) there exist constants \( 0 < c_n < C_n \) such that

\[
c_n \lambda^{n/2 - 1} \leq S_{\lambda} \leq C_n \lambda^{n/2 - 1}.
\]

We start with the decompositions

\[
\sigma_{\lambda}(x) = \int_0^1 e^{2\pi i (|x|^2 - \lambda)t} \, dt
\]

and

\[
e^{-2\pi i \lambda} \sigma_{\lambda}(x) = \int_0^1 e^{2\pi i |x|^2 (t + i\varepsilon)} \ e^{-2\pi i \lambda t} \, dt
\]

and define the modified maximal operator as

\[
\tilde{M}_{\lambda,\varepsilon} f(x) = \sup_{\Lambda \leq \lambda < 2\Lambda} \left| (e^{-2\pi i \lambda} - \varepsilon^{n/2 + 1} \sigma_{\lambda}) \ast f(x) \right|.
\]

From inequality (4) it follows, that if \( \Lambda \leq \varepsilon^{-1} \) and \( f \geq 0, \) we have

\[
M_{\lambda} f(x) \leq c\tilde{M}_{\lambda,\varepsilon} f(x),
\]
for every \( x \), so it is enough to prove (2) for the modified maximal operator. Introducing the convolution operator

\[
S_{t, \varepsilon} f = s_{t, \varepsilon} * f, \quad \text{where} \quad s_{t, \varepsilon}(x) = e^{2\pi i [x] \varepsilon (t + \varepsilon)}.
\]

Minkowski's integral inequality together with formulae (4) and (5) imply

\[
\| \tilde{M}_{\Lambda, \varepsilon} f \|_p \leq c_n \Lambda^{-n/2+1} \int_0^1 \| S_{t, \varepsilon} f \|_p \, dt.
\]

In order to understand the integrand on the right side of inequality (9), we will apply the so-called “circle” method in the variable \( t \) (cf. [3]). First we decompose the interval \([0, 1]\) into neighborhoods of rationals, whose denominator is smaller than a given number \( N \) as follows:

Let \( N > 0 \) be given and consider the set

\[
H = \{ p/q : 1 \leq q \leq N, 0 < p \leq q, (p, q) = 1 \},
\]

and define the neighborhoods

\[
V_{p, q} = \left\{ t \in [0, 1] : \left| t - \frac{p}{q} \right| = \min_{r \in H} \left| t - r \right| \right\}.
\]

From the obvious inequalities:

if \( p/q \neq p_1/q_1 \) then

\[
| \frac{p}{q} - \frac{p_1}{q_1} | \geq \frac{1}{2Nq} + \frac{1}{2Nq_1}.
\]

For every \( t \in [0, 1] \) there exists \( p/q \in H \) such that,

\[
| t - \frac{p}{q} | \leq \frac{1}{Nq},
\]

it follows

\[
W^*_{p, q} \subseteq V_{p, q} \subseteq W_{p, q},
\]

where \( W^*_{p, q} = \{ t : \left| t - \frac{p}{q} \right| < 1/(2Nq) \} \) \( W_{p, q} = \{ t : \left| t - \frac{p}{q} \right| \leq 1/(Nq) \} \).

The crucial point is that one can estimate the Fourier transform \( \hat{s}_{t, \varepsilon}(\xi) \) separately in each neighborhood \( V_{p, q} \) by using Poisson summation and the properties of Gaussian sums, as it is shown below.
2. Fourier transform estimates.

Lemma 1. Let \( t = p/q + \tau, \ t \in V_{p,q}, \) then

\[
\|S_t f\|_{L^2} \leq c_n q^{-n/2} \min \{\varepsilon^{-n/2}, \tau^{-n/2}\} \|f\|_2.
\]

Proof. The Fourier transform of the function \( s_t = s_{t,\varepsilon} \) is defined by

\[
\hat{s}_t(\xi) = \sum e^{2\pi i \langle \xi \rangle (t + i\varepsilon + x\xi)}, \quad \xi \in \Pi^n
\]

and inequality (10) is equivalent to

\[
\sup_{\xi} |\hat{s}_t(\xi)| \leq c_n q^{-n/2} \min \{\varepsilon^{-n/2}, \tau^{-n/2}\}.
\]

Since \( \hat{s}_t(\xi) \) is the product of \( n \) one dimensional functions, i.e. \( \hat{s}_t(\xi) = \Pi_j \hat{s}_t(\xi_j) \) it is enough to prove formula (11) in case when \( n = 1 \). By Poisson summation and substituting \( x = rq + s \), we have

\[
\hat{s}_t(\xi) = \sum_x e^{2\pi i x^2 p/q} s_\tau(x) e^{2\pi i x\xi}
\]

\[
= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_r s_\tau(rq + s) e^{2\pi i (rq + s)\xi}
\]

\[
= \sum_{s=0}^{q-1} e^{2\pi i s^2 p/q} \sum_l \frac{1}{q} e^{-2\pi i s/q} \hat{s}_\tau\left(\frac{l}{q} - \xi\right),
\]

where \( \hat{s}_\tau(\xi) \) is the integral of \( s_\tau(x) e^{-2\pi i x\xi} \) is simply the Fourier transform of \( s_\tau \) as function on \( \mathbb{R} \), which has the simple form

\[
\hat{s}_\tau(\xi) = \int_{\mathbb{R}} e^{2\pi i x^2 (\tau + i\varepsilon) - x\xi} \ dx = (\varepsilon - i\tau)^{-1/2} e^{-\xi^2/(\varepsilon - i\tau)}.
\]

So we have the formula

\[
\hat{s}_t(\xi) = (\varepsilon - i\tau)^{-1/2}
\]

\[
\times \sum_l \left( \frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i (p/q s^2 - l/q s)} \right) e^{-\pi (\xi - l/q)^2/(2(\varepsilon - i\tau))}.
\]
In order to estimate this expression, first we note that because of the properties of Gaussian sums one has
\[
\left| \frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i (s^2 p/q - s l/q)} \right| \leq \sqrt{2} q^{-1/2}.
\]

Now we choose \( \varepsilon = \Lambda^{-1} \), \( N = [\Lambda^{1/2}] \) (where \( [x] \) denotes the integer part of \( x \)), and since \( t = p/q + \tau \in V_{p,q} \) we have \( \tau \leq 1/(Nq) \leq \varepsilon^{1/2} q^{-1} \). It follows
\[
\frac{\varepsilon}{q^2(\varepsilon^2 + \tau^2)} \geq \frac{1}{2 q^2 \varepsilon} \geq \frac{1}{2}, \quad \text{if } \tau \leq \varepsilon
\]
and
\[
\frac{\varepsilon}{q^2(\varepsilon^2 + \tau^2)} \geq \frac{\varepsilon}{2 q^2 \tau^2} = \frac{1}{2} (\varepsilon^{1/2} q^{-1} \tau^{-1})^2 \geq \frac{1}{2}, \quad \text{if } \varepsilon \leq \tau.
\]

Now it is easy to estimate the right hand side of formula (12)
\[
|\hat{\delta}_t(\xi)| \leq |\varepsilon - i\tau|^{-1/2} q^{-1/2} \sum_l e^{-\pi \varepsilon/(2(q\xi-l)^2(q^2+\tau^2))}
\leq c q^{1/2} (\varepsilon + \tau)^{-1/2} \left( \sum_l e^{-\pi/4(q\xi-l)^4} \right)
\leq c q^{-1/2} (\varepsilon + \tau)^{-1/2},
\]
where the constant \( c \) is independent of \( q \) and \( \xi \).

This proves inequality (10) and Lemma 1 follows.

**Proof of Theorem 1.** It is easy to see that
\[
\|S_{t}f\|_1 \leq \|S_{t}\|_1 \|f\|_1 \leq c_n \varepsilon^{-n/2} \|f\|_1.
\]
Let \( 1 < p \leq 2 \) and we choose the number \( \alpha \) such that \( 1/p = \alpha/2 + (1-\alpha) \).
Interpolating between estimates (10) and (13), we have
\[
\|S_{t}f\|_p \leq c_n q^{-\alpha/2} \varepsilon^{-n/2} \min \left\{ 1, \left( \frac{\tau}{\varepsilon} \right)^{-\alpha/2} \right\} \|f\|_p.
\]
This implies
\[
\int_{V_{p,q}} \|S_{t}f\|_p \leq c_n \Lambda^{-n/2+1} q^{-\alpha/2}
\cdot \left( \int_0^\varepsilon e^{-n/2} d\tau + \varepsilon^{-n/2} \int_0^{\infty} \left( \frac{\tau}{\varepsilon} \right)^{-\alpha/2} d\tau \right)
\leq c_n q^{-\alpha/2} (\varepsilon \Lambda)^{-n/2+1}
\leq c'_n q^{-\alpha/2} \|f\|_p.
\]
It follows when $n > 4$, $\alpha > 4/n$ or equivalently $p > n/(n - 2)$

$$
\| \hat{M}_{\Lambda, \varepsilon} f \|_p \leq c_n \left( \sum_{p/q \in H} q^{-\alpha/2} \right) \| f \|_p \\
\leq c_n \| f \|_p \left( \sum_{q=1}^{\infty} q^{-\alpha/2+1} \right) \\
\leq c_n \| f \|_p .
$$

This proves Theorem 1.


We now briefly describe how the $L^2$ estimate generalize to $k$-spheres. The extra complications arise are similar to those of the Waring problem. Indeed we refer to the analysis of Hardy-Littlewood in [3], where it was proved that for $n > 2^{k-1} k$

$$
c_{n,k} \lambda^{n/k-1} \leq S_{\lambda,k} \leq C_{n,k} \lambda^{n/k-1} ,
$$

hence as for $k = 2$ one has

$$
\| M_{\Lambda, k} f \|_p \leq c_{n,k} \Lambda^{-n/k+1} \int_0^1 \| S_t f \|_p dt ,
$$

where the kernel of the operator $S_t$ is $s_t(x) = e^{2\pi i \sum_j |x_j|^k (t + i\varepsilon)}$.

For $t = p/q + \tau$ Poisson summation yields

$$
\hat{s}_t(\xi) = \sum_{t=-\infty}^{\infty} \left( \frac{1}{q} \sum_{s=0}^{q-1} e^{2\pi i (s^k p/q - sl/q)} \right) \tilde{s}_\tau \left( \frac{t}{q} - \xi \right) ,
$$

where $\tilde{s}_\tau(\eta) = \int_{\mathbb{R}} s_\tau(x) e^{-2\pi i \eta} dx$ is the Fourier transform on $\mathbb{R}$.

The decomposition into neighborhoods of rationals for $k > 2$ looks as follows

$$
H_{k,0} = \left\{ \frac{p}{q} : q \leq \Lambda^{1/k} \right\} , \quad H_{k,1} = \left\{ \frac{p}{q} : \Lambda^{1/k} < q \leq \Lambda^{1-1/k} \right\} .
$$

$V_{p,q}$ is called a major arc if $p/q \in H_{k,0}$ and a minor arc if $p/q \in H_{k,1}$. 
The reasoning of Theorem 1 generalizes to the major arcs as is shown in

**Lemma 2.** Let \( p/q \in H_{k,0} \), \( t \in V_{p,q} \). Then we have for \( p > n/(n-k+1) \)

\[
\Lambda^{-n/k+1} \int_{V_{p,q}} \| S_t f \|_p dt \leq c_{n,k,p} q^{-nq/K} \| f \|_p,
\]

where \( \alpha = 2p/(p-1) \).

**Proof.** We make use of the following estimates which are proved in [3] using slightly different notations.

\[
|\hat{s}_\tau(\eta)| \leq c |\tau + i \varepsilon|^{-1/k}
\]

holds uniformly in \( \eta \). Let \( \eta = l - q\xi \) then one has

\[
|\hat{s}_\tau\left(\xi - \frac{l}{q}\right)| \leq c_k |\tau + i \varepsilon|^{-1/2(k-2)}/q^{(k-2)/(2(k-1))}
\]

\[
\cdot |\eta|^{-(k-2)/(2(k-1))} e^{-c|\eta|^{k/(k-1)}}.
\]

From inequalities (16) and (17) it follows

\[
\sum_{l=-\infty}^{\infty} |\hat{s}_\tau\left(\xi - \frac{l}{q}\right)| \leq c_k \left( |\tau + i \varepsilon|^{-1/k} + |\tau + i \varepsilon|^{-1/2(k-1)} q^{(k-2)/(2(k-1))} \right),
\]

where inequality (16) is used when \( |l - q\xi| \) is minimal. Also one has the standard estimate for the Weyl sum

\[
q^{-1}\sum_{s=0}^{q-1} e^{2\pi i(s^k p/q - s l/q)} \leq c q^{-1/K},
\]

which holds uniformly in \( l \), when \( K = 2^{k-1} \). Taking the \( n \)-th power of \( \hat{s}_t \), we obtain (in \( n \)-dimension) on the major arcs

\[
\sup_{\xi} |\hat{s}_t(\xi)|
\]

\[
\leq c_{n,k,q^{-n/K}} \left( |\tau + i \varepsilon|^{-n/k} + |\tau + i \varepsilon|^{-n/(2(k-1))} q^{n(k-2)/(2(k-1))} \right).
\]

Let \( 1/p = \alpha/2 + 1 - \alpha \) and using the trivial estimate

\[
\| s_t \|_1 \leq \left( \sum_{x \in \mathbb{Z}} e^{-2\pi x|x|^k} \right)^n \leq c_n \varepsilon^{-n/k},
\]
we obtain by interpolation

\[ \|S_t\|_{p\to p} \leq c_{n,k,p} q^{-n\alpha/K} e^{-n/k} \]

\[ \cdot \left( \left| i + \frac{\tau}{\varepsilon} \right|^{-n/k} + \left| i + \frac{\tau}{\varepsilon} \right|^{-n/(2(k-1))} (\varepsilon^{1/k} q)^{(n(k-2)/(2(k-1)))} \right)^\alpha. \]

Using the facts that on a major arc \( \varepsilon^{1/k} q = \Lambda^{-1/k} q \leq 1 \) and the simple estimate

\[ \int_{\mathbb{R}} \left| i + \frac{\tau}{\varepsilon} \right|^{-\beta} d\tau \leq c_\beta \varepsilon, \quad \text{for } \beta > 1. \]

Estimate (15) follows when \( n\alpha/(2(k-1)) > 1 \). This proves Lemma 2.

On the minor arcs one can give direct estimates for \( \hat{s}_t(\xi) \) exploiting that the denominator \( q \) is large

**Lemma 3.** Let \( p/q \in H_{k,1}, \ t \in V_{p,q} \) then we have

\[ \sup_{\xi} \hat{s}_{t,\varepsilon}(\xi) \leq c_{n,k} \Lambda^{n/k-n/(4kK)}. \]

**Proof.** It is enough to prove (18) in one dimension. Let \( L = \Lambda^{1/k+\delta} \), for some \( \delta > 0 \), then one has

\[ \hat{s}_t(\xi) = \sum_{x=0}^{\infty} e^{2\pi i (x^k (t+i\varepsilon)+x\xi)} + O \left( \sum_{x>L} e^{-\varepsilon x^k} \right) \]

and

\[ \sum_{x>L} e^{-\varepsilon x^k} \leq e^{-\varepsilon \Lambda^{1+\delta} \varepsilon^{-1/k}} \leq e^{-\Lambda^\delta} \Lambda^{1/k} = O(1). \]

To estimate the main term of formula (19) we use partial summation. Let us define the sums \( s_{l,\xi} = \sum_{x=0}^{l} e^{2\pi i (x^k p/q + x\xi)} \), we have

\[ \sum_{l=0}^{L} (s_{l,\xi} - s_{l-1,\xi}) e^{2\pi i k (\tau+i\varepsilon)} \]

\[ = \sum_{l=0}^{L} s_{l,\xi} \left( e^{2\pi i (\tau+i\varepsilon) k^k} - e^{2\pi i (l-1)k (\tau+i\varepsilon)} \right). \]

Since on the minor arcs \( \tau \leq \Lambda^{1/k-1} q^{-1} \leq \Lambda^{-1} \), it follows

\[ \left| e^{2\pi i (\tau+i\varepsilon) k^k} - e^{2\pi i (\tau+i\varepsilon) l^k} \right| \leq c_k \Lambda^{-1/k+k\delta}, \]
so the sum in the formula (19) is less than equal
\[ |\hat{s}_t(\xi)| \leq c_{k,\delta} \left( \max_{l \leq L} |s_{l,\xi}| \right) \Lambda^{(k+1)\delta}. \]

Using the standard estimate for Weyl sums (c.f. [2, Chapter 6]), one has
\[ \left| \sum_{x=0}^{l} e^{2\pi i (x^p/p + x \xi)} \right| \leq c_{k,\delta} \Lambda^{(1/k)(1-1/(2K)) + 2k\delta} \]
holds uniformly in \( \xi \) and \( p \), when \( \Lambda^{1/k} \leq q \leq \Lambda^{1-1/k}, l \leq \Lambda^{1/k+\delta}. \)

The above estimates imply for \( \delta \leq \delta(k) \) the estimate
\[ |\hat{s}_t,\xi(\xi)| \leq c_{k,\delta} \Lambda^{1/k - 1/(2kK) + 4k\delta} \leq c_{k} \Lambda^{1/k - 1/(4kK)} \leq c_{n,k} \]
holds uniformly in \( \xi \), and Lemma 3 follows.

**Proof of Theorem 2.** Interpolation between the trivial \( L^1 \), and the \( L^2 \) estimate (18), shows that for \( 1/p = 1 - \alpha/2 \) on a minor arc we have
\[ \Lambda^{-n/k+1} \| S_t \|_{p \to p} \leq c_{n,k,p} \Lambda^{n/k - n\alpha/(4Kk)}. \]

Hence for \( n > 4K \), \( p > n/(n - 4Kk) \) one has
\[ \| M_{\Lambda,k} f \|_p \leq c_{n,k,p} \left( \sum_{p/q \in H_0,k} q^{-n\alpha/K} + \Lambda^{-n/k+1+n/k - n\alpha/(4kK)} \right) \| f \|_p \]
\[ \leq c_{n,k,p}' \| f \|_p, \]
since \( n\alpha/K > 2 \) and \( n\alpha/(4Kk) > 1 \). This proves Theorem 2.

We would like to point out how these estimates are connected with the distribution of integer points on spherical caps. More precisely we define the maximal function
\[ s_{\lambda,t}^* = \sup_{\lambda \leq \mu < 2\lambda} |S_{\mu}^{n-1} \cap (x + D_t^n)|, \]
where \( x + D_t^n = \{ u \in \mathbb{Z}^n : |x - u| \leq t^{1/2} \}. \) On the other hand the average number of integer points on a spherical cap of radius \( t^{1/2} \) lying on the sphere of radius \( \lambda^{1/2} \) is clearly
\[ \tilde{S}_{\lambda,t} = t^{(n-1)/2} \lambda^{-1/2}. \]
Theorem 1. implies the following

**Corollary 1.** Let \( \varepsilon > 0 \). If \( l > \lambda^{1-\varepsilon/2} \) then

\[
\lambda^{-n/2} \left| \left\{ x \in D^{n}_{\lambda} : s^{*}_{\lambda,l}(x) > \lambda^{\varepsilon} \tilde{s}_{\lambda,l} \right\} \right| < c_{n,\varepsilon} \lambda^{-\varepsilon n/(2(n-2))}.
\]

**Proof.** Let \( \lambda > 0 \), \( f(x) = |x|^{-n+2} \). First we estimate \( M_{\lambda}f(x) \) from below as follows

\[
M_{\lambda}f(x) \geq c_{n} \lambda^{-n/2+1} \sup_{\lambda \leq \mu < 2\lambda} \sum_{l=1}^{\infty} l^{-n/2+1} \left| \left\{ y \in \mathbb{Z}^{n} : |x-y|^{2} = l, |y|^{2} = \mu \right\} \right|
\]

\[
\geq c_{n} \lambda^{-n/2+1} \left[ \sum_{\lambda \leq \mu < 2\lambda} L^{-n/2+1} \left| \left\{ y \in \mathbb{Z}^{n} : L \leq |x-y|^{2} < 2L, |y|^{2} = \mu \right\} \right| \right]
\]

\[
\geq c_{n} \lambda^{-n/2+1} \left[ \sum_{\lambda \leq \mu < 2\lambda} L^{-n/2+1} \left| \left\{ y \in \mathbb{Z}^{n} : |x-y|^{2} \leq L, |y|^{2} = \mu \right\} \right| \right],
\]

where the last inequality was obtained by partial summation. This immediately implies

\[
M_{\lambda}f(x) \geq c_{n} \lambda^{-n/2+1} \frac{s^{*}_{\lambda,L}(x)}{\tilde{s}_{\lambda,L}} \frac{L^{1/2}}{\lambda^{1/2}},
\]

for every dyadic value \( L = 2^{l} \), but it remains true for every integer \( l \) since the function \( s^{*}_{\lambda,l}(x) \) is monotone increasing in \( l \).

Choosing \( p = n/((n-2)+\eta) \), it follows for \( l > \lambda^{1-\varepsilon/2} \)

\[
\| M_{\lambda}f \|_{p} \geq \lambda^{-\eta(n/2-1)} \lambda^{-n/2} \sum_{x \in D^{n}_{\lambda}} \left( \frac{s^{*}_{\lambda,l}(x)}{\tilde{s}_{\lambda,l}} \right)^{n/(n-2)+\eta} \frac{L^{1/2}}{\lambda^{1/2}} \geq \lambda^{3\eta n/(4(n-2)-\eta n/2)} \lambda^{-n/2} \left| \left\{ x \in D^{n}_{\lambda} : s^{*}_{\lambda,l} > \lambda^{\varepsilon} \tilde{s}_{\lambda,l} \right\} \right|.
\]

Choosing \( \eta \) small enough estimate (23) follows immediately, since \( f \in L^{p}(\mathbb{Z}^{n}) \) and the maximal operator \( M_{\lambda} \) is bounded in \( L^{p}(\mathbb{Z}^{n}) \) by Theorem 1.
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Akos Magyar
Department of Mathematics
Sloan Lab. 253-37
California Institute of Technology
1200 E. California Blvd.
Pasadena, CA 91125, U.S.A.
amagyar@cco.caltech.edu