Approximation and symbolic calculus for Toeplitz algebras on the Bergman space

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Abstract

If \( f \in L^\infty(\mathbb{D}) \) let \( T_f \) be the Toeplitz operator on the Bergman space \( L^2_\mathbb{A} \) of the unit disk \( \mathbb{D} \). For a \( C^* \)-algebra \( A \subset L^\infty(\mathbb{D}) \) let \( \mathfrak{T}(A) \) denote the closed operator algebra generated by \( \{ T_f : f \in A \} \). We characterize its commutator ideal \( \mathfrak{C}(A) \) and the quotient \( \mathfrak{T}(A)/\mathfrak{C}(A) \) for a wide class of algebras \( A \). Also, for \( n \geq 0 \) integer, we define the \( n \)-Berezin transform \( B_nS \) of a bounded operator \( S \), and prove that if \( f \in L^\infty(\mathbb{D}) \) and \( f_n = B_nT_f \) then \( T_{f_n} \rightarrow T_f \).

1. Introduction and preliminaries

Suppose that \( A \) is a \( C^* \)-algebra with unit. The commutator ideal \( \mathfrak{C} \) is the closed bilateral ideal generated by the elements \( [x, y] = xy - yx \), with \( x, y \in A \). The quotient \( A/\mathfrak{C} \) is a commutative \( C^* \)-algebra with unit, which by the Gelfand-Naimark Theorem is isometrically isomorphic to \( C(M) \), the algebra of continuous functions on some compact Hausdorff space \( M \). Following the arrows

\[ A \twoheadrightarrow A/\mathfrak{C} \cong C(M) \]

we can associate to every \( x \in A \) a function \( f_x \in C(M) \), which is the ‘symbol’ referred to in the title of the paper. Since the algebra \( A \) is determined by \( \mathfrak{C} \) and \( C(M) \), the study of these two objects is an important tool for a better understanding of \( A \). The possible advantages of this point of view are that \( C(M) \) can be treated by topological methods, since it depends exclusively on the space \( M \), and that \( \mathfrak{C} \) is usually much smaller than \( A \). Of course, the first step of this journey is to determine \( \mathfrak{C} \) and \( C(M) \). The whole process is known as abelianization, and it can be carried out for a much wider class

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of algebras than \( C^* \)-algebras. In particular, these ideas have been widely studied in the context of Toeplitz algebras acting on the Hardy space \( H^2 \) (see [18, pp. 339-392]). The literature shows some partial attempts to develop a similar scheme for Toeplitz algebras acting on the Bergman space \( L^2_a = L^2_a(dA) \), where \( dA \) is the normalized area measure on \( D \) (see [14, Ch. 4] for a general discussion). We give below a brief summary of known results.

Let \( \mathcal{L}(L^2_a) \) be the algebra of bounded operators on \( L^2_a \). If \( B \subset L^\infty(D) \) is a closed subalgebra, let \( \mathfrak{T}(B) \) be the closed subalgebra of \( \mathcal{L}(L^2_a) \) generated by the Toeplitz operators \( \{ T_a : a \in B \} \) and \( \mathfrak{C}(B) \) be the commutator ideal of \( \mathfrak{T}(B) \).

In [11] Coburn proved that \( \mathfrak{C}(C(D)) \) is the ideal of compact operators and \( \mathfrak{T}(C(D))/\mathfrak{C}(C(D)) \) is isomorphic to \( C(\partial D) \). In [17] McDonald and Sundberg characterized the quotient \( \mathfrak{T}(\mathcal{U})/\mathfrak{C}(\mathcal{U}) \), where \( \mathcal{U} \) is the \( C^* \)-algebra in \( L^\infty(D) \) generated by \( H^\infty \). Later, the two papers by Axler and Zheng ([4], [5]) provided additional information on Coburn’s and McDonald-Sundberg’s theorems by giving characterizations of the respective commutator ideals in terms of the Berezin transform. We give precise statements of these results in Sections 6 and 7. In [20] the author showed that \( \mathfrak{C}(L^\infty(D)) = \mathfrak{T}(L^\infty(D)) \).

Despite these results, no systematic theory of abelianization has been given so far for Toeplitz algebras on the Bergman space. One of the purposes of this paper is to develop a general theory of abelianization for Toeplitz algebras \( \mathfrak{T}(B) \), where \( B \) belongs to a special class of \( C^* \)-algebras in \( L^\infty(D) \) that we call hyperbolic. Our main goal is to explain the underlying phenomenon that is apparently common to Coburn’s and McDonald-Sundberg’s theorems, and to apply it to other hyperbolic algebras.

Let \( \mathcal{A} \subset L^\infty(D) \) be the algebra of functions on \( D \) that are uniformly continuous with respect to the pseudohyperbolic metric. If \( n \) is a nonnegative integer, we define the \( n \)-Berezin transform \( B_n : \mathcal{L}(L^2_a) \to \mathcal{A} \). This is a linear operator, and we show that if \( a \in L^\infty(D) \) and \( a_n = B_n T_a \), then \( T_a \) tends to \( T_a \) in operator norm. In particular, the Toeplitz algebras associated to \( L^\infty(D) \) and \( \mathcal{A} \) coincide. This will allow us to reduce the study of \( \mathfrak{T}(\mathcal{B}) \) and \( \mathfrak{C}(\mathcal{B}) \) for some \( C^* \)-algebras \( \mathcal{B} \subset L^\infty(D) \) that are not hyperbolic, to the case of hyperbolic algebras. Once the reduction is made, we can use the maximal ideal space of \( \mathcal{A} \) as a powerful tool to describe \( \mathfrak{C}(\mathcal{B}) \) and \( \mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B}) \). We begin fixing some notation.

For \( z \in D \) denote
\[
\varphi_z(\omega) = \frac{z - \omega}{1 - \overline{z}\omega}.
\]
The pseudohyperbolic metric on \( D \) is defined as \( \rho(z, \omega) = |\varphi_z(\omega)| \). This metric is invariant under the action of Aut\( (D) \). Sometimes, especially in
estimates involving the triangle inequality, it will be useful to use the hyperbolic metric
\[ h(z, \omega) = \log \frac{1 + \rho(z, \omega)}{1 - \rho(z, \omega)}, \quad z, \omega \in \mathbb{D} \]
instead of \( \rho \). The passage from one metric to the other is justified because
\[ f(x) = \log \frac{1 + x}{1 - x} \]
is a strictly increasing function of \( x \in (0, 1) \). For \( z \in \mathbb{D}, \ r \in (0, 1) \) and \( s \in (0, \infty) \) write
\[ K(z, r) = \{ \omega \in \mathbb{D} : \rho(z, \omega) \leq r \} \quad \text{and} \quad K_h(z, r) = \{ \omega \in \mathbb{D} : h(z, \omega) \leq s \} \]
for the closed pseudohyperbolic (resp. hyperbolic) disk of center \( z \) and radius \( r \) (resp. \( s \)).

Let \( \mathcal{B} \subset L^\infty(\mathbb{D}) \) be a closed subalgebra, where by algebra we always mean a unitary algebra. The maximal ideal space of \( \mathcal{B} \) is
\[ M(\mathcal{B}) = \{ \alpha : \mathcal{B} \to \mathbb{C} : \alpha \text{ is linear, multiplicative and } \alpha(1) = 1 \}, \]
provided with the weak * topology induced by the dual space of \( \mathcal{B} \). It is a compact Hausdorff space. We can look at a function \( f \in \mathcal{B} \) as a continuous function on \( M(\mathcal{A}) \) via the Gelfand transform
\[ \hat{f}(\alpha) = \alpha(f) \quad (\alpha \in M(\mathcal{B})). \]
If \( \mathcal{B} \subset C(\mathbb{D}) \cap L^\infty(\mathbb{D}) \) separates points of \( \mathbb{D} \) then evaluations at points of \( \mathbb{D} \) are members of \( M(\mathcal{B}) \). So, \( \mathbb{D} \) is naturally imbedded into \( M(\mathcal{B}) \), and \( \hat{f} \) is an extension to the whole maximal space of the function \( f \). Unless the contrary is stated we avoid writing the hat for the Gelfand transform and look at \( f \) as a function on \( M(\mathcal{B}) \). The algebra
\[ \mathcal{A} = \{ f \in L^\infty(\mathbb{D}) : f \text{ is uniformly continuous with respect to } \rho \} \]
will be a major protagonist of this paper. It is \( C^* \)-algebra such that \( \mathbb{D} \) is dense in \( M(\mathcal{A}) \). Indeed, there cannot be \( \alpha \in M(\mathcal{A}) \setminus \overline{\mathbb{D}} \), because otherwise there is \( f \in \mathcal{A} \) with \( f(\alpha) = 0 \) while \( |f| \geq \delta > 0 \) on \( \mathbb{D} \) (since \( \mathcal{A} \) is a \( C^* \)-algebra). Since such \( f \) is invertible in \( \mathcal{A} \), it is not in the maximal ideal \( \text{Ker } \alpha \). Further information on \( M(\mathcal{A}) \) can be found in [8].

If \( a \in L^\infty(\mathbb{D}) \) let \( M_a \) be the multiplication operator on \( L^2(\mathbb{D}) \) and \( T_a \) be the Toeplitz operator on \( L^2_a \). That is, \( T_a = P_+ M_a \), where \( P_+ : L^2(\mathbb{D}) \to L^2_a \) is the Bergman projection. It is clear that \( \| M_a \| = \| a \|_\infty \) and \( \| T_a \| \leq \| a \|_\infty \).
A big difference with Toeplitz operators on the Hardy space \( H^2 \) is that the latter inequality is not always an equality, although we still have that \( T_a = 0 \).
only when $a = 0$. For $z \in \mathbb{D}$, the ‘change of variable operator’ is given by $U_z f = (f \circ \varphi_z) \varphi'_z$. That is,

$$(U_z f)(\omega) = f(\varphi_z(\omega)) \frac{|z|^2 - 1}{(1 - \overline{z} \omega)^2}.$$ 

It is easy to prove that $U_z T_a U_z = T_a \circ \varphi_z$ for every $a \in L_\infty(\mathbb{D})$, and since $U_z$ is unitary and self-adjoint, then

$$(T_{a_1} \ldots T_{a_n})_z = (U_z T_{a_1} U_z) \ldots (U_z T_{a_n} U_z) = T_{a_1 \circ \varphi_z} \ldots T_{a_n \circ \varphi_z}$$

for $a_1, \ldots, a_n \in L_\infty(\mathbb{D})$. We will write

$$S_z = U_z T_a U_z \quad \text{for } S \in \mathfrak{L}(L_a^2).$$

The paper is organized as follows. The main results are Theorems 5.7, 6.4 and 6.5. In Section 2 we introduce the $n$-Berezin transform of a bounded operator and study its basic properties. If $a \in L_\infty(\mathbb{D})$, $B_n T_a$ coincides with $B_n(a)$, the more familiar $n$-Berezin transform of a function. In Section 3 we study the maximal ideal space of $\mathcal{A}$ and use some of its features to define the notion of hyperbolic algebra. A characterization of these algebras is obtained in terms of interpolating sequences.

If $S \in \mathfrak{I}(\mathcal{B})$, where $\mathcal{B}$ is a hyperbolic algebra, we construct in Section 4 a continuous map $\Psi^S_{\mathcal{B}}$ from the maximal ideal space of $\mathcal{B}$ into $\mathfrak{I}(\mathcal{B})$, when provided with the strong operator topology, and study its interaction with the $n$-Berezin transform. We prove that $\Psi^S_{\mathcal{B}}$ is multiplicative as a function of $S$, which translates into a kind of asymptotic multiplicative behavior of $B_n$. This will be a fundamental tool for much of what follows.

Theorem 5.7 shows that $T_{B_n(a)}$ tends to $T_a$ for $a \in L_\infty(\mathbb{D})$. As a consequence we obtain that if $B_n(a)$ belongs to a hyperbolic algebra $\mathcal{B}$ for infinitely many values of $n$ then $T_a \in \mathfrak{I}(\mathcal{B})$. This argument will reduce the study of $\mathfrak{I}(C)$ for some non-hyperbolic algebras $C \subset L_\infty(\mathbb{D})$ to the hyperbolic case.

Theorem 6.4 gives a characterization of $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{I}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$ when $\mathcal{B}$ is hyperbolic. If $S$ is a finite sum of finite products of Toeplitz operators with symbols in $L_\infty(\mathbb{D})$ and $\mathcal{B}$ is a hyperbolic algebra, Theorem 6.5 provides a necessary and sufficient condition for $S \in \mathfrak{I}(\mathcal{B})$ and $S \in \mathfrak{C}(\mathcal{B})$.

Section 7 is devoted to applications of the previous results. It is shown that the theorem of McDonald-Sundberg and part of Coburn’s theorem are particular cases of Theorem 6.4. An example will be given to illustrate how Theorems 5.7 and 6.4 can be used to characterize $\mathfrak{C}(C)$ and $\mathfrak{I}(C)/\mathfrak{C}(C)$ for some $C^*$-algebras $C \subset L_\infty(\mathbb{D})$ that are not hyperbolic.

Finally, we give a partial result towards a possible characterization of the center of $\mathfrak{I}(L_\infty(\mathbb{D}))/\mathcal{K}$, where $\mathcal{K}$ denotes the ideal of compact operators. We finish the paper posing some open problems.
2. The $n$-Berezin transform.

If $n$ is a nonnegative integer and $z \in \mathbb{D}$, the function
\[
K_z^{(n)}(\omega) = \frac{1}{(1 - \overline{z}\omega)^{2+n}} \quad (\omega \in \mathbb{D})
\]
is the reproducing kernel of $z$ in the weighted Bergman space $L^2_a(dA_n)$, where $dA_n(\omega) = (n+1)(1 - |\omega|^2)^n dA(\omega)$. The $n$-Berezin transform of an operator $S \in \mathcal{L}(L^2_a)$ is defined as

\[
(B_n S)(z) \overset{\text{def}}{=} (n+1)(1 - |z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle.
\]

It is clear that $B_n S \in C^\infty(\mathbb{D})$ for every $S \in \mathcal{L}(L^2_a)$. Using that
\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j |\omega|^{2j} = (1 - |\omega|^2)^n
\]
we see that if $S = T_a$, with $a \in L^\infty(\mathbb{D})$, then
\[
(B_n a)(z) \overset{\text{def}}{=} (B_n T_a)(z) = (n+1)(1 - |z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \int_\mathbb{D} \frac{a(\omega)|\omega|^{2j}}{|1 - \overline{z}\omega|^{2(2+n)}} dA(\omega)
\]
\[
= \int_\mathbb{D} a(\omega) \frac{(1 - |\omega|^2)^{2+n}}{|1 - \overline{z}\omega|^{2(2+n)}} (n+1)(1 - |\omega|^2)^n dA(\omega)
\]
\[
= \int_\mathbb{D} a(\varphi_z(\zeta))(n+1)(1 - |\zeta|^2)^n dA(\zeta),
\]
where the last equality comes from the change of variables $\omega = \varphi_z(\zeta)$. Since $dA_n(\xi)$ is a probability measure that tends to concentrate its mass at 0 when $n \to \infty$, then $(B_n a)(z)$ is an average of $a$ satisfying $\|B_n(a)\|_\infty \leq \|a\|_\infty$ for all $a \in L^\infty(\mathbb{D})$. A straightforward calculation shows that $B_n$ maps $L^\infty(\mathbb{D})$ into $\mathcal{A}$ for every $n \geq 0$, and we will prove in Corollary 4.6 that the same holds for $\mathcal{L}(L^2_a)$. The last expression in (2.2) clearly shows that $\|B_n(a) - a\|_\infty \to 0$ when $n \to \infty$ for every $a \in \mathcal{A}$. That is, the sequence $\{B_n\}$ works as an approximate identity for $\mathcal{A}$. In particular, $\lim_n \|T_{B_n(a)} - T_a\| = 0$ for $a \in \mathcal{A}$.

The 0-Berezin transform of an operator is the usual Berezin transform, which has been extensively used in recent research (see for instance [2], [4], [5] and [19]). The $n$-Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [6]. Many of the results of this section were
proved by Ahern, Flores and Rudin [2] for $n$-Berezin transforms of functions of several variables. However, the results here do not follow immediately from theirs, because there are \textit{a priori} several ways to define $B_nS$ for $n \geq 1$ and $S \in \mathfrak{L}(L^2_n)$ so that (2.2) holds when $S = T_a$. If for instance $S \in \mathfrak{L}(L^2_n) \cap \mathfrak{L}(L^2_n(dA_n))$, then the usual Berezin transform of $S$ with respect to the weighted Bergman space $L^2_n(dA_n)$ is $(1 - |z|^2)^{2+n} \langle SK_z^{(n)}, K_z^{(n)} \rangle_{dA_n}$, which differs from our definition of $B_nS$. It is precisely because of the results of this section (especially Proposition 2.4) that I convinced myself (and hopefully convince the reader) about (2.1) as the right definition of $B_nS$.

\textbf{Lemma 2.1} Let $S \in \mathfrak{L}(L^2_n)$ and $n \geq 0$. Then
\begin{equation}
(2.3) \quad (n + 2)(1 - |z|^2)B_n(S - T_\omega ST_\omega)(z) = (n + 1)B_{n+1}(T_{1 - \omega_\alpha}ST_{1 - \omega_\alpha})(z)
\end{equation}
for every $z \in \mathfrak{D}$.

\textbf{Proof.} A simple rearrangement of terms gives
\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j \left[ (\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \right] - \langle S(\omega^{j+1} K_z^{(n)}), \omega^{j+1} K_z^{(n)} \rangle
\]
\[
= \langle SK_z^{(n)}, K_z^{(n)} \rangle + (-1)^{n+1} \langle S(\omega^{n+1} K_z^{(n)}), \omega^{n+1} K_z^{(n)} \rangle
\]
\[
+ \sum_{j=1}^{n} \left( \binom{n}{j} + \binom{n}{j-1} \right) (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle
\]
\[
= \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle.
\]
Multiplying by $(n + 2)(n + 1)(1 - |z|^2)^{3+n}$ and using that
\[
T_{1 - \omega_\alpha}(\omega^j K_z^{(n+1)}) = \omega^j K_z^{(n)}
\]
the above equality becomes (2.3).

\textbf{Lemma 2.2} $B_nS_\alpha = (B_nS) \circ \varphi_\alpha$ for every $n \geq 0$, $S \in \mathfrak{L}(L^2_n)$ and $\alpha \in \mathfrak{D}$.

\textbf{Proof.} We shall prove the lemma by induction on $n$. The easy identity
\begin{equation}
(2.4) \quad (1 - \varphi_\alpha(\omega)\bar{\omega})^{-1} = (1 - \bar{\alpha}\omega^{-1})(1 - \varphi_\alpha(\omega))^{-1}
\end{equation}
implies that
\[
(U_\alpha K_z^{(0)})(\omega) = \frac{|\alpha|^2 - 1}{(1 - \bar{\alpha}\omega)(1 - \varphi_\alpha(\omega))} \frac{|\alpha|^2 - 1}{(1 - \alpha\omega)^2} K_z^{(0)}(\varphi_\alpha(z)\omega).
\]
Thus
\[
\langle B_0 \varphi_\alpha(z) | B_0 \varphi_\alpha(z) \rangle = (1 - |\varphi_\alpha(z)|^2) \langle SK_z^{(0)}, K_z^{(0)} \rangle = (B_0S)(\varphi_\alpha(z))
\]
This takes care of $n = 0$.\qed
The main tool for the inductive step will be formula (2.3), that we rewrite as

\[(B_{n+1}S)(z) = c_n(1 - |z|^2)B_n[T_{(1-\varphi z)}^{-1}(S - T_{\varpi ST_\omega})T_{(1-\omega z)}^{-1}](z),\]

where \(c_n = (n + 2)/(n + 1).\) By (1.1) then

\[T_{(1-\varphi z)}^{-1}(U_\alpha SU_\alpha - T_{\varpi}U_\alpha SU_\alpha T_\omega)T_{(1-\omega z)}^{-1}\]

\[= U_\alpha T_{(1-\varphi_\alpha(z)\varpi)^{-1}}[S - T_{\varphi_\alpha(z)\varpi}ST_{\varphi_\alpha(z)\omega}]T_{(1-\varphi_\alpha(z)\omega)}^{-1}U_\alpha = J.\]

Then (2.4) yields

\[J = |1 - \alpha \varpi|^2 U_\alpha T_{(1-\varphi_\alpha(z)\varpi)^{-1}}[T_{1-\alpha \varpi}ST_{1-\varpi z} - T_{\varpi}ST_{1-\varpi z}]T_{(1-\varphi_\alpha(z)\omega)}^{-1}U_\alpha\]

\[(2.6) = \frac{1 - |\alpha|^2}{|1 - \alpha \varpi|^2} U_\alpha T_{(1-\varphi_\alpha(z)\varpi)^{-1}}[S - T_{\varphi_\alpha(z)\varpi}ST_{\varphi_\alpha(z)\omega}]T_{(1-\varphi_\alpha(z)\omega)}^{-1}U_\alpha.\]

Hence,

\[(B_{n+1}S_\alpha)(z) = c_n(1 - |z|^2)B_n(J)(z)\]

\[= c_n(1 - |\varphi_\alpha(z)|^2)B_n(U_\alpha T_{(1-\varphi_\alpha(z)\varpi)^{-1}}[S - T_{\varphi_\alpha(z)\varpi}ST_{(1-\varphi_\alpha(z)\omega)}^{-1}U_\alpha](z)\]

\[= c_n(1 - |\varphi_\alpha(z)|^2)B_n(T_{(1-\varphi_\alpha(z)\omega)^{-1}}[S - T_{\varphi z}ST_{\varphi z}]T_{(1-\varphi_\alpha(z)\omega)}^{-1})(\varphi_\alpha(z))\]

\[= B_{n+1}(S)(\varphi_\alpha(z)),\]

where the first equality comes from (2.5) with \(U_\alpha SU_\alpha\) instead of \(S,\) the second from (2.6), the third by inductive hypothesis and the last one from (2.5) with \(\varphi_\alpha(z)\) instead of \(z.\)

\[\]
The (conformally) invariant Laplacian is \( \hat{\Delta} = (1 - |z|^2)^2 \partial \bar{\partial} \), where \( \partial \) and \( \bar{\partial} \) are the traditional Cauchy-Riemann operators. So, when \( f \) is analytic on \( \mathbb{D} \), \( \partial f = f' \), \( \bar{\partial} f = 0 \), and \( \bar{\partial} \bar{f} = 0 \). It is easy to check that \( \hat{\Delta} f \circ \psi = \hat{\Delta} (f \circ \psi) \) for every \( \psi \in \text{Aut}(\mathbb{D}) \).

**Proposition 2.4** Let \( S \in \mathcal{L}(L^2_a) \) and \( n \geq 0 \). Then

\[
(2.7) \quad \hat{\Delta} B_n S = 4(n + 1)(n + 2)(B_n S - B_{n+1} S).
\]

**Proof.** By Lemma 2.2 and the conformal invariance of \( \hat{\Delta} \) it is enough to prove that the equality holds at \( z = 0 \). Using the mentioned properties of \( \partial \) and \( \bar{\partial} \), a tedious but straightforward calculation gives

\[
(2.8) \quad \hat{\Delta} [(1 - |z|^2)^{n+2} \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle](0) = 4(n + 2)(-\langle S \omega^j, \omega^j \rangle + (n + 2)\langle S \omega^{j+1}, \omega^{j+1} \rangle)
\]

for every \( 0 \leq j \leq n \). So, writing \( X_j = (-1)^j \langle S \omega^j, \omega^j \rangle \), we have

\[
\hat{\Delta} (B_n S)(0) = 4(n + 1)(n + 2) \sum_{j=0}^{n} \binom{n}{j} [-X_j - (n + 2)X_{j+1}]
\]

\[
= 4(n + 1)(n + 2) \left\{-X_0 - (n + 2)X_{n+1} - \sum_{j=1}^{n} \left[ \binom{n}{j} + (n + 2)\binom{n}{j-1} \right] X_j \right\}.
\]

On the other hand,

\[
(B_n S - B_{n+1} S)(0) = -(n + 2)X_{n+1} + \sum_{j=0}^{n} \left[ (n + 1)\binom{n}{j} - (n + 2)\binom{n + 1}{j} \right] X_j.
\]

A comparison of the coefficients for each \( X_j \) gives the result. \( \Box \)

**Corollary 2.5** If \( S \in \mathcal{L}(L^2_a) \) and \( n \geq 1 \) then

\[
(2.9) \quad B_n S = \left( 1 - \frac{\hat{\Delta}}{4n(n + 1)} \right) B_{n-1} S
\]

and

\[
(2.10) \quad B_n S = G_n(\hat{\Delta}) B_0 S,
\]

where

\[
G_n(\lambda) = \prod_{k=1}^{n} \left( 1 - \frac{\lambda}{4k(k + 1)} \right).
\]

**Proof.** Formula (2.9) is a rewriting of (2.7), while (2.10) follows immediately from (2.9). \( \Box \)
Lemma 2.6 If $S \in \mathcal{L}(L^2_a)$ and $n \geq 0$ then $\hat{\Delta}B_0(B_nS) = B_0\hat{\Delta}(B_nS)$.

Proof. If $f = B_nS$, Corollary 2.3 and (2.7) imply that $f$ and $\hat{\Delta}f$ are bounded. Hence, Lemma 1 of [1] says that $\hat{\Delta}B_0f = B_0\hat{\Delta}f$. ■

Corollary 2.7 Let $S \in \mathcal{L}(L^2_a)$ and $k,j \geq 0$. Then $(B_kB_j)(S) = (B_jB_k)(S)$.

Proof. Combine (2.10) with the previous lemma. ■

3. Algebras related to the maximal ideal space of $\mathcal{A}$

For the next two subsections, if $E \subset M(\mathcal{A})$ then $\overline{E}$ denotes the closure of $E$ in the space $M(\mathcal{A})$.

Since the $M(\mathcal{A})$-topology agrees with the Euclidean topology on $\mathbb{D}$, $\overline{E}$ has the same meaning in both topologies when $E \subset r\mathbb{D}$ for some $0 < r < 1$. Later on, we will have to distinguish between closures in different spaces. A sequence $\{z_n\} \subset \mathbb{D}$ is separated if $\rho(z_n,z_k) \geq \delta > 0$ for $n \neq k$.

3.1. One-to-one maps from $\mathbb{D}$ into $M(\mathcal{A})$

Lemma 3.1 Let $E,F \subset \mathbb{D}$. Then $E \cap F = \emptyset$ if and only if $\rho(E,F) > 0$.

Proof. If $E \cap F = \emptyset$ then there is $f \in \mathcal{A}$ such that $f \equiv 1$ on $E$ and $f \equiv 0$ on $F$. The uniform $\rho$-continuity of $f$ implies that

$$\rho(E,F) = \rho(E \cap \mathbb{D}, F \cap \mathbb{D}) > 0.$$ 

Now suppose that $\rho(E,F) \geq \alpha > 0$ and consider the function

$$f(z) = \begin{cases} 
1 & \text{if } \rho(z,E) \leq \alpha/2 \\
0 & \text{if } \rho(z,E) > \alpha/2
\end{cases}$$

Simple estimates show that $B_n(f) \to 1$ uniformly on $\{z : \rho(z,E) < \alpha/4\}$ and $B_n(f) \to 0$ uniformly on $\{z : \rho(z,F) < \alpha/4\}$. Since $B_n(f) \in \mathcal{A}$, it separates $\overline{E}$ from $\overline{F}$ for $n$ big enough, showing that they are disjoint. ■

Let $x \in M(\mathcal{A})$ and suppose that $(z_n)$ is a net in $\mathbb{D}$ that tends to $x$. We can think of $(\varphi_{z_n})$ as a net in the product space $M(\mathcal{A})^\mathbb{D}$. By compactness there is a convergent subnet $(\varphi_{z_{n_\beta}})$, meaning that there is some function $\varphi : \mathbb{D} \to M(\mathcal{A})$ such that $f \circ \varphi_{z_{n_\beta}} \to f \circ \varphi$ pointwise on $\mathbb{D}$ for every $f \in \mathcal{A}$.

We aim to show that the whole net $(z_n)$ tends to $\varphi$ and that $\varphi$ does not depend on the net. So, suppose that $(\omega_\gamma)$ is another net in $\mathbb{D}$ converging to $x$ such that $\varphi_{\omega_\gamma}$ tends to some $\psi \in M(\mathcal{A})^\mathbb{D}$. If $\varphi \neq \psi$ there is
some $\xi \in \mathbb{D}$ such that $\varphi(\xi) \neq \psi(\xi)$. Then there are closed disjoint neighborhoods $U, V \subset M(A)$ of $\varphi(\xi)$ and $\psi(\xi)$, respectively. Since $\varphi_{z_n}(\xi) \rightarrow \varphi(\xi)$ and $\varphi_{\omega_n}(\xi) \rightarrow \psi(\xi)$, there are tails of both nets satisfying

$$E = \{\varphi_{z_n}(\xi) : \beta \geq \beta_0\} \subset U \quad \text{and} \quad F = \{\varphi_{\omega_n}(\xi) : \gamma \geq \gamma_0\} \subset V.$$  

By Lemma 3.1 then $\rho(E, F) \geq \rho(U \cap \mathbb{D}, V \cap \mathbb{D}) > 0$. Since for every $z, \omega \in \mathbb{D}$ there is a constant $c_\xi > 0$ such that

$$\rho(\varphi_z(\xi), \varphi_\omega(\xi)) < c_\xi \rho(z, \omega),$$

then

$$\rho(E, F) \leq c_\xi \inf\{\rho(z_{\alpha,\beta}, \omega_\gamma) : \beta \geq \beta_0, \gamma \geq \gamma_0\} = 0,$$

where the last equality holds because both nets $(z_{\alpha,\beta})$ and $(\omega_\gamma)$ tend to $x$. We obtain a contradiction and consequently $\varphi = \psi$. The map $\varphi$ will be denoted $\varphi_\xi$, and notice that $\varphi_\xi(0) = \lim \varphi_{z_n}(0) = \lim z_n = x$.

The following lemma is in [20, Lemma 2.1].

**Lemma 3.2** Let $S$ be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $S = S_1 \cup \ldots \cup S_N$ such that for every $1 \leq j \leq N$: $\rho(z, \omega) > \sigma$ for all $z \neq \omega$ in $S_j$.

**Lemma 3.3** Every $x \in M(A)$ is in the closure of some separated sequence.

**Proof.** Suppose that $x \in M(A)$ and let $(\omega_\alpha)$ be a net in $\mathbb{D}$ such that $\omega_\alpha \rightarrow x$. Take a separated sequence $S$ such that $\rho(z, S) < 1/8$ for every $z \in \mathbb{D}$, and for each $\omega_\alpha$ pick some $z_\alpha$ in $S$ such that $\rho(z_\alpha, \omega_\alpha) < 1/8$ for every $\alpha$. Therefore there is $\xi_\alpha \in 8^{-1}\mathbb{D}$ so that $\omega_\alpha = \varphi_{z_\alpha}(\xi_\alpha)$. Taking subnets we can assume that $\xi_\alpha \rightarrow \xi$ with $|\xi| \leq 1/8$. We claim that $\varphi_{z_\alpha}(\xi)$ tends to $x$. Indeed, if $f \in A$ then

$$|f(\varphi_{z_\alpha}(\xi)) - f(x)| \leq |f(\varphi_{z_\alpha}(\xi)) - f(\varphi_{z_\alpha}(\xi_\alpha))| + |f(\omega_\alpha) - f(x)|,$$

where the first summand tends to 0 because $\rho(\varphi_{z_\alpha}(\xi), \varphi_{z_\alpha}(\xi_\alpha)) = \rho(\xi, \xi_\alpha) \rightarrow 0$, and the second summand tends to 0 because $\omega_\alpha \rightarrow x$. Thus, $x$ is in the closure of the sequence $T = \{\varphi_{z_\alpha}(\xi) : z_\alpha \in S\}$. By Lemma 3.2 we can split $S = S_1 \cup \ldots \cup S_N$, where for each $1 \leq j \leq N$, $\rho(z_1, z_2) > 1/2$ when $z_1, z_2 \in S_j$ are different. We also have the corresponding decomposition $T = T_1 \cup \ldots \cup T_N$, where $T_j = \{\varphi_{z}(\xi) : z \in S_j\}$. Hence, there is at least one $j_0$ such that $x$ is in the closure of $T_{j_0}$. The lemma will follow if we show that $T_{j_0}$ is a separated sequence. If $z_1, z_2 \in S_{j_0}$ are different then

$$\rho(z_1, z_2) \leq \rho(z_1, \varphi_{z_1}(\xi)) + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) + \rho(\varphi_{z_2}(\xi), z_2) = 2|\xi| + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)).$$

So, $\rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) \geq (1/2) - 2|\xi| \geq 1/4$, proving our claim. 

$\blacksquare$
Lemma 3.4 Let \((z_n)\) be a net in \(\mathbb{D}\) converging to \(x \in M(\mathcal{A})\). Then

(i) \(\varphi_x\) is a continuous one-to-one map from \(\mathbb{D}\) into \(M(\mathcal{A})\),

(ii) \(f \circ \varphi_x \in \mathcal{A}\) for every \(f \in \mathcal{A}\),

(iii) \(f \circ \varphi_{z_n} \rightarrow f \circ \varphi_x\) uniformly on compact sets of \(\mathbb{D}\) for every \(f \in \mathcal{A}\).

Proof. Suppose that \(\omega \in \mathbb{D}\) and \(f \in \mathcal{A}\). Given \(\varepsilon > 0\) there is \(\delta > 0\) such that \(|f(u) - f(v)| < \varepsilon\) if \(\rho(u, v) < \delta\). Take \(\omega_1 \in K(\omega, \delta)\). Since \(\rho(\varphi_{z_n}(\omega_1), \varphi_{z_n}(\omega)) = \rho(\omega_1, \omega) < \delta\) then \(|f(\varphi_{z_n}(\omega_1)) - f(\varphi_{z_n}(\omega))| < \varepsilon\) for every \(\alpha\). Then

\[
|f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))| \\
\leq |f(\varphi_x(\omega_1)) - f(\varphi_{z_n}(\omega))| + |f(\varphi_{z_n}(\omega)) - f(\varphi_{z_n}(\omega_1))| + |f(\varphi_{z_n}(\omega_1)) - f(\varphi_x(\omega_1))| + \varepsilon
\]

for every \(\alpha\). Taking limits in \(\alpha\) we get \(|f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))| \leq \varepsilon\) when \(\rho(\omega_1, \omega) < \delta\). This proves the continuity of \(\varphi_x\) and (ii).

To prove that \(\varphi_x\) is one-to-one, for an arbitrary \(0 < r < 1\) we will construct a function \(f \in \mathcal{A}\) (depending on \(r\)) such that \((f \circ \varphi_x)(\omega) = \omega\) when \(|\omega| < r\). It is convenient to deal with the hyperbolic metric \(\rho\) instead of \(\rho\). Write \(s = \log \frac{1+r}{1-r}\). By Lemma 3.2 there is a sequence \(\{z_n\}\) in \(\mathbb{D}\) whose closure contains \(x\) and such that \(h(z_n, z_m) > 5s\) if \(n \neq m\). Therefore

\[
(3.1) \quad h(K_h(z_n, 2s), K_h(z_m, 2s)) \geq s \quad \text{if} \quad n \neq m.
\]

Take \(g \in C(\mathbb{D})\) such that \(g(\omega) = \omega\) if \(h(\omega, 0) < s\) (i.e.: if \(|\omega| < r\)) and \(g(\omega) = 0\) if \(h(\omega, 0) > 2s\). Thus \(g \circ \varphi_{z_n}\) is supported in \(K_h(z_n, 2s)\) and

\[
f = \sum_{n \geq 1} (g \circ \varphi_{z_n}) \in C(\mathbb{D}).
\]

Since \(g\) is uniformly continuous with respect to the Euclidean metric then it is \(h\)-uniformly continuous. Hence, given \(\varepsilon > 0\) there is \(\delta\), with \(0 < \delta < s/2\), such that

\[
(3.2) \quad |g(\xi_1) - g(\xi_2)| < \varepsilon \quad \text{if} \quad h(\xi_1, \xi_2) < \delta.
\]

Let \(\omega_1, \omega_2 \in \mathbb{D}\) such that \(h(\omega_1, \omega_2) < \delta\). By (3.1) \(K_h(\omega_1, \delta)\) cuts at most one of the disks \(K_h(z_n, 2s)\). If it doesn’t cut any, then \(f(\omega_1) = f(\omega_2) = 0\). If it cuts \(K_h(z_{n_0}, 2s)\), then \(f(\omega_1) - f(\omega_2) = g(\varphi_{z_{n_0}}(\omega_1)) - g(\varphi_{z_{n_0}}(\omega_2))\), and since \(h(\varphi_{z_{n_0}}(\omega_1), \varphi_{z_{n_0}}(\omega_2)) = h(\omega_1, \omega_2) < \delta\) then (3.2) says that \(|f(\omega_1) - f(\omega_2)| < \varepsilon\). Thus \(f \in \mathcal{A}\).
If $k$ is any positive integer and $|\omega| < r$ then $h(0, \omega) < s$ and $\varphi_{z_k}(\omega) \in K_h(z_k, s)$. So, (3.1) and the inclusion: $\text{supp} (g \circ \varphi_{z_n}) \subset K_h(z_n, 2s)$ imply that $(g \circ \varphi_{z_n})(\varphi_{z_k}(\omega)) = 0$ for $n \neq k$. Consequently

$$f(\varphi_{z_k}(\omega)) = (g \circ \varphi_{z_k})(\varphi_{z_k}(\omega)) = g(\omega) = \omega.$$ 

Thus, if $(z_n)$ is a net of points in $\{z_n\}$ that tends to $x$ then $(f \circ \varphi_{z_n})(\omega) = \omega$ for every $\alpha$ and every $\omega \in r\mathbb{D}$. Therefore $(f \circ \varphi_x)(\omega) = \omega$ when $\omega \in r\mathbb{D}$.

Suppose that (iii) fails. This means that there are $f \in A$, $0 < r < 1$ and $\varepsilon > 0$ such that $|(f \circ \varphi_{z_n})(\xi_\alpha) - (f \circ \varphi_x)(\xi_\alpha)| > \varepsilon$ for some points $\xi_\alpha \in r\mathbb{D}$.

We can also assume that $\xi_\alpha \rightarrow \xi$. Since $(f \circ \varphi_{z_n})(\xi_\alpha) \rightarrow (f \circ \varphi_x)(\xi)$, this contradicts the uniform $\rho$-continuity of $f$. ■

3.2. The hyperbolic parts

Definition. If $x, y \in M(A)$ define $\rho(x, y) = \sup \rho(S, T)$, where $S$ and $T$ run over all the separated sequences in $\mathbb{D}$ so that $x \in \mathbb{S}$ and $y \in \mathbb{T}$. Defining $h(x, y)$ in analogous fashion, we have

$$h(x, y) = \log \frac{1 - \rho(x, y)}{1 + \rho(x, y)}.$$

Lemma 3.5 Let $x, y \in M(A) \setminus \mathbb{D}$. Then

1. $\rho(x, y) = \alpha < 1$ if and only if $y = \varphi_x(\omega)$ for some $\omega$ with $|\omega| = \alpha$.

2. $y = \varphi_x(\xi)$ with $\xi \in \mathbb{D}$ if and only if every separated sequences $S, T$ such that $x \in \mathbb{S}$ and $y \in \mathbb{T}$ satisfy $\rho(T, \{\varphi_x(\xi) : z_n \in S\}) = 0$.

3. $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) = h(\xi_1, \xi_2)$ for every $\xi_1, \xi_2 \in \mathbb{D}$.

4. $h$ is a $[0, +\infty]$-valued metric on $M(A)$.

Proof. (1) Suppose that $\rho(x, y) = \alpha < 1$ and take $b \in (a, 1)$. The continuity of $\varphi_x$ implies that $\varphi_x(\mathbb{D})$ is compact. So, if $y \notin \varphi_x(\mathbb{D})$ there are closed disjoint neighborhoods $U$ of $\varphi_x(\mathbb{D})$ and $V$ of $y$. Let $S$ and $T$ be separated sequences in $\mathbb{D}$ such that $x \in \mathbb{S}$ and $y \in \mathbb{T}$. If $(z_n)$ is a net in $S$ that tends to $x$ then $\varphi_{z_n}(\xi) \rightarrow \varphi_x(\xi)$ for every $\xi \in \mathbb{D}$. By a compactness argument $\varphi_{z_n}(\mathbb{D}) \subset U$ for a tail $(z_n)_{n \geq n_0}$ of the original net. Let $S_1 = \{z_n \in S : z_n = z_\alpha \text{ for some } \alpha \geq \alpha_0\}$. Then $x \in \mathbb{S}_1$ and $\varphi_{z_n}(\mathbb{D}) \subset U$ for every $z_n \in S_1$. This means that

$$K(z_n, b) \subset U \text{ for every } z_n \in S_1.$$
On the other hand, since \( V \) is a neighborhood of \( y \) then

\[
y \in \overline{T}_1, \quad \text{where} \quad T_1 = \{ z \in T : z \in V \}. \tag{3.4}
\]

Since \( U \) and \( V \) are disjoint, (3.3) and (3.4) say that \( \rho(S_1, T_1) \geq b > a \), contradicting the definition of \( \rho(x, y) = a \). Since \( b \in (a, 1) \) is arbitrary then \( y \in \varphi_x(\partial \mathbb{D}) \), so \( y = \varphi_x(\omega) \) with \( |\omega| \leq a \).

Reciprocally, suppose that \( y = \varphi_x(\omega) \) with \( |\omega| = a \), and let \( S, T \) be separated sequence in \( \mathbb{D} \) such that \( x \in S \) and \( y \in \mathcal{T} \). If \( (z_n) \) is a net in \( S \) that tends to \( x \) then \( \varphi_{z_n}(\omega) \to y \). Thus \( y \in \overline{T}_1 \), where \( T_1 = \{ \varphi_{z_n}(\omega) : z_n \in S \} \). So, \( y \in T_1 \cap T \neq \emptyset \) and by Lemma 3.1, \( \rho(T_1, T) = 0 \). That is, given \( \varepsilon > 0 \) there are \( z_n \in S \) and \( \omega_n \in \mathcal{T} \) such that \( \rho(\varphi_{z_n}(\omega), \omega_n) < \varepsilon \), which yields

\[
\rho(z_n, \omega_n) \leq \rho(z_n, \varphi_{z_n}(\omega)) + \rho(\varphi_{z_n}(\omega), \omega_n) < |\omega| + \varepsilon = a + \varepsilon.
\]

So, \( \rho(S, T) \leq a \) and by definition \( \rho(x, y) \leq a \).

(2). The necessity follows from Lemma 3.1. If \( y \neq \varphi_x(\xi) \) then \( \rho(y, \varphi_x(\xi)) \neq 0 \) and there are separated sequences \( T_1, T_2 \) such that \( \varphi_x(\xi) \in \overline{T}_1, y \in \overline{T}_2 \) and \( \rho(T_1, T_2) \geq \delta > 0 \). Let \( S \) be a separated sequence such that \( x \in \mathcal{S} \). Therefore \( x \) is in the closure of \( S_1 = \{ z_n : \rho(\varphi_{z_n}(\xi), T_1) < \delta/2 \} \), because if \( x \in \mathcal{S} \setminus S_1 \) then

\[
\varphi_x(\xi) \in \{ \varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus S_1 \} \cap \overline{T}_1
\]

while

\[
\rho(\{ \varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus S_1 \}, T_1) \geq \delta/2,
\]

which contradicts Lemma 3.1. So, for \( z_n \in S_1 \), \( \rho(\varphi_{z_n}(\xi), T_2) \geq \delta/2 \).

(3). Fix \( \xi_1, \xi_2 \in \mathbb{D} \). By Lemma 3.2 there is a separated sequence \( S = \{ z_k \} \) such that \( x \in \mathcal{S} \) and \( h(z_n, z_m) \geq h(\xi_1, \xi_2) + h(0, \xi_1) + h(0, \xi_2) \) if \( n \neq m \). Since

\[
\begin{align*}
\quad h(z_n, z_m) & \leq h(z_n, \varphi_{z_n}(\xi_1)) + h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) + h(\varphi_{z_n}(\xi_2), z_m) \nonumber \\
& = h(0, \xi_1) + h(0, \xi_2) + h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)),
\end{align*}
\]

then \( h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) \geq h(\xi_1, \xi_2) \) if \( n \neq m \). Therefore

\[
h(\{ \varphi_{z_n}(\xi_1) \}_{n \geq 1}, \{ \varphi_{z_n}(\xi_2) \}_{m \geq 1}) = h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) = h(\xi_1, \xi_2),
\]

implying that \( h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \geq h(\xi_1, \xi_2) \). For the other inequality let \( T_1, T_2 \) be separated sequences such that \( \varphi_x(\xi_1) \in \overline{T}_1 \) and \( \varphi_x(\xi_2) \in \overline{T}_2 \). For a separated sequence \( S \) such that \( x \in \mathcal{S} \) and \( \varepsilon > 0 \) write

\[
S' = \{ z_n \in S : h(\varphi_{z_n}(\xi_1), T_1) < \varepsilon, \ h(\varphi_{z_n}(\xi_2), T_2) < \varepsilon \}
\]

and \( S'' = S \setminus S' \).
By \((2)\) \(x \notin \overline{\mathcal{S}}\). So, \(x \in \overline{\mathcal{S}}\) and
\[
h(T_1, T_2) \leq h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) + 2\epsilon = h(\xi_1, \xi_2) + 2\epsilon.
\]
That is, \(h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \leq h(\xi_1, \xi_2) + 2\epsilon\). \((4)\). We must prove only that given \(x, y, z \in M(\mathcal{A})\),
\[
(3.5) \quad h(x, y) \leq h(x, z) + h(z, y)
\]
The inequality is obvious if its right member is infinite. Otherwise \((1)\) says that \(x = \varphi_x(\xi_1)\) and \(y = \varphi_x(\xi_2)\) for some \(\xi_1, \xi_2 \in \mathcal{D}\). Then \((3.5)\) becomes
\[
h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \leq h(\varphi_x(\xi_1), \varphi_x(0)) + h(\varphi_x(0), \varphi_x(\xi_2)),
\]
which holds by \((3)\).

**Definition.** If \(x \in M(\mathcal{A})\) define the hyperbolic part of \(x\) as
\[
H(x) = \{\varphi_x(\omega) : \omega \in \mathcal{D}\}.
\]
Observe that \((1)\) of Lemma 3.5 implies that
\[
H(x) = \{y \in M(\mathcal{A}) : \rho(x, y) < 1\} = \{y \in M(\mathcal{A}) : h(x, y) < \infty\}
\]
and by \((4)\) of the same lemma, \(\{H(x) : x \in M(\mathcal{A})\}\) is a partition of \(M(\mathcal{A})\).

In fact if \(z \in H(x) \cap H(y)\) then for any \(u \in H(x)\),
\[
h(u, y) \leq h(u, x) + h(x, z) + h(z, y) < \infty.
\]
So, \(H(x) \subset H(y)\) and by symmetry they coincide.

**Lemma 3.6** The map \(x \mapsto \varphi_x\) from \(M(\mathcal{A})\) into \(M(\mathcal{A})^\mathcal{D}\) is continuous.

**Proof.** Let \((x_\alpha)\) be a net in \(M(\mathcal{A})\) that tends to \(x\) and \(\xi \in \mathcal{D}\). We must show that if \((x_\beta)\) is a subnet such that \(\varphi_{x_\beta}(\xi) \rightarrow y\) then \(y = \varphi_x(\xi)\). Let \(\mathcal{S} = \{z_n\}\) and \(\mathcal{T} = \{\omega_n\}\) be separated sequences such that \(x \in \overline{\mathcal{S}}\) and \(y \in \overline{\mathcal{T}}\).

For \(\delta > 0\) write
\[
U = \bigcup_{n \geq 1} K(z_n, \delta) \quad \text{and} \quad V = \bigcup_{n \geq 1} K(\omega_n, \delta).
\]
Since there is \(f \in \mathcal{A}\) such that \(f(z_n) = 0\) for all \(n\) and \(f \equiv 1\) on \(\mathcal{D} \setminus U\) then \(\overline{U} \supset \{m \in M(\mathcal{A}) : |f(m)| < 1/2\}\), a neighborhood of \(x\). So, \(\overline{U}\) is a neighborhood of \(x\) and by the same reason \(\overline{V}\) is a neighborhood of \(y\). Since \(x_\beta \rightarrow x\) and \(\varphi_{x_\beta}(\xi) \rightarrow y\), there is \(\beta_0\) such that for every \(\beta \geq \beta_0\),
\begin{align*}
(i) \quad & \varphi_{x_\beta}(\xi) \in \overline{V}, \quad \text{and} \\
(ii) \quad & x_\beta \in \overline{\mathcal{S}_\beta}, \quad \text{where} \ \mathcal{S}_\beta = \{z_\beta(\beta)\}_{n \geq 1} \text{ is a separated sequence in } U.
\end{align*}
Assume that $\beta \geq \beta_0$. Since
$$\varphi_{x_{n}}(\xi) \in \overline{\{\varphi_{z_{n}}(\beta)\}_{n \geq 1} \cap \bigcup_n K(\omega_n, \delta)}$$
then Lemma 3.1 says that $\rho({\varphi_{z_{n}}, (\beta), T}) \leq \delta$. So, there is $n_0$ such that
$\rho(\varphi_{z_{n}}, (\beta), T) < 2\delta$. On the other hand, by definition of $U$ and (ii) there is some $z_{k_0} \in S$ such that $\rho(z_{k_0}, z_{n_0}, (\beta)) \leq \delta$. Since there is $c_{\xi} > 0$ such that
$$\rho(\varphi_{z_{k_0}}, (\beta), T) \leq c_{\xi}\rho(z_{k_0}, z_{n_0}, (\beta)) \leq c_{\xi}\delta,$$
then $\rho(\varphi_{z_{k_0}}, (\beta), T) \leq (c_{\xi} + 2)\delta$. This shows that
$$\rho(\{\varphi_{z_{n}}(\xi) : z_n \in S\}, T) \leq (c_{\xi} + 2)\delta,$$
and since $\delta > 0$ is arbitrary, $\rho(\{\varphi_{z_{n}}(\xi) : z_n \in S\}, T) = 0$. Since $S$ and $T$ are arbitrary separated sequences such that $x \in S$ and $y \in T$ then (2) of Lemma 3.5 tells us that $y = \varphi_{x}(\xi)$.

3.3. Hyperbolic algebras

A closed self-adjoint subalgebra $B$ of $A$ that separates the points of $D$ and contains the constants will be called a prehyperbolic algebra. For such $B$, Theorem 4.28 of [13] implies that whenever $b \in B$ is invertible in $A$ then the inverse belongs to $B$. Hence, the disk is dense in $M(B)$, because if there exists $y \in M(B)$ that is not in the closure of $D$ then there is $f \in B$ such that $f(y) = 0$ and $|f| \geq \delta > 0$ on $D$. Since clearly $f$ is invertible in $A$, then so is in $B$ and consequently $f$ cannot vanish anywhere in $M(B)$, a contradiction.

The inclusion of $B$ in $A$ induces by transposition a projection $\pi : M(A) \rightarrow M(B)$. Since $\pi(D) = D$ is dense in $M(B)$ then $\pi$ is onto. For a set $E \subset D$ we write $\overline{E}_{M}$, with $M = M(A)$ or $M(B)$, to distinguish between closures in the corresponding space. No distinction will be made for the closure of sets in $C$.

A closed set $F \subset M(A)$ will be called saturated if $H(x) \subset F$ whenever $x \in F$. If $\pi : M(A) \rightarrow M(B)$ is the natural projection, write
$$G_{B} = \{y \in M(B) : \pi^{-1}(y) \text{ is a singleton}\}$$
and
$$\Gamma_{B} = \{y \in M(B) : \pi^{-1}(y) \text{ is saturated}\}.$$
That is, if $y \in M(B)$ then $y \in G_{B}$ if and only if $B$ separates every $x \in \pi^{-1}(y)$ from any other point of $M(A)$ (so $\pi^{-1}(y) = \{x\}$), and $y \in \Gamma_{B}$ if and only if $b \circ \varphi_{x}$ is constant for all $x \in \pi^{-1}(y)$ and $b \in B$. Since no single point is a saturated set then $G_{B} \cap \Gamma_{B} = \emptyset$. In addition, there could be points in $M(B)$ that are not in $G_{B} \cup \Gamma_{B}$. We will be interested in the cases that exclude the last possibility.
Definition. A prehyperbolic algebra $\mathcal{B}$ will be called hyperbolic if $M(\mathcal{B}) = G_B \cup \Gamma_B$. That is, if $\Gamma^{-1}(\pi(x)) = \{x\}$ or contains $H(x)$ for every $x \in M(\mathcal{A})$.

Lemma 3.7 Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra. Then

1. $\Gamma_B$ is closed,
2. the restriction $\pi_0 : \Gamma^{-1}(\Gamma_B) \to G_B$ of $\pi$ is an onto homeomorphism.

Proof. (1) If $x$ is in the closure of $\Gamma^{-1}(\Gamma_B)$ take a net $(x_{\alpha})$ in $\Gamma^{-1}(\Gamma_B)$. Hence, if $y \in D$ and $f \in \mathcal{B}$, Lemma 3.6 gives

$$f(x) - f(\varphi_x(\omega)) = \lim f(x_{\alpha}) - f(\varphi_{x_{\alpha}}(\omega)) = 0,$$

implying that $f \circ \varphi_x \equiv f(x)$, so $x \in \Gamma^{-1}(\Gamma_B)$. That is, $\Gamma^{-1}(\Gamma_B)$ is closed in $M(\mathcal{A})$ and then $\pi(\Gamma^{-1}(\Gamma_B))$ is closed in $M(\mathcal{B})$.

(2) By definition of $G_B$, $\pi_0$ is one-to-one and onto, so we must show that $\pi_0 : G_B \to \Gamma_B$ is continuous. Let $(y_{\alpha})$ be a net in $G_B$ such that $y_{\alpha} \to y \in G_B$ and let $x_{\alpha} \in \pi^{-1}(G_B)$ such that $\pi(x_{\alpha}) = y_{\alpha}$. If $(x_{\alpha_{\beta}})$ is a convergent subnet of $(x_{\alpha})$, say to $x$, then $y_{\alpha_{\beta}} = \pi(x_{\alpha_{\beta}}) \to \pi(x) = y$. So, $x \in \Gamma^{-1}(y)$, but since $y \in G_B$ then $\pi^{-1}(y) = \{x\}$. Hence every convergent subnet of $(x_{\alpha})$ tends to $x$, and then $x_{\alpha} \to x$.

Proposition 3.8 Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra and $y \in M(\mathcal{B})$. The following conditions are equivalent

1. $y \in \Gamma_B$.
2. $f \circ \varphi_{z_{\alpha}} \to c \in \mathbb{C}$ uniformly on compact sets for every net $(z_{\alpha})$ in $\mathbb{D}$ tending to $y$ and every $f \in \mathcal{B}$.
3. For every separated sequence $\mathcal{S}$ such that $y \in \mathcal{S}^{M(\mathcal{B})}$ and every $f \in \mathcal{B}$ there is a subsequence $\{z_n\}$ of $\mathcal{S}$ (depending on $f$) such that $f \circ \varphi_{z_n} \to c \in \mathbb{C}$ pointwise on $\mathbb{D}$.

Proof. (a1) $\Rightarrow$ (a2). If $y \in \Gamma_B$ then $\pi^{-1}(y)$ is saturated. Let $(z_{\alpha})$ be a net in $\mathbb{D}$ such that $z_{\alpha} \to y$ in $M(\mathcal{B})$ and take a subnet $(z_{\alpha_{\beta}})$ that converges in $M(\mathcal{A})$, say to $x$. Thus $\pi(z_{\alpha_{\beta}}) \to \pi(x) = y$ in $M(\mathcal{B})$, saying that $x \in \pi^{-1}(y)$. Since $H(x) \subset \pi^{-1}(y)$ (because it is saturated) then

$$f(\varphi_x(\xi)) = \lim f(\varphi_{z_{\alpha_{\beta}}}(\xi)) = \text{const.} = \lim f(\varphi_{z_{\alpha_{\beta}}}(0)) = \lim f(z_{\alpha_{\beta}}) = f(y)$$

for every $f \in \mathcal{B}$ and $\xi \in \mathbb{D}$. This proves that whenever $(z_{\alpha_{\beta}})$ is a subnet of $(z_{\alpha})$ that converges in $M(\mathcal{A})$ then $f \circ \varphi_{z_{\alpha_{\beta}}} \to f(y)$ pointwise. By Lemma 3.4 the convergence is also uniform on compact sets, and consequently $f \circ \varphi_{z_{\alpha}} \to f(y)$ in that way.
(a2)⇒(a3). If \( y \in \overline{\mathcal{S}M(B)} \) there is a net \((z_n)\) in \( \mathcal{S} \) such that \( z_n \to y \) in \( M(B) \). If \( f \in B \) then by (a2), \( f \circ \varphi_{z_n} \to c \in \mathbb{C} \) uniformly on compact sets. Therefore for any positive integer \( n \) there is some \( z_n \) (that we rename as \( z_n \)) such that
\[
\sup \{ |(f \circ \varphi_{z_n})(\omega) - c| : |\omega| \leq 1 - n^{-1} \} \leq n^{-1}.
\]
Therefore \( \{ z_n \} \) is a subsequence of \( \mathcal{S} \) that satisfies (a3).

(a3)⇒(a1). We will show that (a3) fails when (a1) fails. If \( y \notin \Gamma_B \) there is \( x \in \pi^{-1}(y) \) such that \( H(x) \not\subset \pi^{-1}(y) \). Therefore there is \( f \in B \) such that \( f \circ \varphi_x \neq \text{const.} \), or what is the same, \( (f \circ \varphi_x)(\omega) \neq f(x) \) for some \( \omega \in \mathbb{D} \). Put \( \eta = |(f \circ \varphi_x)(\omega) - f(x)| > 0 \). If \( \mathcal{S} \) is any separated sequence such that \( x \in \overline{\mathcal{S}M(A)} \) and we take
\[
S_1 = \{ z \in \mathcal{S} : |(f \circ \varphi_{z_n})(\omega) - f(z_n)| \geq \eta/2 \}
\]
then \( x \in \overline{\mathcal{S}_1M(B)} \). Hence \( y = \pi(x) \in \overline{\mathcal{S}_1M(B)} \) and (a3) fails for \( S_1 \) and \( f \).

Suppose that \( f \) is a continuous function from \( M(A) \) into a topological space \( T \). If \( B \) is a hyperbolic algebra, the restriction \( f|_D \) admits a continuous extension from \( M(B) \) into \( T \) if and only if \( f(\pi^{-1}(y)) = \text{const.} \) for every \( y \in \Gamma_B \). In particular, for \( T = \mathbb{C} \) we obtain that \( f \in A \) belongs to \( B \) if and only if \( f(\pi^{-1}(y)) = \text{const.} \) for every \( y \in \Gamma_B \).

Let \( B \subset L^\infty(\mathbb{D}) \) be a closed algebra. A sequence \( \{ z_n \} \subset \mathbb{D} \) is called interpolating for \( B \) if for every \( \{ \eta_n \} \in \ell^\infty \) there exists \( f \in B \) such that \( f(z_n) = \eta_n \) for every \( n \). It is clear that if \( B \) is a subalgebra of \( A \) then every interpolating sequence for \( B \) must be separated and that every separated sequence is interpolating for \( A \). We say that \( f \in A \) separates two sets \( E, F \subset M(A) \) when \( \overline{f(E)} \cap \overline{f(F)} = \emptyset \).

**Proposition 3.9** Let \( B \subset A \) be a prehyperbolic algebra. For \( y \in M(B) \) consider the following conditions

(a) \( y \in G_B \).

(b) There is an interpolating sequence \( \mathcal{S} = \{ z_n \} \) for \( B \), whose closure in \( M(B) \) contains \( y \), such that for every \( \delta > 0 \) sufficiently small there exists \( f \in B \) that separates \( \{ z_n \} \) from \( \mathbb{D} \setminus \bigcup_n K(z_n, \delta) \).

Then (b) implies (a), and if \( B \) is hyperbolic, (a) implies (b).

**Proof.** (b)⇒(a). Let \( y \in M(B) \) and \( \mathcal{S} \) as in (b). We claim that \( \pi^{-1}(y) \subset \overline{\mathcal{S}M(A)} \), because otherwise there is \( x \in \pi^{-1}(y) \) and a separated sequence \( T \subset \mathbb{D}, \) with \( x \in T^{M(A)} \), such that \( \rho(\mathcal{S}, T) \geq \alpha > 0 \). The continuity of \( \pi \) implies that \( y = \pi(x) \in T^{M(B)} \), but this is not possible because by hypothesis there is \( f \in B \) such that \( \overline{f(\mathcal{S})} \cap \overline{f(T)} = \emptyset \), which contradicts \( y \in \overline{\mathcal{S}M(B)} \cap \overline{T^{M(B)}} \).
Now suppose that there are two different points \( x_1, x_2 \in \pi^{-1}(y) \). Then there is a disjoint decomposition \( \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \), where

\[
x_1 \in \overline{\mathcal{S}_1^M(\mathcal{A})} \quad \text{and} \quad x_2 \in \overline{\mathcal{S}_2^M(\mathcal{A})}.
\]

Since \( \mathcal{S} \) is interpolating for \( \mathcal{B} \) there exists \( f \in \mathcal{B} \) that separates \( \mathcal{S}_1 \) from \( \mathcal{S}_2 \), leading to the same contradiction obtained before. Hence, \( \pi^{-1}(y) \) is a single point.

(b) \( \Rightarrow \) (b) for \( \mathcal{B} \) hyperbolic. If \( y \in G_{\mathcal{B}} \) then \( \pi^{-1}(y) = \{ x \} \) for some \( x \in M(\mathcal{A}) \). Since \( \pi^{-1}(\Gamma_{\mathcal{B}}) \) is closed in \( M(\mathcal{A}) \) (by Lemma 3.7) and \( x \not\in \pi^{-1}(\Gamma_{\mathcal{B}}) \) then there is a closed neighborhood \( F \) of \( x \) in \( M(\mathcal{A}) \) such that \( F \cap \pi^{-1}(\Gamma_{\mathcal{B}}) = \emptyset \). Hence there is \( f \in \mathcal{A} \) such that \( f \equiv 1 \) on \( F \) and \( f \equiv 0 \) on \( \pi^{-1}(\Gamma_{\mathcal{B}}) \).

Let \( T \subset \mathbb{D} \) be a separated sequence such that \( x \in \overline{T}^M(\mathcal{A}) \). Since \( f \equiv 1 \) on a neighborhood of \( x \) then \( x \in \overline{\mathcal{S}^M(\mathcal{A})} \), where

\[
\mathcal{S} = \{ z \in T : f(z) = 1 \} = \{ z_n \}.
\]

Hence, \( y = \pi(x) \in \overline{\mathcal{S}^M(\mathcal{B})} \). Observe also that \( \overline{\mathcal{S}^M(\mathcal{A})} \subset F \subset \pi^{-1}(G_{\mathcal{B}}) \).

Let \( \{ \eta_n \} \) be an arbitrary sequence in \( \ell^\infty \) and take \( g \in \mathcal{A} \) such that \( g(z_n) = \eta_n \) for every \( n \). Since \( f \equiv 0 \) on \( \pi^{-1}(\Gamma_{\mathcal{B}}) \) then so is \( h = fg \in \mathcal{A} \), and consequently \( h \in \mathcal{B} \). In addition, \( h(z_n) = f(z_n)g(z_n) = \eta_n \) for every \( n \), which shows that \( \mathcal{S} \) is interpolating for \( \mathcal{B} \). Since \( f \) is \( \rho \)-uniformly continuous and \( f(z_n) = 1 \) for all \( n \) then

\[
\bigcup_n K(z_n, \delta) \subset \{ z : |f(z)| > 1/2 \}
\]

when \( \delta > 0 \) is small enough. Take \( a \in \mathcal{A} \) such that

\[
(3.6) \quad a(z_n) = 1 \quad \text{for all} \quad n, \quad \text{and} \quad a \equiv 0 \quad \text{on} \quad \mathbb{D} \setminus \bigcup_n K(z_n, \delta).
\]

Since \( f \equiv 0 \) on \( \pi^{-1}(\Gamma_{\mathcal{B}}) \) then

\[
\pi^{-1}(\Gamma_{\mathcal{B}}) \subset \{ z : |f(z)| < 1/4 \}^M(\mathcal{A}) \subset \mathbb{D} \setminus \bigcup_n K(z_n, \delta),
\]

implying that \( a \equiv 0 \) on \( \pi^{-1}(\Gamma_{\mathcal{B}}) \). Hence \( a \in \mathcal{B} \) and (3.6) says that it separates \( \mathcal{S} \) from \( \mathbb{D} \setminus \bigcup_n K(z_n, \delta) \). So (b) holds.

Propositions 3.8 and 3.9 provide criteria to decide whether a given prehyperbolic algebra is hyperbolic or not. Let us summarize these criteria in the next corollary.

**Corollary 3.10** A prehyperbolic algebra \( \mathcal{B} \) is hyperbolic if and only if every \( y \in \mathcal{M}(\mathcal{B}) \) satisfies some of the conditions \((a_1), (a_2), (a_3)\) or some of the conditions \((b_1), (b_2)\).
4. Operator-valued compact maps

We recall that if \( S \in \mathcal{L}(L^2_\alpha) \) and \( z \in \mathbb{D} \) then \( S_z = U_z S U_z \), where \( U_z f = (f \circ \varphi_z) \varphi_z' \). Consider the map \( \Psi_S : \mathbb{D} \to \mathcal{L}(L^2_\alpha) \) given by \( \Psi_S(z) = S_z \). We will study the possibility to extend \( \Psi_S \) continuously to \( M(\mathcal{A}) \) when \( \mathcal{L}(L^2_\alpha) \) is provided with the weak or the strong operator topology (WOT and SOT, respectively). We will also look for a possible extension to \( M(\mathcal{B}) \), where \( \mathcal{B} \) is an arbitrary hyperbolic algebra.

**Theorem 4.1** Let \((E,d)\) be a metric space and \( f : \mathbb{D} \to E \) be a continuous map. Then \( f \) admits a continuous extension from \( M(\mathcal{A}) \) into \( E \) if and only if \( f \) is uniformly \((\rho,d)\) continuous and \( \overline{f(\mathbb{D})} \) is compact.

**Proof.** Suppose that \( f \in C(M(\mathcal{A}),E) \). Since \( \mathbb{D} \) is dense in the compact space \( M(\mathcal{A}) \) then \( \overline{f(\mathbb{D})} = f(M(\mathcal{A})) \) is compact. If \( f \) is not uniformly \((\rho,d)\) continuous there are two sequences \( z_n, \omega_n \in \mathbb{D} \) such that \( \rho(z_n, \omega_n) \to 0 \) and \( d(f(z_n), f(\omega_n)) \geq \delta > 0 \) for every \( n \). By the continuity of \( f \) on \( \mathbb{D} \) the sequence does not accumulate on \( \mathbb{D} \). Let \( x \in \{z_n \setminus M(\mathcal{A}) \setminus \mathbb{D} \} \) and \( (z_n) \) be a subnet of \( \{z_n\} \) that tends to \( x \). Since every \( z_\alpha \) is some \( z_{n(\alpha)} \), writing \( \omega_\alpha = \omega_{n(\alpha)} \) we have a subnet \( (\omega_\alpha) \) of the sequence \( \{\omega_n\} \) such that

\[
\rho(\omega_\alpha, \omega_\alpha) \to 0 \quad \text{and} \quad d(f(z_\alpha), f(\omega_\alpha)) \geq \delta \quad \text{for all} \quad \alpha.
\]

The first condition in (4.1) implies that \( g(\omega_\alpha) \to g(x) \) for every \( g \in \mathcal{A} \), meaning that \( \omega_\alpha \to x \) in \( M(\mathcal{A}) \). Since \( f \) is continuous on \( M(\mathcal{A}) \) then \( \lim f(\omega_\alpha) = f(x) = \lim f(z_\alpha) \), contradicting (4.1).

Now assume that \( f \) is uniformly \((\rho,d)\) continuous on \( \mathbb{D} \) and \( \overline{f(\mathbb{D})} \) is compact. For \( x \in M(\mathcal{A}) \) write

\[
F(x) \overset{\text{def}}{=} \{ \lambda \in E : f(z_\alpha) \to \lambda, \text{ for some net } z_\alpha \to x, \ z_\alpha \in \mathbb{D} \}.
\]

The compactness of \( \overline{f(\mathbb{D})} \) assures that \( F(x) \) is nonempty. Then \( F \) is a multivalued function defined on \( M(\mathcal{A}) \), and a standard diagonal argument shows that \( f \) can be extended continuously to \( M(\mathcal{A}) \) if and only if \( F(x) \) is single-valued for every \( x \in M(\mathcal{A}) \). So, let \( x \in M(\mathcal{A}) \) and assume that there are \( \lambda_1, \lambda_2 \in F(x) \) such that \( d(\lambda_1, \lambda_2) = \alpha > 0 \). Let \( B(\lambda, r) \) denote the open ball in \( E \) of center \( \lambda \in E \) and radius \( r > 0 \), and consider the sets

\[
V_i = \{ z \in \mathbb{D} : f(z) \in B(\lambda_i, \alpha/4) \}, \quad i = 1, 2.
\]

Since \( \lambda_i \in F(x) \) then \( x \in V_i^{M(\mathcal{A})} \) for \( i = 1, 2 \). Lemma 3.1 then tells us that \( \rho(V_1, V_2) = 0 \). On the other hand,

\[
d(f(V_1), f(V_2)) \geq d(B(\lambda_1, \alpha/4), B(\lambda_2, \alpha/4)) \geq \alpha/2.
\]

By the uniform \((\rho,d)\)-continuity of \( f \), the last inequality implies that \( \rho(V_1, V_2) > 0 \), a contradiction. \( \blacksquare \)
Lemma 4.2 For $z, \alpha \in \mathbb{D}$ put $\lambda = \lambda(z, \alpha) = (\alpha \bar{z} - 1)/(1 - \bar{z}\alpha)$. Then $U_{\varphi_{z}((\alpha))} U_{z} = V_{\lambda} U_{\alpha}$, where $(V_{\lambda} f)(\omega) = \lambda f(\lambda \omega)$ for $f \in L^{2}_{\alpha}$.

Proof. Since the function $\varphi_{\varphi_{z}((\alpha))} \circ \varphi_{z} \circ \varphi_{\alpha}$ is an automorphism that fixes the origin, it must be a rotation. A little bit of algebra shows that this function maps $\lambda$ to 1. Since $\varphi_{\varphi_{z}((\alpha))}$ is its own inverse then $\varphi_{z} \circ \varphi_{\alpha}(\lambda \omega) = \varphi_{\varphi_{z}((\alpha))}(\omega)$. Therefore

$$(U_{\varphi_{z}((\alpha))} U_{z} f)(\omega) = (f \circ \varphi_{z} \circ \varphi_{\varphi_{z}((\alpha))}(\omega)) \varphi'_{z}(\varphi_{\varphi_{z}((\alpha))}(\omega)) \varphi'_{\varphi_{z}((\alpha))}(\omega)$$

$$= (f \circ \varphi_{z} \circ \varphi_{z} \circ \varphi_{\alpha}(\lambda \omega)) \varphi'_{z}(\varphi_{\alpha}(\lambda \omega)) \varphi'_{\alpha}(\lambda \omega) \lambda$$

$$= (f \circ \varphi_{\alpha})(\lambda \omega) \varphi'_{\alpha}(\lambda \omega) \lambda = (V_{\lambda} U_{\alpha} f)(\omega),$$

where the third equality holds because since $\varphi_{z} \circ \varphi_{z} = \text{id}$ then $(\varphi'_{z} \circ \varphi_{z}) \varphi'_{z} = 1$.

Lemma 4.3 Let $f \in L^{2}_{\alpha}$ and $\varepsilon > 0$. Then there is $\delta = \delta(f, \varepsilon) > 0$ such that $\rho(z_{1}, z_{2}) < \delta \Rightarrow \| U_{z_{1}} f - U_{z_{2}} f \| < \varepsilon$.

Proof. Since the polynomials are dense in $L^{2}_{\alpha}$ and $\| U_{z} \| = 1$ for every $z \in \mathbb{D}$, it is enough to assume that $f$ is a polynomial. If $\rho(z_{1}, z_{2}) < \delta$ then $z_{2} = \varphi_{z_{1}}(\alpha)$ with $|\alpha| < \delta$. By the previous lemma,

$$(I - U_{\varphi_{z_{1}}((\alpha))} U_{z_{1}}) f(\omega) = f(\omega) - f \left( \alpha - \lambda \omega \right) \left( \left| \alpha \right|^{2} - 1 \right) \frac{\lambda \omega}{1 - \bar{\alpha} \lambda \omega} \lambda,$$

where $\lambda$ comes from the lemma. When $\alpha \to 0$ we have $\lambda(z_{1}, \alpha) \to -1$ uniformly in $z_{1}$, so the above expression tends to 0 uniformly in $z_{1}$ and $\omega$. Hence,

$$\| U_{z_{1}} f - U_{\varphi_{z_{1}}((\alpha))} f \| = \| (U_{\varphi_{z_{1}}((\alpha))} U_{z_{1}} - I) f \| < \varepsilon$$

if $|\alpha|$ is small enough. That is, if $\delta$ is small enough.

Proposition 4.4 Let $S \in \mathcal{L}(L^{2}_{\alpha})$. Then the map $\Psi : \mathbb{D} \to (\mathcal{L}(L^{2}_{\alpha}), \text{WOT})$ extends continuously to $M(A)$.

Proof. The closed the ball $B(0, \| S \|) \subset \mathcal{L}(L^{2}_{\alpha})$ of center 0 and radius $\| S \|$ is compact and metrizable with the WOT-topology. Since $\Psi_{S}(\mathbb{D})$ is contained in $B(0, \| S \|)$, Theorem 4.1 reduces the problem to show that $\Psi_{S}$ is uniformly continuous from the disk with the pseudo-hyperbolic metric into $B(0, \| S \|)$ with the weak operator topology. This amounts to see that for every $f, g \in L^{2}_{\alpha}$, the function $z \mapsto \langle S_{z} f, g \rangle$ is uniformly continuous from $(\mathbb{D}, \rho)$ into $(\mathbb{C}, | |)$. For $z_{1}, z_{2} \in \mathbb{D}$ we have

$$U_{z_{1}} S U_{z_{1}} - U_{z_{2}} S U_{z_{2}} = U_{z_{1}} S (U_{z_{1}} - U_{z_{2}}) + (U_{z_{1}} - U_{z_{2}}) S U_{z_{2}} = A + B.$$

If $f, g \in L^{2}_{\alpha}$ then $|\langle A f, g \rangle| \leq \| U_{z_{1}} S \| \| (U_{z_{1}} - U_{z_{2}}) f \|_{2} \| g \|_{2}$ and $|\langle B f, g \rangle| = |\langle f, B^{*} g \rangle| \leq \| f \|_{2} \| U_{z_{2}} S^{*} \| \| (U_{z_{1}} - U_{z_{2}}) g \|_{2}$. By Lemma 4.3 both expressions can be made small if we take $\rho(z_{1}, z_{2})$ small enough.
Theorem 4.5 Let \( S \in \mathfrak{T}(\mathcal{A}) \). Then the map
\[
\Psi_S : \mathbb{D} \to (\mathfrak{L}(L^2_a), \text{SOT})
\]
extends continuously to \( M(\mathcal{A}) \). In addition, \( \Psi_S(M(\mathcal{A})) \subset \mathfrak{T}(\mathcal{A}) \).

Proof. First suppose that \( S = T_a \), with \( a \in \mathcal{A} \). If \( z \in \mathbb{D} \) tends to \( x \in M(\mathcal{A}) \), Lemma 3.4 says that \( a \circ \varphi_z \to a \circ \varphi_x \) uniformly on compact sets. Thus, if \( f \in L^2_a \) and \( 0 < r < 1 \),
\[
\| (T_{a \circ \varphi_z} - T_{a \circ \varphi_x}) f \|^2 \leq \sup_{rB} |a \circ \varphi_z - a \circ \varphi_x|^2 \| f \|_2^2 + 2 \| a \|_\infty^2 \int_{D \setminus rD} |f|^2 dA.
\]
We can choose some \( r = r(f, \|a\|_{\infty}) \) close enough to 1 so that the second term is smaller than a given \( \varepsilon > 0 \), and for such \( r \) the first term tends to 0 as \( z \to x \). Since
\[
\Psi_{S+T} = \Psi_S + \Psi_T,
\]
the case of a polynomial in Toeplitz operators reduces to the case \( S = T_{a_1} \cdots T_{a_k} \), where \( a_j \in \mathcal{A} \) and \( \|a_j\|_{\infty} \leq 1 \) for \( j = 1, \ldots, k \). Consider the operators
\[
S_j = \begin{cases} T_{a_1 \circ \varphi_z} \cdots T_{a_{j-1} \circ \varphi_z} T_{a_j \circ \varphi_x} \cdots T_{a_k \circ \varphi_x} & \text{if } 1 \leq j \leq k \\ T_{a_1 \circ \varphi_z} \cdots T_{a_k \circ \varphi_x} & \text{if } j = k + 1 \end{cases}
\]
If \( f \in L^2_a \) then
\[
\| (S_{k+1} - S_1) f \| \leq \sum_{j=1}^{k} \| (S_{j+1} - S_j) f \|
\]
and since we have proved that \( T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x} \to 0 \) in the strong operator topology as \( z \to x \), then
\[
\| (S_{j+1} - S_j) f \| = \| T_{a_1 \circ \varphi_z} \cdots T_{a_{j-1} \circ \varphi_z} (T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x}) T_{a_{j+1} \circ \varphi_x} \cdots T_{a_k \circ \varphi_x} f \| \leq \| (T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x}) T_{a_{j+1} \circ \varphi_x} \cdots T_{a_k \circ \varphi_x} f \| \to 0
\]
when \( z \to x \). Finally, if \( S \in \mathfrak{T}(\mathcal{A}) \) is arbitrary, given \( \varepsilon > 0 \) there is a polynomial in Toeplitz operators with symbols in \( \mathcal{A} \), say \( T \), such that \( \|S - T\| < \varepsilon \). By Proposition 4.4 there is some \( S_x \in \mathfrak{L}(L^2_a) \) (i.e.: \( S_x = \Psi_S(x) \)) such that
\[
S_z - T_z \to S_x - T_x \quad \text{weakly when } z \to x.
\]
Weak limits do not increase norms, so \( \|S_x - T_x\| \leq \varepsilon \). The result follows because \( \|S_z - T_z\| < \varepsilon \) for all \( z \in \mathbb{D} \) and \( T_z \to T_x \) strongly when \( z \to x \). \( \blacksquare \)
Corollary 4.6 If $S \in \mathcal{L}(L^2_a)$ and $n \geq 0$ is an integer then $B_n S \in \mathcal{A}$. Besides, $B_n S_x = (B_n S) \circ \varphi_x$ for every $x \in M(A)$.

Proof. By (2.1) and Lemma 2.2

$$(B_n S)(z) = ((B_n S) \circ \varphi_z)(0) = (B_n S_z)(0) = (n+1) \sum_{j=0}^{n} \binom{n}{j} (-1)^j \langle S_z^{(n)} \omega^j, \omega^j \rangle.$$

Since by Proposition 4.4 the map $z \mapsto \langle S_z^{(n)} \omega^j, \omega^j \rangle$ extends continuously to $M(A)$, it belongs to $\mathcal{A}$ for every $0 \leq j \leq n$. For the second assertion take a net $(z_\alpha)$ in $\mathbb{D}$ that tends to $x$ and then take limits in the equality $(B_n S_{z_\alpha})(\xi) = (B_n S)(\varphi_{z_\alpha}(\xi))$ for each fixed $\xi \in \mathbb{D}$. The first term tends to $(B_n S_x)(\xi)$ because Proposition 4.4 says that $z \mapsto \langle S_z^{(n)} \omega^j, \omega^j \rangle$ extends continuously to $M(A)$, and the second term tends to $(B_n S)(\varphi_x(\xi))$ because $B_n S \in \mathcal{A}$. ■

Corollary 4.7 If $S \in \mathcal{L}(L^2_a)$ and $x \in M(A)$ the following conditions are equivalent

(i) $S_u = \lambda I$ for every $u \in H(x)$

(ii) $S_u = \lambda I$ for some $u \in H(x)$

(iii) $B_0 S \equiv \lambda$ on $H(x)$.

Proof. Since $H(u) = H(x)$ when $u \in H(x)$ then every $v \in H(x)$ has the form $v = \varphi_u(\omega)$ for some $\omega \in \mathbb{D}$. By the previous corollary

$$(B_0 S)(v) = (B_0 S)(\varphi_u(\omega)) = (B_0 S_u)(\omega).$$

This identity and the fact that $B_0$ acts in a one-to-one fashion on $\mathcal{L}(L^2_a)$ give all the equivalences. ■

Since for $a \in A$ we have

$$(T_a)_x^* = T_{a \circ \varphi_x} \rightarrow T_{a \circ \varphi_x} = (T_{a})_x^*$$

in the SOT-topology when $z \rightarrow x$, then also $(T_z)^* \rightarrow (T_x)^*$ in the SOT-topology for all $T \in \mathfrak{F}(A)$. Also, since the product of a WOT-convergent and a SOT-convergent net in $\mathcal{L}(L^2_a)$ tends weakly to the product of the limits, Proposition 4.4 and Theorems 4.5 imply that

$$(4.2) \quad S_x T_x = (ST)_x \quad \text{and} \quad T_x S_x = (TS)_x$$

for every $S \in \mathcal{L}(L^2_a), T \in \mathfrak{F}(A)$ and $x \in M(A)$. This fails if we only assume $S, T \in \mathcal{L}(L^2_a)$. 
Indeed, consider the operator defined by $Sf(\omega) = f(-\omega)$. Since $S^2 = I$ then $(S^2)_x = I$ for every $x \in M(\mathcal{A})$. On the other hand, since $SK^{(0)}_x = K^{(0)}_x$ then

$$(B_0S)(z) = (1 - |z|^2)^2(K^{(0)}_{z^2}, K^{(0)}_z) = \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2}.$$ 

So $(B_0S)(z) \to 0$ when $|z| \to 1$, and then $(B_0S)(x) = 0$ for all $x \in M(\mathcal{A}) \setminus \mathbb{D}$. Corollary 4.7 then tells us that $S_x = 0$ for $x \in M(\mathcal{A}) \setminus \mathbb{D}$, making (4.2) impossible for this choice of $S$ and $T = S$.

**Lemma 4.8** Let $S \in \mathcal{L}(L^2_\alpha)$ and $x \in M(\mathcal{A})$. Suppose that there is some $n_0 \geq 0$ such that $(B_{n_0}S) \circ \varphi_x = g$ harmonic. Then $(B_nS) \circ \varphi_x = g$ for every $n \geq 0$.

**Proof.** By Corollary 4.6, $\tilde{\Delta}(B_{n_0}S_x) = \tilde{\Delta}g = 0$, which together with (2.7) yields $B_{n_0+1}S_x = B_{n_0}S_x = g$. Then $B_nS_x = g$ for every $n \geq n_0$. Thus $B_0(B_nS_x) = B_0g = g$ for $n \geq n_0$, implying that

$$g = \lim_{n \to \infty} B_0B_nS_x = \lim_{n \to \infty} B_nB_0S_x = B_0S_x,$$

where the second equality follows from Corollary 2.7 and the last one because since $B_0S_x \in \mathcal{A}$ by Corollary 4.6, then $B_n(B_0S_x) \to B_0S_x$ uniformly. Taking $n_0 = 0$, we have proved above that $B_nS_x = g$ for every $n \geq 0$. $lacksquare$

By the lemma we can add two more equivalences to Corollary 4.7, saying that $B_nS \equiv \lambda$ on $H(x)$ for every (or for some) $n \geq 0$.

**Theorem 4.9** Let $S \in \mathfrak{S}(\mathcal{A})$ and $\mathcal{B}$ be a hyperbolic algebra. Then the following conditions are equivalent,

1. $S_x = \lambda I$ when $x \in \pi^{-1}(y)$ for every $y \in \Gamma_\mathcal{B}$, where $\lambda \in \mathbb{C}$ depends only on $y$,

2. there is a continuous map $\Psi^\mathcal{B}_S : M(\mathcal{B}) \to (\mathfrak{S}(\mathcal{A}), SOT)$ such that $\Psi^\mathcal{B}_S \circ \pi = \Psi_S$,

3. $B_nS \in \mathcal{B}$ for some $n \geq 0$,

4. $B_nS \in \mathcal{B}$ for all $n \geq 0$.

If $S \in \mathcal{L}(L^2_\alpha)$ the theorem holds replacing $(\mathfrak{S}(\mathcal{A}), SOT)$ by $(\mathcal{L}(L^2_\alpha), WOT)$ in (2).

**Proof.** If (1) holds then for every $y \in M(\mathcal{B})$ and $x \in \pi^{-1}(y)$, $S_x$ is an operator that only depends on $y$. Hence $\Psi^\mathcal{B}_S(y) = S_x$ is well defined and satisfies the equality in (2). The continuity of $\Psi^\mathcal{B}_S$ from $M(\mathcal{B})$ into any of the metric spaces $(\mathfrak{S}(\mathcal{A}), SOT)$ or $(\mathcal{L}(L^2_\alpha), WOT)$ (according to the hypothesis) follows from the respective continuity of $\Psi_S$, which is given by Theorem 4.5 and Proposition 4.4.
Now suppose that (2) holds. This means that if \( y \in M(B) \) then \( S_x \) is the same operator \( T \) for every \( x \in \pi^{-1}(y) \). Since \( \varphi_x(D) \subset \pi^{-1}(y) \) for \( y \in \Gamma_B \), then \( S_{\varphi_x(\omega)} = T \) for every \( \omega \in \mathbb{D} \). Corollary 4.6 then says that

\[
(B_0S)(\varphi_x(\omega)) = (B_0S_{\varphi_x(\omega)})(0) = (B_0T)(0)
\]

for every \( x \in \pi^{-1}(y) \) and \( \omega \in \mathbb{D} \). Writing \( \lambda = (B_0T)(0) \), we obtain that \( B_0S \equiv \lambda \) on \( H(x) \) for every \( x \in \pi^{-1}(y) \). Hence \( B_0S \) is constant on \( \pi^{-1}(y) \) for every \( y \in \Gamma_B \), meaning that \( (B_0S)|_{D} \) extends continuously to \( M(B) \). Since the Gelfand-Naimark Theorem identifies \( B \) with \( C(M(B)) \) then \( B_0S \in B \). This proves (3) for \( n = 0 \). If (3) holds for some \( n_0 \geq 0 \) then \( B_{n_0}S = \lambda_y \in \mathbb{C} \) on \( \pi^{-1}(y) \) for every \( y \in \Gamma_B \). Lemma 4.8 then implies that the same happens with \( B_nS \) for all \( n \geq 0 \). This proves (4). Finally, if (4) holds then \( (B_0S)|_{\pi^{-1}(y)} = \lambda_y \in \mathbb{C} \) for \( y \in \Gamma_B \). In particular, \( B_0S \equiv \lambda_y \) on \( H(x) \) for every \( x \in \pi^{-1}(y) \). Then (1) follows from Corollary 4.7.

If \( S \in \mathfrak{L}(L^2_a) \) satisfies the conditions of the theorem then the map \( z \mapsto S_z \) admits a continuous extension to \( M(B) \) given by \( \Psi^B_S \). Write

\[
\Psi^B_S(y) = \hat{S}^B_y
\]

when \( y \in M(B) \). If \( B = A \) we keep the previous notation \( \Psi_S(y) = S_y \) for \( y \in M(A) \). Also, since it is clear that we can identify \( \hat{S}^B_z \) with \( S_z \) when \( z \in \mathbb{D} \), we do not make this notation distinction for \( z \in \mathbb{D} \). Observe that if \( y \in M(B) \) and \( (z_\alpha) \) is a net in \( \mathbb{D} \) that tends to \( y \) in \( M(B) \), then \( \hat{S}^B_y \) admits the two alternative and equivalent expressions

\[
\hat{S}^B_y = \lim_\alpha S_{z_\alpha},
\]

a WOT-limit in general and a SOT-limit if \( S \in \mathfrak{T}(A) \), or

\[
\hat{S}^B_y = S_x \text{ for some (or all) } x \in \pi^{-1}(y),
\]

where \( \pi : M(A) \to M(B) \) is the natural projection. Also, if \( b \in B \) we can look at \( b \) as a continuous function on \( M(B) \) or on \( M(A) \). If \( B \neq A \) we write \( \hat{b}^B \) when we need to distinguish the domain of the function, otherwise \( b \) will be looked as a function on \( M(A) \). Of course, if \( z \in \mathbb{D} \) then \( b(z) \) has the same meaning either way.

If \( B \) is a hyperbolic algebra, \( b \in B \) and \( y \in \Gamma_B \), then for every \( x \in \pi^{-1}(y) \) we have

\[
(T_b)_x = T_{b_0\varphi_x} = \lambda
\]

with \( \lambda \in \mathbb{C} \) depending only on \( y \) (actually \( \lambda = \hat{b}^B(y) \)). Since \( \mathfrak{T}(B) \) is generated by these Toeplitz operators, the same holds for every \( S \in \mathfrak{T}(B) \). Theorem 4.9 then says that \( B_nS \in B \) when \( S \in \mathfrak{T}(B) \), for every nonnegative integer \( n \).
5. Approximation and truncation by Toeplitz operators

If $A \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $\mathfrak{T}_0(A)$ for the algebra generated by the Toeplitz operators $T_a$, with $a \in A$, without taking closure. In [4] Axler and Zheng found simple but very ingenious estimates for the norm of operators in $\mathfrak{T}_0(L^\infty(\mathbb{D}))$. The present section (especially Lemmas 5.2 and 5.5) makes heavy use of their method.

5.1. Norm estimates and truncation

The following lemma is a particular case of Lemma 4.2.2 in [21].

**Lemma 5.1** If $c < 0$ and $t > -1$ then

$$J_{c,t}(z) = \int_D \frac{(1 - |\omega|^2)^t}{1 - z\overline{\omega}} dA(\omega), \quad z \in \mathbb{D},$$

is bounded.

The next result appeared in [4] for $p = 6$. The proof sketched here is a standard modification of that proof involving Lemma 5.1.

**Lemma 5.2** Let $p > 4$. Then there is a constant $C_p < \infty$ such that if $S \in \mathfrak{L}(L^2_a)$, then

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |w|^2}} dA(w) \leq \frac{C_p \|S_z 1\|_p}{\sqrt{1 - |z|^2}}$$

for all $z \in \mathbb{D}$ and

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |z|^2}} dA(z) \leq \frac{C_p \|S_w^* 1\|_p}{\sqrt{1 - |w|^2}}$$

for all $w \in \mathbb{D}$.

**Proof.** To prove (5.1) let $S \in \mathfrak{L}(L^2_a)$ and fix $z \in \mathbb{D}$. Since

$$U_z 1 = (|z|^2 - 1)K_z^{(0)} \quad \text{and} \quad U_z U_z = I$$

then

$$U_z S_z 1 = (|z|^2 - 1)SK_z^{(0)}.$$

Thus

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |w|^2}} dA(w) = \frac{1}{1 - |z|^2} \int_D \frac{|(S_z 1)(\varphi_z(w))| |\varphi_z'(w)|}{\sqrt{1 - |w|^2}} dA(w).$$
Making the substitution \( w = \varphi_z(\lambda) \) in the last integral and using Holder’s inequality with \( 1/p + 1/q = 1 \), we obtain

\[
\int_D \frac{|(SK_z(0)^{(0)})(w)|}{\sqrt{1 - |w|^2}} dA(w) = \frac{1}{\sqrt{1 - |z|^2}} \int_D \frac{|(S_z(1)(\lambda))|}{\sqrt{1 - |\lambda|^2}} dA(\lambda)
\leq \frac{\|S_z1\|_p}{\sqrt{1 - |z|^2}} \left( \int_D \frac{dA(\lambda)}{|1 - \bar{z}\lambda|^q(1 - |\lambda|^2)^{q/2}} \right)^{1/q}
\leq \frac{\|S_z1\|_p}{\sqrt{1 - |z|^2}} J(z)^{1/q},
\]

where

\[
J(z) = \int_D \frac{(1 - |\lambda|^2)^{-q/2}}{|1 - \bar{z}\lambda|^{2-q/2}+(3/2)q-2} dA(\lambda).
\]

Since \( p > 4 \) then \( q < 4/3 \), which yields \( q/2 < 2/3 < 1 \) and \( (3/2)q - 2 < 0 \). By Lemma 5.1 there is \( J_q > 0 \) such that \( J(z) \leq J_q \) for every \( z \in \mathbb{D} \). This proves (5.1) with \( C_p = J_q^{1/q} \). Replace \( S \) with \( S^* \) and interchange the roles of \( w \) and \( z \) in (5.1) to obtain

\[
\int_D \frac{|(S^*K_w^{(0)}(0))(w)|}{\sqrt{1 - |w|^2}} dA(z) \leq \frac{C_p\|S^*1\|_p}{\sqrt{1 - |z|^2}}.
\]

Then use the equality \((S^*K_w^{(0)}(0))(z) = (SK_z^{(0)})(w)\) to obtain (5.2). \( \square \)

**Lemma 5.3** Let \( S \in L(L^2_\mathbb{D}) \), \( a, b \in L^\infty(\mathbb{D}) \) and \( p > 4 \). Then

\[
\|T_bST_a\|_{L(L^2_\mathbb{D})} \leq C_p (\|a\|_\infty \|b\|_\infty)^{1/2} \sup_{\omega \in D} \{\|S_z1\|_p |a(z)|\}^{1/2} \sup_{\omega \in D} \{\|S^*_w1\|_p |b(w)|\}^{1/2},
\]

where \( C_p \) is the constant of Lemma 5.2.

**Proof.** For \( f \in L^2_\mathbb{D} \) and \( w \in D \), we have

\[
(ST_a f)(w) = \langle ST_a f, K_w^{(0)} \rangle = \langle af, S^*K_w^{(0)} \rangle
= \int_D f(z)a(z)(S^*K_w^{(0)})(z) dA(z)
= \int_D f(z)a(z)(SK_z^{(0)})(w) dA(z).
\]

Thus, if \( M_b \) denotes the multiplication operator,

\[
(M_bST_a f)(w) = \int_D f(z)a(z)b(w)(SK_z^{(0)})(w) dA(z).
\]
If \( \Phi(z, w) = |a(z)b(w)(SK_z^0)(w)| \) and \( h(z) = (1 - |z|^2)^{-1/2} \) then (5.1) yields
\[
\int_D \Phi(z, w)h(w) \, dA(w) \leq C_p\|b\|_\infty \|S_z\|_p |a(z)| \, h(z)
\leq C_p\|b\|_\infty \sup_{z \in D}\{\|S_z\|_p |a(z)|\} \, h(z),
\]
and by (5.2)
\[
\int_D \Phi(z, w)h(z) \, dA(w) \leq C_p\|a\|_\infty \|S_w^*\|_p |b(w)| \, h(w)
\leq C_p\|a\|_\infty \sup_{\omega \in D}\{\|S_w^*\|_p |b(w)|\} \, h(w).
\]
By Schur’s theorem (see the proof in [21, p. 42]) the operator \( M_bST_a \) satisfies an inequality as in the lemma. The result follows because
\[
\|(T_bST_a)f\|_{L^2} \leq \|(M_bST_a)f\|_{L^2}
\]
for every \( f \in L^2_a \).

Suppose that \( 1 < p < p' < \infty, f \in L^p(\mathbb{D}) \) and \( 0 < r < 1 \). Split the integral
\[
\|f\|_p^p = \|f\chi_{D \setminus rD}\|_p^p + \|f\chi_{rD}\|_p^p,
\]
where \( \chi_E \) denotes the characteristic function of the set \( E \). Taking \( \alpha = p'/p \) and \( \beta = p'/(p' - p) \) we have \( \alpha^{-1} + \beta^{-1} = 1 \). By Holder’s inequality
\[
\|f\chi_{D \setminus rD}\|_p^p \leq \|f\|_{\alpha p}^p \|\chi_{D \setminus rD}\|_\beta = \|f\|_{p'}^p (1 - r^2)^{1 - \frac{p'}{p}},
\]
and consequently
\[
\|f\|_p^p \leq \|f\|_{p'}^p (1 - r^2)^{1 - \frac{p'}{p}} + \|f\chi_{rD}\|_p^p.
\]

**Proposition 5.4** Suppose that \( S \in \mathcal{T}_0(L^\infty(\mathbb{D})) \) and \( F \subset M(\mathcal{A}) \) is a closed saturated set such that \( B_0S \equiv 0 \) on \( F \). Given \( \varepsilon > 0 \) there is an open neighborhood \( \Omega \) of \( F \) in \( M(\mathcal{A}) \) such that if \( U \subset \Omega \cap \mathbb{D} \) is measurable, then
\[
\|T_{a\chi_U}S\|_{L^2(\mathbb{D})} < \varepsilon \quad \text{and} \quad \|ST_{a\chi_U}\|_{L^2(\mathbb{D})} < \varepsilon
\]
for every \( a \in L^\infty(\mathbb{D}) \) with \( \|a\|_\infty \leq 1 \).

**Proof.** Since \( F \) is saturated and \( B_0S \equiv 0 \) on \( F \), Proposition 4.4 and Corollary 4.7 say that \( S_z \xrightarrow{\text{wot}} S_z = 0 \) when \( z \to x \in F \), with \( z \in \mathbb{D} \). Thus \( S_z1 \to 0 \) weakly in \( L^2_a \) and consequently
\[
S_z1 \to 0 \quad \text{uniformly on compact sets as} \quad z \to x \quad (z \in \mathbb{D})
\]
for every \( x \in F \).
Write

\[ S = \sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j}, \]

with \( a_j \in L^\infty(\mathbb{D}) \), and fix \( p, p' \) with \( 4 < p < p' \). Then

\[ \|S_1\|_{p'} = \left\| \sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j} \varphi_{z_1} \right\|_{p'} \leq \sum_{i=1}^{m} \prod_{j=1}^{n_i} c_{p'} \|a_j\|_{\infty} = c, \]

where \( c_{p'} \) is the norm of the analytic projection \( P_+ \) acting on \( L^{p'}(\mathbb{D}) \). For \( 0 < r < 1 \), (5.3) yields

\[ \|S_1\|_p \leq \|S_1\|_{p'} (1 - r)^{1 - \frac{p}{p'}} + \|(S_1)\chi_{rD}\|_{p'}. \]

By (5.6) there is \( r \) close enough to 1 so that the first member of the sum is smaller than \( \varepsilon/2 \), while (5.5) and the compactness of \( F \) imply that there is a neighborhood \( \Omega \) of \( F \) so that the second member is smaller than \( \varepsilon/2 \) for every \( z \in \Omega \cap \mathbb{D} \). In particular, if \( U \subset \Omega \cap \mathbb{D} \) this holds for every \( z \in U \). Since \( \|a\|_{\infty} \leq 1 \), Lemma 5.3 gives

\[ \|ST_{a_{\chi_{U}}}\|^2 \leq C_{p}^2 \sup_{D} \{\|S_1\|_p : z \in U\} \|S_{\omega}^{*}1\|_p \leq C_{p}^2 c \varepsilon^{1/p}, \]

where \( c \) comes from (5.6) with \( S^{*} \) instead of \( S \), and \( C_{p} \) is the constant of Lemma 5.3. To prove the first inequality of (5.4) observe that \( B_{0}S^{*} = B_{0}S \) also satisfies the hypothesis of the proposition and \( \|T_{a_{\chi_{U}}}S^{*}\| = \|S_{\omega}T_{a_{\chi_{U}}}\| \). □

### 5.2. Approximation properties of the \( k \)-Berezin transforms

**Lemma 5.5** Suppose that \( \{S_{k}\} \) is a bounded sequence in \( \mathcal{L}(L^{2}_{a}) \) such that \( \|B_{0}S_{k}\|_{\infty} \to 0 \) when \( k \to \infty \). Then

\[ \sup_{z \in D} |(S_{k})_z| \to 0 \]

uniformly on compact subsets of \( \mathbb{D} \) when \( k \to \infty \).

**Proof.** Since there is a constant \( C \) such that \( \|S_{k}\| \leq C \) for every \( k \), then it is enough to prove that for every \( S \in \mathcal{L}(L^{2}_{a}) \), \( \eta > 0 \) and \( r \in (0, 1) \), there is a function \( c(r, \eta) > 0 \), independent of \( S \), such that

\[ \sup_{z \in D} |(S_1)(u)| \leq c(r, \eta)\|B_{0}S\|_{\infty} + \eta\|S\| \]

when \( u \in r\mathbb{D} \).
Since
\[ K^{(0)}_z(w) = \sum_{m=0}^{\infty} (m+1)z^m \omega^m, \]
then for \( z, \lambda \in \mathbb{D} \) we have
\[ (B_0S)(\varphi_z(\lambda)) = (B_0S_z)(\lambda) = (1 - |\lambda|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1)\langle S_z \omega^j, \omega^m \rangle \tilde{\lambda}^j \lambda^m, \]
where the first equality comes from Lemma 2.2. Then, for \( 0 < \delta < 1/2 \) (to be chosen later) we obtain
\[
\int_{\delta D} \frac{(B_0S)(\varphi_z(\lambda))\tilde{\lambda}^n}{(1 - |\lambda|^2)^2} \, dA(\lambda) = \sum_{j,m=0}^{\infty} (j+1)(m+1)\langle S_z \omega^j, \omega^m \rangle \int_{\delta D} \tilde{\lambda}^{j+n} \lambda^{m} \, dA(\lambda)
\]
\[ = \sum_{j=0}^{\infty} (j+1)\langle S_z \omega^j, \omega^{j+n} \rangle \delta^{2j+2n+2}
\]
\[ = \delta^{-n+2}\left(\langle S_z 1, \omega^n \rangle + \sum_{j=1}^{\infty} (j+1)\langle S_z \omega^j, \omega^{j+n} \rangle \delta^{2j}\right). \]

Since \( 0 < \delta < 1/2 \) and \( \|\omega^j\| = (j+1)^{-1/2} \) then
\[ |\langle S_z 1, \omega^n \rangle| \leq \frac{1}{\delta^{2n+2}}\|B_0S\|_\infty \int_{\delta D} \delta^n \, dA(\lambda) \left(1 - |\lambda|^2\right)^2 + \|S\| \sum_{j=1}^{\infty} (j+1)\|\omega^j\| \|\omega^{j+n}\| \delta^{2j} \]
\[ \leq 2\delta^{-n}\|B_0S\|_\infty + \delta \|S\|, \]
where the last inequality holds because \( \sum_{j=1}^{\infty} \delta^{2j} \leq \delta \) when \( 0 < \delta < 1/2 \). By (5.8)
\[ (S_z 1)(u) = \langle S_z 1, K_u^{(0)} \rangle = \sum_{n \geq 0} (n+1)\langle S_z 1, \omega^n \rangle u^n, \]
implying that
\[ |(S_z 1)(u)| \leq \sum_{0 \leq n \leq N-1} (n+1)|\langle S_z 1, \omega^n \rangle| + \sum_{n \geq N} (n+1)^{1/2}\|S_z\| r^n \]
for \( z \in \mathbb{D}, \ u \in r\mathbb{D} \) and \( N \geq 1 \). Since \( r \in (0,1) \) we can fix some integer \( N = N(r, \eta) \) big enough so that the second sum is bounded by \( (\eta/2)\|S\| \). Using (5.9) in (5.10) we get
\[ |(S_z 1)(u)| \leq N \sum_{0 \leq n \leq N-1} |\langle S_z 1, \omega^n \rangle| + (\eta/2)\|S\|
\]
\[ \leq 2N^2\delta^{-N}\|B_0S\|_\infty + N^2\delta\|S\| + (\eta/2)\|S\| \]
for \( z \in \mathbb{D} \) and \( u \in r\mathbb{D} \). Choosing \( \delta = \delta(r, \eta) < \min\{\eta/2N^2, 1/2\} \) we obtain (5.7) with \( c(r, \eta) = 2N^2\delta^{-N} \). \( \blacksquare \)
Lemma 5.6 Let \( \{S_k\} \) be a sequence in \( \mathcal{L}(L^2_a) \) such that for some \( p' > 4 \),

\[
\|B_0 S_k\|_\infty \to 0, \quad \text{when } k \to \infty,
\]

\[
\sup_{z \in D} \|(S_k)_z 1\|_{p'} \leq C \quad \text{and} \quad \sup_{\omega \in D} \|(S_k^*)_\omega 1\|_{p'} \leq C,
\]

where \( C > 0 \) does not depend on \( k \). Then

\[
\|S_k\|_{\mathcal{L}(L^2_a)} \to 0 \quad \text{when } k \to \infty.
\]

Proof. By (5.12) and Lemma 5.3 with \( a = b = 1 \),

\[
\|S_k\|_{\mathcal{L}(L^2_a)} \leq C_{p'} \sup_{z \in D} \|(S_k)_z 1\|_{p'}^{1/2} \sup_{\omega \in D} \|(S_k^*)_\omega 1\|_{p'}^{1/2} \leq C_{p'} C.
\]

Hence, \( \{S_k\} \) is a bounded sequence in \( \mathcal{L}(L^2_a) \) that satisfies (5.11). Under these conditions Lemma 5.5 says that

\[
\sup_{z \in D} \|(S_k)_z 1\| \to 0 \quad \text{uniformly on compact sets of } \mathbb{D}.
\]

Let \( p \) with \( 4 < p < p' \). By (5.3)

\[
\sup_{z \in D} \|(S_k)_z 1\|_p \leq \sup_{z \in D} \|(S_k)_z 1\|_{p'} (1 - r)^{1 - \frac{p}{p'}} + \sup_{z \in D} \|(S_k)_z 1\chi_r D\|_p
\]

for every \( 0 < r < 1 \). By (5.12) the first member of the sum is bounded by

\[
C_{p'} (1 - r)^{1 - \frac{p}{p'}},
\]

which can be made small by taking \( r \) close to 1, and by (5.13) the second member of the sum tends to 0 as \( k \to \infty \). Therefore,

\[
\sup_{z \in D} \|(S_k)_z 1\|_p \to 0 \quad \text{when } k \to \infty
\]

for every \( p \in (4, p') \). Using again Lemma 5.3, this time with \( p \) instead of \( p' \), we obtain

\[
\|S_k\|_{\mathcal{L}(L^2_a)} \leq C_p \sup_{z \in D} \|(S_k)_z 1\|_p^{1/2} \sup_{\omega \in D} \|(S_k^*)_\omega 1\|_{p'}^{1/2} \leq C_p \sup_{z \in D} \|(S_k)_z 1\|_p^{1/2} C^{1/2} \to 0
\]

when \( k \to \infty \), where the last inequality holds by (5.12), since \( \|\cdot\|_p \leq \|\cdot\|_{p'} \). ■
Theorem 5.7 If \( a \in L^\infty(\mathbb{D}) \) then \( T_{B_k(a)} \to T_a \) in operator norm when \( k \to \infty \). In particular, \( \mathcal{T}(A) = \mathcal{T}(L^\infty(\mathbb{D})) \).

Proof. Write \( S_k = T_{B_k(a)} - T_a \). Since Corollary 2.7 says that \( B_0B_k = B_kB_0 \) on \( \Sigma(L^2) \) then

\[
B_0S_k = B_0T_{B_k(a)} - B_0T_a = B_0B_k(a) - B_0(a) = B_kB_0(a) - B_0(a),
\]

which tends uniformly to 0 when \( k \to \infty \) because \( B_0(a) \in A \). That is, \( \{S_k\} \) satisfies (5.11). On the other hand, if \( p' > 4 \) then

\[
\|(S_k)z1\|_{p'} = \|P_+M(B_k(a) - a)\phi_z,1\|_{p'} \leq c_{p'}(\|B_k(a)\|_\infty + \|a\|_\infty) \leq 2c_{p'}\|a\|_\infty,
\]

where \( c_{p'} \) is the norm of the analytic projection \( P_+ \) acting on \( L^{p'}(\mathbb{D}) \) (see [21, p. 54]). Since

\[
(S_k^*)z = P_+M(B_k(a) - a)\phi_z
\]

then also

\[
\|(S_k^*)z1\|_{p'} \leq 2c_{p'}\|a\|_\infty.
\]

So, \( \{S_k\} \) satisfies (5.12) and Lemma 5.6 then says that \( \|S_k\|_{\Sigma(L^2)} \to 0 \) as \( k \to \infty \). \(\blacksquare\)

Remark 5.8 An obvious consequence of the theorem is that Theorems 4.5 and 4.9 hold for \( S \in \mathcal{T}(L^\infty(\mathbb{D})) \). The argument of Theorem 5.7 works word by word for any \( S \in \Sigma(L^2) \) such that \( T_{B_k}S = S \) satisfies (5.12) for some \( p' > 4 \). So, \( T_{B_k} \to S \) for such \( S \). Maybe this holds for every \( S \in \mathcal{T}_0(L^\infty(\mathbb{D})) \), which would imply that \( \mathcal{T}(L^\infty(\mathbb{D})) \) is the closure of \( \{T_a : a \in A\} \).

6. Abelianization

Lemma 6.1 Let \( F \subset M(A) \setminus \mathbb{D} \) be a closed saturated set, \( \Omega \subset M(A) \) an open neighborhood of \( F \) and \( k \geq 0 \) an integer. Write \( U = \Omega \cap \mathbb{D} \) and \( \mathfrak{F} = \{a \in L^\infty(\mathbb{D}) : a \equiv 0 \text{ on } U\} \). Then

\[
B_ka \equiv 0 \text{ on } F \text{ for every } a \in \mathfrak{F}.
\]

In particular, if \( B \) is a hyperbolic algebra and \( F = \pi^{-1}(\Gamma_B) \) then \( B_ka \in B \) and \( T_a \in \Sigma(B) \).

Proof. By Lemma 4.8 it is enough to prove the lemma for \( k = 0 \). Let \( x \in F \) and take a net \( \{z_\alpha\} \) in \( \mathbb{D} \) such that \( z_\alpha \to x \). We claim that for every \( r \in (0, 1) \) there is \( \alpha_0 = \alpha_0(r) \) such that \( \phi_{z_\alpha}(r\mathbb{D}) \subset \Omega \) for \( \alpha \geq \alpha_0 \). Otherwise there is a subnet \( \{z_\alpha\} \) and points \( \xi_\beta \in r\mathbb{D} \) such that \( \phi_{z_\alpha}(\xi_\beta) \not\in \Omega \) for all \( \beta \). We can assume that \( \xi_\beta \to \xi_0 \), with \( \|
\xi_0\| \leq r \). If \( f \in A \), the inequality

\[
|f(\phi_{z_\alpha}(\xi_\beta)) - f(\phi_x(\xi_0))| \leq |f(\phi_{z_\alpha}(\xi_\beta)) - f(\phi_{z_\alpha}(\xi_0))| + |f(\phi_{z_\alpha}(\xi_0)) - f(\phi_x(\xi_0))|
\]

and the uniform \( \rho \)-continuity of \( f \) imply that \( f(\phi_{z_\alpha}(\xi_\beta)) \to f(\phi_x(\xi_0)) \).
Therefore
\[ \varphi_{z_{a \beta}}(\xi) \to \varphi_{z}(\xi_0) \in H(x) \subset F, \]
and since \( \Omega \) is a neighborhood of \( F \) then \( \varphi_{z_{a \beta}}(\xi) \in \Omega \) for \( \beta \geq \beta_0 \), a contradiction. So, if \( a \in F \) and \( 0 < r < 1 \), there is \( \alpha_0 \) such that \( (a \circ \varphi_{z_a})(\omega) = 0 \) for \( |\omega| < \rho \) and \( \alpha \geq \alpha_0 \). Hence for \( \alpha \geq \alpha_0 \),
\[
|(B_0a)(z_a)| \leq \int_D |(a \circ \varphi_{z_a})(\omega)| dA(\omega) = \int_{D \setminus rD} |(a \circ \varphi_{z_a})(\omega)| dA(\omega) \leq \|a\|_\infty (1 - r^2),
\]
which can be made arbitrarily small by taking \( r \) close enough to 1. Therefore \( (B_0a)(z_a) \to 0 \), but since also \( (B_0a)(z_a) \to (B_0a)(x) \) then \( (B_0a)(x) = 0 \), and this happens for all \( x \in F \).

Now suppose that \( F = \pi^{-1}(\Gamma_B) \), with \( B \) a hyperbolic algebra. Since \( B_kx \in A \) identically vanishes on \( \pi^{-1}(\Gamma_B) \) then \( B_kx \in B \). Consequently \( (B_kx) \in \mathfrak{T}(B) \), and since by Theorem 5.7, \( T_{B_kx} \to T_a \) as \( k \to \infty \), then so is \( T_a \).

Let \( F \subset M(A) \) be a closed set. A set \( U \subset \mathbb{D} \) will be called a relative neighborhood of \( F \) if there is some open neighborhood \( \Omega \subset M(A) \) of \( F \) such that \( U = \Omega \cap \mathbb{D} \). Since the disk is dense in \( M(A) \) and \( \Omega \) is open, it is clear that \( \overline{U}_M(A) \) contains \( \Omega \), and consequently it is a neighborhood of \( F \). Also, for \( V \subset \mathbb{D} \) we will denote \( V^c = \mathbb{D} \setminus V \).

**Lemma 6.2** Let \( S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_{ij}} \), with \( a_{ij} \in L^\infty(\mathbb{D}) \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n_i \), and \( F \subset M(A) \) be a closed saturated set such that \( B_0S \equiv 0 \) on \( F \). Then given \( \varepsilon > 0 \) there exist relative neighborhoods \( U, V \) of \( F \) such that
\[
\left\| S - \left( \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_{ij}} \right) T_{xUV} \right\| < \varepsilon.
\]

**Proof.** Without loss of generality we can assume that \( \|a_{ij}\|_\infty \leq 1 \) for every \( i, j \). By Proposition 5.4 there is a relative neighborhood \( \hat{U} \) of \( F \) such that \( (6.1) \)
\[
\|S - ST_{xUV}\| = \|ST_{xUV}\| < \varepsilon.
\]
By Lemma 6.1 and (4.2), for \( 1 \leq i \leq m \) each of the operators
\[
S_k^i \overset{\text{def}}{=} \left( \prod_{j=k}^{n_i} T_{a_{ij}} \right) T_{xUV}, \quad 1 \leq k \leq n_i, \quad S_{n_i+1} = T_{xUV}
\]
satisfies \( B_0S_k^i = 0 \) on \( F \). Hence, a new use of Proposition 5.4 provides a relative neighborhood \( V \) of \( F \) such that
\[
\|T_{a_{ij}}S_k^i \| \leq \varepsilon
\]
for every \( 1 \leq i \leq m \) and \( 1 \leq k \leq n_i \).
Indeed, the proposition says that there are relative neighborhoods $V_k^i$ of $F$ that satisfy the inequality for each $i$ and $k$, but it also says that their intersection satisfies the inequality. Therefore

$$\|T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e S_k^i - T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e S_{k+1}^i\|
$$

$$= \|T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e T_{a_k}^i S_{k+1}^i - T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e S_k^i\|
$$

$$\leq \|T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e \| \|T_{a_k}^i - T_{a_k}^i \chi V e\| S_{k+1}^i
$$

$$\leq \|T_{a_k}^i \chi V e S_{k+1}^i\| < \varepsilon,$$

which leads to

$$\|T_{a_1}^i \cdots T_{a_n}^i \chi V e - T_{a_1}^i \chi V e \cdots T_{a_n}^i \chi V e T_{\chi V e}\|
$$

$$\leq \sum_{k=1}^{n_i} \|T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e S_k^i - T_{a_1}^i \chi V e \cdots T_{a_{k-1}}^i \chi V e S_{k+1}^i\| < n_i \varepsilon.$$ 

Therefore

$$\left\| \left( \sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j}^i \right) \chi V e - \left( \sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j}^i \chi V e \right) \chi V e \right\| \leq \sum_{i=1}^{m} n_i \varepsilon.$$ 

Since $ST_{\chi V e} = (\sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j}^i)T_{\chi V e}$ and $\varepsilon > 0$ is arbitrary, the lemma follows from (6.1) and the above inequality.

If $B \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $C_0(B)$ for the bilateral ideal of $\mathfrak{T}_0(B)$ generated by commutators $[T_a, T_b] = T_a T_b - T_b T_a$, with $a, b \in B$. Therefore, $C(B)$ is the closure of $C_0(L^\infty(\mathbb{D}))$.

**Lemma 6.3.** Let $B$ be a hyperbolic algebra. If $S \in C_0(L^\infty(\mathbb{D}))$ is such that $B_0S \in B$ and $\overline{B_0S}^B \equiv 0$ on $\Gamma_B$ then $S \in C(B)$.

**Proof.** By hypothesis

$$S = \sum_{i=1}^{m} T_{b_1^i} \cdots T_{b_{n_i}^i} [a_1^i, a_2^i] T_{c_1^i} \cdots T_{c_{k_i}^i},$$

where $n_i, k_i$ and $m$ are some positive integers and all the symbols are in $L^\infty(\mathbb{D})$. If $\overline{B_0S}^B \equiv 0$ on $\Gamma_B$, Lemma 6.2 says that given $\varepsilon > 0$ there are relative neighborhoods $U, V$ of $\Gamma_B$ such that if

$$R = \sum_{i=1}^{m} T_{b_1^i \chi V e} \cdots T_{b_{n_i}^i \chi V e} [a_1^i \chi V e, a_2^i \chi V e] T_{c_1^i \chi V e} \cdots T_{c_{k_i}^i \chi V e} T_{\chi V e}$$

then $\|S - R\| < \varepsilon$. By Lemma 6.1 every Toeplitz operator involved in the last expression is in $\mathfrak{T}(B)$. So, $R \in C(B)$ and then so is $S$. 

\[\blacksquare\]
It is well known that if $B$, $D$ are $C^*$-algebras and $\phi$ is a $*$-homomorphism from $B$ to $D$, then $\|\phi\| \leq 1$ and $\phi$ is an isometry if and only if $\phi$ is one-to-one [13, p. 100].

**Theorem 6.4** If $B$ is a hyperbolic algebra then

1. $\mathcal{C}(B) = \{S \in \mathcal{T}(B) : \hat{B}_0S^B \equiv 0 \text{ on } \Gamma_B\} = \{S \in \mathcal{T}(B) : \hat{S}^B_y = 0 \text{ for all } y \in \Gamma_B\}$.

2. $S - T_{B_0S} \in \mathcal{C}(B)$ for every $S \in \mathcal{T}(B)$.

3. The $C^*$-algebras $\mathcal{T}(B)/\mathcal{C}(B)$ and $C(\Gamma_B)$ are isomorphic via $\phi : S + \mathcal{C}(B) \mapsto \hat{B}_0S^B|_{\Gamma_B}$.

**Proof.** (1). The equality of the last two sets follows from Corollary 4.7. Suppose first that $S \in \mathcal{C}_0(B)$, so

$$S = \sum_{1 \leq i \leq n} A_i [T_{a_i}, T_{b_i}] B_i,$$

where $a_i, b_i \in B$ and $A_i, B_i \in \mathcal{T}_0(B)$. If $x \in \pi^{-1}(\Gamma_B)$ then $a_i \circ \varphi_x$ and $b_i \circ \varphi_x$ are constant functions for all $1 \leq i \leq n$. By (4.2) then

$$S_x = \sum_{1 \leq i \leq n} (A_i)_x [T_{a_i \circ \varphi_x}, T_{b_i \circ \varphi_x}] (B_i)_x = 0.$$

Since every $S \in \mathcal{C}(B)$ can be approximated by operators of this form, then $S_x = 0$ for every $x \in \pi^{-1}(\Gamma_B)$. By Corollary 4.7 then $B_0S \equiv 0$ on $\pi^{-1}(\Gamma_B)$, which is another way to say that $\hat{B}_0S^B \equiv 0$ on $\Gamma_B$. This proves the inclusion of the first set into the second one.

Suppose now that $S \in \mathcal{T}(B)$ and $\hat{B}_0S^B \equiv 0$ on $\Gamma_B$. We can assume that $\|S\| = 1$. Let $0 < \varepsilon < 1$ and take $Q \in \mathcal{T}_0(B)$ such that $\|Q - S\| < \varepsilon$. Since $Q \in \mathcal{T}(B)$ then $\hat{Q}^B_y = \lambda I$ and $\hat{B}_0Q^B(y) = \lambda$ for every $y \in \Gamma_B$, where $\lambda \in \mathbb{C}$ depends on $y$. Thus

$$(\hat{B}_0Q^B)_y = \lim_{z \to y} T_{(B_0Q)\circ \varphi_z} = T_{(B_0Q)^B(y)} = \lambda I.$$

Then

$$B_0(Q - T_{B_0Q})^B \equiv 0 \quad \text{on } \Gamma_B$$

by Corollary 4.7, and since $\hat{B}_0S^B \equiv 0$ on $\Gamma_B$ then

$$B_0(T_{B_0S})^B \equiv 0 \quad \text{on } \Gamma_B$$

by the same corollary.
So, if
\[ S_1 = Q - T_{B_0} + T_{B_0}S \]
then \( \hat{B}_0 S_1^B \equiv 0 \) on \( \Gamma_B \) and
\[
\|S_1 - S\| \leq \|Q - S\| + \|T_{B_0} - T_{B_0}Q\| \leq 2\|Q - S\| < 2 \varepsilon.
\]
In [20, Thm. 1.1] it is proved that
\[ \mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{T}(L^\infty(\mathbb{D})) \]
so it contains the identity \( I \).

Since Theorem 5.7 implies that
\[ \mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{C}(A) \]
then \( I \in \mathfrak{C}(A) \). Consequently there is \( R \in \mathfrak{C}_0(A) \) such that \( \|R - I\| < \varepsilon \). Thus
\[
\|RS_1 - S_1\| \leq \|R - I\| \|S_1\| < \varepsilon(\|S\| + 2 \varepsilon) < 3 \varepsilon.
\]
Since \( B_0 S_1 \equiv 0 \) on \( \pi^{-1}(\Gamma_B) \), Corollary 4.7 says that \( (S_1)_{x} = 0 \) for all \( x \in \pi^{-1}(\Gamma_B) \). By (4.2) then \( (RS_1)_{x} = R_{x}(S_1)_{x} = 0 \) for all \( x \in \pi^{-1}(\Gamma_B) \), which means that
\[ B_0(RS_1) \in \mathcal{B} \quad \text{and} \quad B_0(RS_1)^B \equiv 0 \quad \text{on } \Gamma_B. \]
But since \( R \in \mathfrak{C}_0(A) \) and \( S_1 \in \mathfrak{T}_0(A) \) then \( RS_1 \in \mathfrak{C}(A) \), which together with Lemma 6.3 gives \( RS_1 \in \mathfrak{C}(B) \). By (6.2) and (6.3), \( \|RS_1 - S\| < 5 \varepsilon \) and (1) follows.

(2). Let \( y \in \Gamma_B \). Since \( S \in \mathfrak{T}(\mathcal{B}) \) then \( \hat{S}_y^B = \lambda I \). Thus
\[
(\hat{B}_0 S)^B(y) = \lambda \quad \text{and} \quad (\hat{T}_{B_0}^B S)_y = T_{(B_0 S)^B(y)} = \lambda I.
\]
The result then follows from (1).

(3). By (1) the map \( \phi \) is well-defined and one-to-one. It is clear that \( \phi \) is *-linear. Suppose that \( S, T \in \mathfrak{T}(\mathcal{B}) \) and \( y \in \Gamma_B \). Then
\[ \hat{S}_y^B = \lambda_S I \quad \text{and} \quad \hat{T}_y^B = \lambda_T I \]
for some \( \lambda_S, \lambda_T \in \mathbb{C} \) that depend on \( y \). Hence
\[
\hat{B}_0(ST)^B(y) = \lim_{z \to y} \langle S_z T_z, 1, 1 \rangle = \langle \hat{S}_y^B \hat{T}_y^B, 1, 1 \rangle = \langle \lambda_S \lambda_T, 1, 1 \rangle = \lambda_S \lambda_T = \langle (B_0 S)^B(y), (B_0 T)^B(y) \rangle,
\]
and \( \phi \) is multiplicative. If \( f \in C(\Gamma_B) \) we can extend \( f \) to a continuous function \( F \) on \( M(B) \). Therefore \( F \in \mathcal{B} \) and
\[ \phi(T_F + \mathfrak{C}(\mathcal{B})) = \hat{B}_0 F^B|_{\Gamma_B} = f. \]
So, \( \phi \) is onto.

\[ \blacksquare \]
Theorem 6.5 Let \( B \) be a hyperbolic algebra and \( S \in \mathfrak{T}_0(L^\infty(\mathbb{D})) \). Then

1. \( S \in \mathfrak{T}(B) \) if and only if \( B_0S \in B \).
2. \( S \in \mathfrak{C}(B) \) if and only if \( \hat{B}_0S^B \equiv 0 \) on \( \Gamma_B \).

Proof. (1). We know the necessity from Theorem 4.9. Suppose that \( S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i} \), where all \( a_j^i \in L^\infty(\mathbb{D}) \), and \( B_0S \in B \). Then \( T_{B_0S} \in \mathfrak{T}(B) \) and

\[ B_0(S - T_{B_0S})^B \equiv 0 \quad \text{on} \quad \Gamma_B. \]

Consequently Lemma 6.2 tells us that given \( \varepsilon > 0 \) there are relative neighborhoods \( U, V \) of \( \Gamma_B \) such that

\[ \|S - T_{B_0S} - \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}\chi_U \cdot T_{(B_0S)^B}\chi_U\| < \varepsilon. \]

By Lemma 6.1,

\[ T_{a_j^i}\chi_U, T_{(B_0S)^B}\chi_U \in \mathfrak{T}(B) \]

for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n_i \). Therefore \( S \in \mathfrak{T}(B) \).

(2). The necessity follows from (1) of Theorem 6.4. For the sufficiency, observe that it is implicit in the condition \( \hat{B}_0S^B \equiv 0 \) on \( \Gamma_B \) that \( B_0S \in B \).

By the previous assertion then \( S \in \mathfrak{T}(B) \). So, (1) of Theorem 6.4 says that \( S \in \mathfrak{C}(B) \). \( \blacksquare \)

If \( B \) is a hyperbolic algebra and \( a \in A \), then \( a \in B \) if and only if \( B_0a \in B \). Therefore the theorem says that \( T_a \in \mathfrak{T}(B) \) if and only if \( a \in B \) and that \( T_a \in \mathfrak{C}(B) \) if and only if \( a \equiv 0 \) on \( \pi^{-1}(\Gamma_B) \).

The algebra \( C(\overline{\mathbb{D}}) \), of continuous functions on the closed disk is hyperbolic, its maximal ideal space identifies with \( \overline{\mathbb{D}} \), and it is immediate that \( \Gamma_C(\overline{\mathbb{D}}) = \partial\mathbb{D} \) via this identification. Since by Coburn’s theorem \( \mathfrak{C}(C(\overline{\mathbb{D}})) \) is the ideal of compact operators, then part (2) of the theorem says that \( S \in \mathfrak{T}_0(L^\infty(\mathbb{D})) \) is compact if and only if

\[ (B_0S)(z) \rightarrow 0 \quad \text{as} \quad |z| \rightarrow 1^- \]

That is, we recover the theorem of Axler and Zheng [4, Thm. 2.2]. It is clear that the above condition is equivalent to \( S_z = 0 \) for all \( x \in M(A) \setminus \mathbb{D} \), or what is the same, \( S_z \rightarrow 0 \) in the SOT-topology when \( |z| \rightarrow 1 \).
7. Applications

7.1. Continuous functions up to a boundary set

Suppose that $E \subset \partial \mathbb{D}$ is a closed set and consider the algebra $C_E$ formed by the functions of $\mathcal{A}$ that extend continuously to $E$. Then $C_E$ is a hyperbolic algebra. If $id \in \mathcal{A}$ denotes the function $id(z) = z$ and for $\lambda \in \partial \mathbb{D}$ we write $M_\lambda = \{ x \in M(\mathcal{A}) : id(x) = \lambda \}$ for the fiber of $\lambda$ over $M(\mathcal{A})$, then $M(\mathcal{C}_E)$ consists of $M(\mathcal{A}) / \sim$, where $\sim$ is the equivalence relation that collapses $M_\lambda$ to a single point (depending on $\lambda$) for each $\lambda \in E$. Thus, $\Gamma_{C_E}$ can be identified with $E$. Theorem 6.4 then says that $C(\mathcal{C}_E) = \{ S \in \mathfrak{T}(C_E) : \lim_{z \to E} (B_0 S)(z) = 0 \}$ and $\mathfrak{T}(C_E) / C(\mathcal{C}_E) \simeq C(E)$.

As mentioned before, when $E = \partial \mathbb{D}$, the above isomorphism is part of Coburn’s theorem. Now consider the algebra $CL^\infty_E$ formed by the functions in $L^\infty(\mathbb{D})$ that extend continuously to $E$. Since $CL^\infty_E \not\subset \mathcal{A}$, it is not a hyperbolic algebra. So, at a first sight it is not possible to apply our results to this algebra. Fortunately, Theorem 5.7 gives us a way to overcome this apparent difficulty. In fact, it is easy to prove that if $f \in CL^\infty_E$ then $B_k f \in C_E$ for every $k \geq 0$ and $(B_k f)(\lambda) = f(\lambda)$ for $\lambda \in E$. By Theorem 5.7 then $\mathfrak{T}(C_E) = \mathfrak{T}(CL^\infty_E)$ and $\mathfrak{C}(C_E) = \mathfrak{C}(CL^\infty_E)$.

7.2. The McDonald-Sundberg Theorem

Let $\mathcal{U}$ be the $C^*$-subalgebra of $L^\infty(\mathbb{D})$ generated by $H^\infty = \{ f \in L^\infty(\mathbb{D}) : f \text{ is analytic} \}$. The celebrated corona theorem of Carleson [10] states that $\mathbb{D}$ is dense in $M(H^\infty)$, the maximal ideal space of $H^\infty$. This translates into the alternative description of $\mathcal{U}$ as $C(M(H^\infty))$. Since Schwarz Lemma implies that $H^\infty \subset \mathcal{A}$ then $\mathcal{U} \subset \mathcal{A}$. Therefore $\mathcal{U}$ is a prehyperbolic algebra and we aim to prove that it is hyperbolic.

Clearly, every interpolating sequence for $H^\infty$ is interpolating for $\mathcal{U}$. The interpolating sequences for $H^\infty$ were characterized by Carleson in [9]. Suppose that $x \in M(H^\infty) \setminus \mathbb{D}$ is in the closure of some interpolating sequence $\{z_n\}$ for $H^\infty$, where we can assume that $z_n \neq 0$ for all $n \geq 1$. It is known that the infinite product

$$b(\omega) = \prod_{n \geq 1} \frac{|z_n|}{z_n} \varphi_{z_n}(\omega)$$

represents a function $b \in H^\infty$ such that $b(z_n) = 0$ for all $n \geq 1$. This $b$ is called an interpolating Blaschke product.
We also know (see [15, p. 404]) that if $\delta \in (0, 1)$ then there is $\varepsilon(\delta) > 0$ such that

$$|b(\omega)| \geq \varepsilon(\delta) \quad \text{for every} \quad \omega \in \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta).$$

Thus $x$ satisfies condition $(b_2)$ of Proposition 3.9. On the other hand, if $x \in M(H^\infty) \setminus \mathbb{D}$ is not in the closure of any interpolating sequence for $H^\infty$, it is known that for every net $(z_\alpha)$ in $\mathbb{D}$ that tends to $x$,

$$f \circ \varphi_{z_\alpha} \to \lambda \in \mathbb{C}$$

uniformly on compact sets for every $f \in H^\infty$ (see [15, Ch. X]). Since $\mathcal{U}$ is the $C^*$-algebra generated by $H^\infty$ the same holds for every $f \in \mathcal{U}$. Thus $x$ satisfies $(a_2)$ of Proposition 3.8. Consequently Corollary 3.10 tells us that $\mathcal{U}$ is hyperbolic and that $\Gamma_\mathcal{U}$ is formed by the points $x \in M(H^\infty)$ that are not in the closure of any interpolating sequence for $H^\infty$. Such points are usually called ‘trivial points’ because they can be characterized as the $x \in M(H^\infty)$ whose Gleason part (with respect to $H^\infty$) is just $\{x\}$. For the definition and further information on Gleason parts the reader may consult the original paper of Hoffman [16] or Garnett’s book [15, Ch. X].

Theorem 6.4 now tells us that $\mathcal{E}(\mathcal{U}) / \mathcal{E}(\mathcal{U}) \simeq C(\Gamma_\mathcal{U})$, a result obtained by McDonald and Sundberg in [17]. Theorem 6.4 also says that $\mathcal{E}(\mathcal{U}) = \{S \in \mathcal{E}(\mathcal{U}) : \overline{B_0S^\mathcal{U}} \equiv 0 \text{ on } \Gamma_\mathcal{U}\}$ and $S - T_{B_0S} \in \mathcal{E}(\mathcal{U})$, which are recent additions to the McDonald-Sundberg Theorem discovered by Axler and Zheng [5].

### 7.3. The algebra of nontangential limits

Consider the algebra $\mathcal{N} = \{f \in \mathcal{A} : f \text{ has nontangential limits a.e. on } \partial \mathbb{D}\}$. It is clear that $\mathcal{N}$ is prehyperbolic, and we are going to use Corollary 3.10 to show that it is hyperbolic. To do so we need to characterize the interpolating sequences for $\mathcal{N}$. For $u \in \partial \mathbb{D}$ and $0 < \alpha < \pi/2$ let $\Lambda_\alpha(u) = \{u - \omega : |\arg \omega - \arg u| < \alpha, \text{ and } 0 < |u - \omega| < 1\}$ be an angular region with vertex $u$ of total opening $2\alpha$. If $V \subset \mathbb{D}$ set

$$\text{NT}_\alpha(V) = \{u \in \partial \mathbb{D} : u \in \overline{V \cap \Lambda_\alpha(u)}\} \quad \text{and} \quad \text{NT}(V) = \bigcup_{0 < \alpha < \pi/2} \text{NT}_\alpha(V).$$

Geometrically, $\text{NT}(V)$ is the subset of $\partial \mathbb{D}$ that can be approached nontangentially by points of $V$. If $u \in \partial \mathbb{D}$, $0 < r < 1$ and $0 < \alpha < \pi/2$, there is some $0 < \beta < \pi/2$ depending on $\alpha$ and $r$ such that the $r$-pseudohyperbolic neighborhood of $\Lambda_\alpha(u)$ is contained in $\Lambda_\beta(u)$. Thus

$$(7.1) \quad \text{NT}(V) = \text{NT}(\{z \in \mathbb{D} : \rho(z, V) \leq r\}).$$

We write $|E|$ for the one-dimensional Lebesgue measure of $E \subset \partial \mathbb{D}$. 

Lemma 7.1 A separated sequence \( S = \{z_n\} \) is interpolating for \( N \) if and only if \( |NT(S)| = 0 \). If that is the case, for any \( r > 0 \) sufficiently small there exists \( f \in N \) that separates \( S \) from \( \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, r) \).

Proof. Suppose that \( |NT(S)| = 0 \) and \( \rho(z_n, z_m) \geq \delta > 0 \) for \( n \neq m \). By (7.1) then \( |NT(\bigcup_{n \geq 1} K(z_n, \delta/4)| = 0 \). Take \( f \in A \) such that

\[
f(z_n) = 1 \quad \text{for all } n \quad \text{and } f \equiv 0 \quad \text{on } \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta/4).
\]

So, \( f \) has null nontangential limit a.e. on \( \partial \mathbb{D} \). Thus \( f \in N \) and separates \( S \) from \( \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta/4) \). If \( \{\eta_n\} \) is an arbitrary sequence and we take \( g \in A \) such that \( g(z_n) = \eta_n \) for every \( n \) then \( fg \in N \) and \( f(z_n)g(z_n) = \eta_n \) for every \( n \). So, \( S \) is interpolating for \( N \).

Now suppose that \( |NT(S)| > 0 \). If \( 0 < \alpha_k < \alpha_{k+1} \to \pi/2 \) is a strictly increasing sequence, then \( NT(S) = \bigcup_k NT_{\alpha_k}(S) \). So, there is some \( \alpha_k = \alpha \) such that \( |NT_{\alpha}(S)| > 0 \), and consequently there exists a compact set \( E \subset NT_{\alpha}(S) \) of positive measure. That is, \( u \in \Lambda_{\alpha}(u) \cap S \) for every \( u \in E \).

So, if \( u \in E \) there is some \( z_n \in \Lambda_{\alpha}(u) \cap S \). Since \( \Lambda_{\alpha}(u) \) is open, it is geometrically clear that there is a an open neighborhood \( I_n \) of \( u \) in \( \partial \mathbb{D} \) such that \( z_n \in \Lambda_{\alpha}(u) \cap S \) for every \( v \in I_n \). By the compactness of \( E \) there is a finite set \( \mathcal{R}_1 \) in \( S \) such that \( \Lambda_{\alpha}(u) \cap \mathcal{R}_1 \neq \emptyset \) for every \( u \in E \). If \( r_1 = \max\{|z| : z \in \mathcal{R}_1\} \) and \( S_1 = \{z \in S : |z| \leq r_1\} \) then we also have \( \Lambda_{\alpha}(u) \cap S_1 \neq \emptyset \) for every \( u \in E \). We can repeat this process with \( S \setminus S_1 \) instead of \( S \) to obtain \( r_2 \in (r_1, 1) \) such that \( S_2 = \{z \in S : r_1 < |z| \leq r_2\} \) then \( \Lambda_{\alpha}(u) \cap S_2 \neq \emptyset \) for every \( u \in E \). We keep going to construct a sequence \( 0 < r_1 < \cdots < r_n < \cdots < 1 \) such that \( S_n = \{z \in S : r_n-1 < |z| \leq r_n\} \) then

\[
\Lambda_{\alpha}(u) \cap S_n \neq \emptyset \quad \text{for every } u \in E.
\]

The sequence \( \{r_n\} \) must tend to 1 because if \( r_n \leq r < 1 \) for every \( n \) then \( \{z : |z| \leq r\} \cap S \) is infinite, which is not possible because \( S \) is separated.

Now take

\[
\mathcal{T}_1 = \bigcup_{j \text{ odd}} S_j \quad \text{and} \quad \mathcal{T}_2 = \bigcup_{j \text{ even}} S_j.
\]

Since (7.2) holds for all \( n \geq 1 \) then \( E \subset NT_{\alpha}(\mathcal{T}_1) \cap NT_{\alpha}(\mathcal{T}_2) \), and since \( |E| > 0 \), the interpolation problem

\[
f(z_n) = \begin{cases} 1 & \text{for } z_n \in \mathcal{T}_1 \\ 0 & \text{for } z_n \in \mathcal{T}_2 \end{cases}
\]

cannot be solved by a function with nontangential limits almost everywhere on \( E \).
Theorem 7.2 The algebra \( \mathcal{N} \) is hyperbolic. In addition, \( y \in M(\mathcal{N}) \) is in \( G_{\mathcal{N}} \) if and only if \( y \) is in the closure of some interpolating sequence for \( \mathcal{N} \).

Proof. Let \( y \in M(\mathcal{N}) \). If \( y \) is in the closure of an interpolating sequence for \( \mathcal{N} \) the previous lemma says that \( y \) satisfies condition \((b_2)\) of Proposition 3.8, and consequently \( y \in \Gamma_{\mathcal{N}} \). By Corollary 3.10 then \( \mathcal{N} \) is hyperbolic.

The nontangential limit function of \( f \in \mathcal{N} \) will be denoted \( \tilde{f} \). So, \( \tilde{f} \in L^\infty(\partial\mathbb{D}) \).

Lemma 7.3 Let \( f \in \mathcal{N} \). Then \( \tilde{f}^\mathcal{N} \equiv 0 \) on \( \Gamma_{\mathcal{N}} \) if and only if \( \tilde{f} = 0 \).

Proof. If there is \( y \in \Gamma_{\mathcal{N}} \) such that \( |\tilde{f}^\mathcal{N}(y)| = \delta > 0 \) and \( \mathcal{S} \) is a separated sequence such that \( y \in \mathcal{S}^{M(\mathcal{N})} \), then \( y \) is in the \( M(\mathcal{N}) \)-closure of

\[
\mathcal{S}_1 = \{z \in \mathcal{S} : |f(z)| > \delta/2\}.
\]

Since \( y \in \Gamma_{\mathcal{N}} \) then Theorem 7.2 and Lemma 7.1 imply that \( |\text{NT}(\mathcal{S}_1)| > 0 \), and since \( |\tilde{f}| \geq \delta/2 \) for almost every point of \( \text{NT}(\mathcal{S}_1) \), the sufficiency holds.

Now suppose that \( \tilde{f} \neq 0 \), so there is some \( \delta > 0 \) such that \( |\tilde{f}| > \delta \) on a set of positive measure. It is easy then to construct a separated sequence \( \mathcal{S} \) such that \( |\text{NT}(\mathcal{S})| > 0 \) and \( |f(z)| > \delta/2 \) for every \( z \in \mathcal{S} \). The necessity will follow if we show that \( \mathcal{S}^{M(\mathcal{N})} \cap \Gamma_{\mathcal{N}} \neq \emptyset \), because for any \( y \) in the intersection we would have \( |\tilde{f}^\mathcal{N}(y)| \geq \delta/2 \).

Since \( \mathcal{N} \) is hyperbolic, if \( \mathcal{S}^{M(\mathcal{N})} \cap \Gamma_{\mathcal{N}} = \emptyset \) then \( \mathcal{S}^{M(\mathcal{N})} \subset G_{\mathcal{N}} \). So, Proposition 3.9 says that for every \( y \in \mathcal{S}^{M(\mathcal{N})} \setminus \mathcal{S} \) there is an interpolating sequence \( T_y \) for \( \mathcal{N} \), such that \( y \in T_y^{M(\mathcal{N})} \). Hence, for every \( 0 < r < 1 \) the \( M(\mathcal{N}) \)-closure of \( \bigcup_{z \in T_y} K(z, r) \) is a neighborhood of \( y \) (by Lemma 7.1). By the
compactness of $\mathfrak{F}^{M(N)} \setminus \mathcal{S}$ there are finitely many interpolating sequences $T_1, \ldots, T_N$ for $\mathcal{N}$ such that the closure of

$$U \overset{\text{def}}{=} \bigcup_{1 \leq j \leq N} \bigcup_{z \in T_j} K(z, r)$$

is a neighborhood of $\mathfrak{F}^{M(N)} \setminus \mathcal{S}$. Thus there is $0 < \rho < 1$ so that $\mathcal{S} \cap \{z \in \mathbb{D} : |z| \geq \rho\}$ is contained in $U$. Together with (7.1) this yields

$$\text{NT}(\mathcal{S}) \subset \bigcup_{1 \leq j \leq N} \text{NT}(\bigcup_{z \in T_j} K(z, r)) = \bigcup_{1 \leq j \leq N} \text{NT}(T_j),$$

which is impossible because $|\text{NT}(\mathcal{S})| > 0$ while $|\text{NT}(T_j)| = 0$ for $j = 1, \ldots, N$.

\textbf{Lemma 7.4} If $S \in \mathfrak{T}(\mathcal{N})$ then for almost every $u \in \partial \mathbb{D}$ there is $\lambda(u) \in \mathbb{C}$ such that $S_z \overset{\text{SOT}}{\longrightarrow} \lambda(u)I$ when $z \overset{\text{nt}}{\rightarrow} u$.

\textbf{Proof.} Let $a \in \mathcal{N}$ and suppose that $u \in \partial \mathbb{D}$ is such that $a(z) \rightarrow \lambda \in \mathbb{C}$ when $z \overset{\text{nt}}{\rightarrow} u$. If $0 < \alpha < \pi/2$ and $0 < r < 1$ there is $\beta = \beta(\alpha, r)$ in $(\alpha, \pi/2)$ such that $\varphi_z(\omega) \in \Lambda_\beta(u)$ when $z \in \Lambda_\alpha(u)$ and $|\omega| \leq r$. Therefore $a \circ \varphi_z \rightarrow \lambda$ uniformly on $r \mathbb{D}$ when $z \rightarrow u$ inside $\Lambda_\alpha(u)$. Since $r$ is arbitrary the convergence is uniform on compact sets, implying that $(T_u)_z = T_{u \circ \varphi_z} \rightarrow \lambda I$ in the SOT-topology when $z \rightarrow u$ inside $\Lambda_\alpha(u)$. Since $\alpha$ is arbitrary and the product of operators is continuous with respect to the SOT-topology, the lemma holds for every $S \in \mathfrak{T}_0(\mathcal{N})$. If $S \in \mathfrak{T}(\mathcal{N})$ take a sequence $\{S_n\}$ in $\mathfrak{T}_0(\mathcal{N})$ that converges to $S$. So, for every $n \geq 1$ there is a set $E_n \subset \partial \mathbb{D}$ of full measure such that

$$(S_n)_z \overset{\text{SOT}}{\longrightarrow} \lambda_n(u)I \quad \text{when} \quad z \overset{\text{nt}}{\rightarrow} u \in E_n.$$ 

Therefore the set $E = \cap E_n$ has full measure, and given $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that if $u \in E$,

$$|\lambda_n(u) - \lambda_m(u)| \leq \lim_{z \overset{\text{nt}}{\rightarrow} u} \|\lambda_n(u)I - \lambda_m(u)I\| = \|\lambda_n(u) - \lambda_m(u)\| < \varepsilon$$

for all $n, m \geq n_0$. This implies that there is some $\lambda(u) \in \mathbb{C}$ such that $\lambda_n(u) \rightarrow \lambda(u)$ for every $u \in E$. If $f \in L^2_a$ has norm 1, $u \in E$ and $n \geq n_0$, (7.3) yields

$$\|S_z f - \lambda(u)f\| \leq \|S_z f - (S_n)_z f\| + \|(S_n)_z f - \lambda_n(u)f\| + |\lambda_n(u) - \lambda(u)| \|f\| \leq \|S - S_n\| + |\lambda_n(u) - \lambda(u)| + \|(S_n)_z f - \lambda_n(u)f\| \leq 2\varepsilon + \|(S_n)_z f - \lambda_n(u)f\| \rightarrow 2\varepsilon$$

when $z \overset{\text{nt}}{\rightarrow} u$. Thus $S_z f \rightarrow \lambda(u)f$ in $L^2_a$ when $z \overset{\text{nt}}{\rightarrow} u \in E$ and the lemma holds for $S$. 

\hfill \blacksquare
Theorem 7.5 \( \mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \cong L^\infty(\partial \mathbb{D}) \) and

\[
(7.4) \quad \mathfrak{C}(\mathcal{N}) = \{ S \in \mathfrak{T}(\mathcal{N}) : \widetilde{B}_0 S = 0 \}
\]

\[
(7.5) \quad = \{ S \in \mathfrak{T}(\mathcal{N}) : S_z \overset{SOT}{\rightarrow} 0, \text{ when } z \overset{nt}{\rightarrow} u \text{ for a.e. } u \in \partial \mathbb{D} \}.
\]

**Proof.** Equality (7.4) follows immediately from Theorem 6.4 and Lemma 7.3.

By Lemma 7.4, for every \( S \in \mathfrak{T}(\mathcal{N}) \) there is a set \( E_S \subset \partial \mathbb{D} \) of full measure and \( \lambda_S : E_S \rightarrow \mathbb{C} \) such that

\[
(7.6) \quad S_z \overset{SOT}{\rightarrow} \lambda_S(u)I \text{ when } z \overset{nt}{\rightarrow} u \in E_S.
\]

Then \( (B_0 S)(z) = (B_0 S_z)(0) = \langle S_z 1, 1 \rangle \rightarrow \lambda_S(u) \) when \( z \overset{nt}{\rightarrow} u \in E_S \), which means that \( (B_0 S)(u) = \lambda_S(u) \) for every \( u \in E_S \). This proves (7.5).

Let \( \Phi : \mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \rightarrow L^\infty(\partial \mathbb{D}) \) given by \( \Phi(S + \mathfrak{C}(\mathcal{N})) = \widetilde{B}_0 S \). By (7.4) \( \Phi \) is well-defined and one-to-one. It is also clear that \( \Phi \) is *-linear. To prove that \( \Phi \) is multiplicative let \( S, T \in \mathfrak{T}(\mathcal{N}) \) and use (7.6) to obtain

\[
\widetilde{B}_0(ST)(u) = \lim_{z \overset{nt}{\rightarrow} u} \langle S_z T_z 1, 1 \rangle = \lambda_S(u)\lambda_T(u) = \langle B_0 S(u), B_0 T(u) \rangle
\]

for every \( u \in E_S \cap E_T \). Hence \( \phi \) is a *-homomorphism and we only need to show that it is onto. Let \( a \in L^\infty(\partial \mathbb{D}) \) and consider the Poisson integral

\[
A(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-it}|^2} a(e^{it}) \, dt.
\]

So, \( A \) is a bounded harmonic function such that \( \hat{A} = a \). Since \( A \) is uniformly continuous with respect to \( \rho \) then \( A \in \mathcal{N} \). So, \( T_A \in \mathfrak{T}(\mathcal{N}) \) and

\[
\Phi(T_A + \mathfrak{C}(\mathcal{N})) = \widetilde{B}_0 T_A = B_0 A = \hat{A} = a.
\]

Let \( \mathcal{U} \) be the algebra of the McDonald-Sundberg Theorem. Since every \( f \in H^\infty \) has nontangential limits a.e. then \( \mathcal{U} \subset \mathcal{N} \subset \mathcal{A} \). Therefore

\[
\mathfrak{C}(\mathcal{U}) \subset \mathfrak{C}(\mathcal{N}) \subset \mathfrak{C}(\mathcal{A}).
\]

We shall show that both inclusions are proper. The function

\[
a = \sin \left( \log \frac{1 + |z|}{1 - |z|} \right)
\]

is in \( \mathcal{A} \) but has no nontangential limit at any point of \( \partial \mathbb{D} \) [8]. Hence,

\[
T_a \in \mathfrak{C}(\mathcal{A}) \setminus \mathfrak{T}(\mathcal{N}).
\]
The Shilov boundary of $H^\infty$, denoted $\partial H^\infty$, is the smallest closed set $F \subset M(H^\infty)$ such that
\[ \|f\|_\infty = \sup_{x \in F} |\hat{f}(x)| \quad \text{for every } f \in H^\infty. \]

It is known that $\partial H^\infty$ is properly contained in $\Gamma_U$ [15, p. 438], and that a function $f \in U$ satisfies $\hat{f}_U \equiv 0$ on $\partial H^\infty$ if and only if its nontangential function vanishes a.e. on $\partial D$ (see [3, Thm. 7] and [7, Coro. 1.3]). So, take $y \in \Gamma_U \setminus \partial H^\infty$ and $f \in U$ such that $\hat{f}_U \equiv 0$ on $\partial H^\infty$ and $\hat{f}_U(y) = 1$. Since $f(z)$ has trivial nontangential limits almost everywhere then $T_f \in \mathcal{C}(\mathcal{N})$ but since $\hat{f}_U \not\equiv 0$ on $\Gamma_U$ then $T_f \not\in \mathcal{C}(U)$.

Let $\mathcal{N}L^\infty$ be the algebra of functions in $L^\infty(D)$ that have nontangential limits a.e. on $\partial D$. From the paragraph preceding (7.1) it easily follows that if $f \in \mathcal{N}L^\infty$ then $B_k f$ has the same nontangential limits as $f$ a.e. on $\partial D$ for every $k \geq 0$. Thus Theorem 5.7 tells us that
\[ \mathfrak{H}(\mathcal{N}) = \mathfrak{H}(\mathcal{N}L^\infty) \quad \text{and} \quad \mathfrak{C}(\mathcal{N}) = \mathfrak{C}(\mathcal{N}L^\infty). \]

Moreover, let $E \subset D$ be a set of positive measure. Then all of the above can be generalized (with similar proofs) for the algebras
\[ \mathcal{N}L^\infty_E = \{ f \in L^\infty(D) : f \text{ has nontangential limits a.e. on } E \} \]
and
\[ \mathcal{N}_E = \mathcal{N}L^\infty_E \cap \mathcal{A}. \]
Hence, we obtain a version of Theorem 7.5, where $\mathcal{N}$ is replaced by $\mathcal{N}_E$ or $\mathcal{N}L^\infty_E$ and $\partial D$ is replaced by $E$.

### 7.4. Constant on hyperbolic parts

**Definition.** If $F \subset M(\mathcal{A}) \setminus D$ is a closed saturated set, define
\[ \text{CO}(F) = \{ f \in \mathcal{A} : f|_F = \text{const.} \}. \]
and
\[ \text{COH}(F) = \{ f \in \mathcal{A} : f|_{H(x)} = \text{const. for every } x \in F \}. \]
These notations stand for ‘constant on $F$’ and ‘constant on hyperbolic parts of $F$’, respectively. It is clear that $\text{CO}(F)$ and $\text{COH}(F)$ are hyperbolic algebras and that
\[ F = \pi_1^{-1}(\Gamma_{\text{CO}(F)}) = \pi_2^{-1}(\Gamma_{\text{COH}(F)}), \]
where $\pi_1$ and $\pi_2$ are the projections from $M(\mathcal{A})$ onto the respective maximal ideal spaces.
If \( B \) is a hyperbolic algebra and \( \pi : M(\mathcal{A}) \rightarrow M(\mathcal{B}) \) is the usual projection then

\[
\{ S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_B)} = 0 \} \subset \mathfrak{C}(\mathcal{B}) \subset \{ S \in \mathfrak{T}(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_B)} = 0 \},
\]

where the first inclusion follows from Theorem 6.5 and the second from Theorem 6.4. Observe that since the first set contains \( \mathfrak{C}_0(\mathcal{B}) \), it is dense in \( \mathfrak{C}(\mathcal{B}) \). The significance of \( \text{CO}(F) \) and \( \text{COH}(F) \) is given by the following

**Proposition 7.6** Let \( \mathcal{B} \) be a hyperbolic algebra and \( F \subset M(\mathcal{A}) \) be a closed saturated set. Then the following conditions are equivalent

1. \( F = \pi^{-1}(\Gamma_B) \),
2. \( \mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\text{COH}(F)) \),
3. \( \text{CO}(F) \subset \mathcal{B} \subset \text{COH}(F) \).

**Proof.** We prove first the equivalence between (1) and (2). If (1) holds then the comment following (7.7) says that \( \{ S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_{F} = 0 \} \) is dense in both \( \mathfrak{C}(\mathcal{B}) \) and \( \mathfrak{C}(\text{COH}(F)) \), so they must coincide. If (2) holds, (7.7) implies that

\[
\{ S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_B)} = 0 \} \subset \{ S \in \mathfrak{T}(\mathcal{A}) : B_0 S|_{F} = 0 \}.
\]

Therefore \( F \subset \pi^{-1}(\Gamma_B) \), and a symmetrical argument gives the other inclusion, so (1) holds.

If (1) holds the functions of \( \text{CO}(F) \) are continuous on \( M(\mathcal{B}) \) and the functions of \( \mathcal{B} \) are continuous on \( M(\text{COH}(F)) \). Since these are all \( C^* \)-algebras, (3) holds. If (3) holds then

\[
\mathfrak{C}(\text{CO}(F)) \subset \mathfrak{C}(\mathcal{B}) \subset \mathfrak{C}(\text{COH}(F)),
\]

so the proof of (2) reduces to show that \( \mathfrak{C}(\text{CO}(F)) = \mathfrak{C}(\text{COH}(F)) \). But this equality is a special case of the equivalence between (1) and (2). \( \blacksquare \)

Let us write \( \text{COH} \) for \( \text{COH}(M(\mathcal{A}) \setminus \mathcal{D}) \). In this case the last proposition says that \( \mathfrak{C}(\text{COH}) = \mathfrak{C}(\mathcal{C}(\mathcal{D})) \), and this is the ideal of compact operators \( \mathcal{K} \). Then Theorem 6.4 tells us that \( S - T_{B_0} S \in \mathcal{K} \) for every \( S \in \mathfrak{T}(\text{COH}) \). In particular,

\[
\mathfrak{T}(\text{COH})/\mathcal{K} = \{ T_b + \mathcal{K} : b \in \text{COH} \}.
\]

The center of an algebra \( \mathcal{B} \) is formed by the elements that commute with all the members of \( \mathcal{B} \). Our next result relates \( \mathfrak{T}(\text{COH})/\mathcal{K} \) with the center of \( \mathfrak{T}(L^{\infty}(\mathcal{D}))/\mathcal{K} \).
Suppose that \( S \in \mathcal{K} \) and for \( z \in \mathbb{D} \) let \( k_z^0 = (1 - |z|^2)K_z^{(0)} \). Since \( \|k_z^0\| = 1 \) and \( k_z^0 \to 0 \) weakly as \( |z| \to 1 \), then

\[
|(B_0S)(z)| \leq \|Sk_z^0\| \to 0 \quad \text{when } |z| \to 1.
\]

Therefore \( S_x = 0 \) for every \( x \in M(A) \setminus \mathbb{D} \).

**Theorem 7.7** Let \( \mathcal{I} = \{S \in \mathfrak{T}(L^\infty(\mathbb{D})): S_x = 0 \text{ for } x \in M(A) \setminus \mathbb{D}\} \). Then

\[
\{T_b + \mathcal{K} : b \in \text{COH}\} \subset \text{Center}(\mathfrak{T}(L^\infty(\mathbb{D})))/\mathcal{K}) \subset \{T_b + \mathcal{I} : b \in \text{COH}\}
\]

**Proof.** We prove first that if \( S \in \mathfrak{T}(L^\infty(\mathbb{D})) \) and \( b \in \text{COH} \) then \( [S, T_b] \in \mathcal{K} \). Let \( S_n \in \mathfrak{T}_0(A) \) such that \( S_n \to S \). Since \((S_nT_b - T_bS_n) \to (ST_b - T_bS)\) we can assume that \( S \in \mathfrak{T}_0(A) \). By (4.2),

\[
(ST_b - T_bS)_x = S_x(T_b)_x - (T_b)_xS_x \quad \text{for every } x \in M(A),
\]

and since \((T_b)_x\) is a constant operator for every \( x \in M(A) \setminus \mathbb{D} \), then

\[
[S, T_b]_x = 0 \quad \text{for } x \in M(A) \setminus \mathbb{D}.
\]

The comment after Theorem 6.5 then says that \([S, T_b] \) is compact. This proves that \( \{T_b + \mathcal{K} : b \in \text{COH}\} \) is contained in the center of \( \mathfrak{T}(L^\infty(\mathbb{D})))/\mathcal{K} \).

Now suppose that \( S \in \mathfrak{T}(L^\infty(\mathbb{D})) \) is such that

\[
S + \mathcal{K} \subset \text{Center}(\mathfrak{T}(L^\infty(\mathbb{D})))/\mathcal{K}.
\]

This means that \( ST_a - T_aS \in \mathcal{K} \) for every \( a \in L^\infty(\mathbb{D}) \). So,

\[
S_x(T_a)_x - (T_a)_xS_x = 0 \quad \text{for every } x \in M(A) \setminus \mathbb{D},
\]

or equivalently,

\[
S_x(T_a)_x - (T_a)_xS_x \overset{\text{SOT}}{\to} 0 \quad \text{as } |z| \to 1.
\]

Let \( x \in M(A) \setminus \mathbb{D} \) and take a net \((z_n)\) in \( \mathbb{D} \) converging to \( x \). The closed ball of center 0 and radius \( \|S\| \) in \( \mathfrak{L}(L_a^2) \) admits a metric \( d \) with the SOT-topology. Since \( S_{z_n} \overset{\text{SOT}}{\to} S_x \) then for every integer \( n \geq 1 \) there is some point of the net, that we rename as \( z_n \), such that \( d(S_{z_n}, S_x) < 1/n \). So,

\[
S_{z_n} \overset{\text{SOT}}{\to} S_x.
\]

If \( \{r_n\} \) is a sequence in \((0, 1)\) that tends to 1, we can assume (taking a subsequence of \( \{z_n\} \) if needed) that \( K(z_n, r_n) \cap K(z_j, r_j) = \emptyset \) if \( n \neq j \). For an arbitrary \( a \in L^\infty(\mathbb{D}) \) consider the function

\[
b(\omega) = \sum_{j \geq 1} (a \circ \varphi_{z_j})(\omega) \chi_{K(z_j, r_j)}(\omega).
\]
Hence \((T_b)_{\phi z_n} = T_{b \circ \phi z_n}\), where
\[
(b \circ \phi z_n)(\omega) = a(\omega)\chi_{K(0,r_n)}(\omega) + \sum_{j : j \neq n} (a \circ \phi z_j)(\phi z_n(\omega))\chi_{K(\phi z_n(z_j),r_j)}(\omega)
\]
\[
= g_n(\omega) + h_n(\omega).
\]
Since the support of \(h_n\) is disjoint from \(K(0,r_n) = r_n\mathbb{D}\) then
\[
|h_n(\omega)| \leq \|a\|_{\infty} \chi_{D \setminus r_nD}(\omega) \quad \text{for all } \omega \in \mathbb{D}.
\]
Since \(r_n \to 1\), it is clear that \(T_{h_n} \xrightarrow{\text{SOT}} 0\) and \(T_{g_n} \xrightarrow{\text{SOT}} T_a\). Thus
\[
(7.10) \quad (T_b)_{\phi z_n} = T_{g_n} + T_{h_n} \xrightarrow{\text{SOT}} T_a.
\]
By (7.8)
\[
S_{\phi z_n}(T_a)_{\phi z_n} - (T_a)_{\phi z_n} S_{\phi z_n} \xrightarrow{\text{SOT}} 0,
\]
which together with (7.9) and (7.10) gives \(S_x T_a - T_a S_x = 0\). This means that \(S_x\) commutes with every Toeplitz operator with symbol in \(L^\infty(\mathbb{D})\).

The concept of center plays an important role when studying localizations of \(C^*\)-algebras (see [13, Th. 7.47]). I believe that the ideal \(\mathcal{I}\) in Theorem 7.7 is \(\mathcal{K}\), so the inclusions of the theorem should be equalities. If \(S \in \mathcal{L}(L^2_a)\), the essential spectrum \(\sigma_e(S)\) is the spectrum of \(S + \mathcal{K}\) in the Calkin algebra \(\mathcal{L}(L^2_a)/\mathcal{K}\). Let \(\sigma(S)\) denote the usual spectrum of \(S\). Is it true that
\[
\sigma_e(S) = \bigcup_{x \in M(A) \setminus \mathbb{D}} \sigma(S_x) \quad \text{for every } S \in \mathcal{L}(L^\infty(\mathbb{D}))?
\]
There is strong evidence to support an affirmative answer. This holds for \(S \in \mathcal{L}(\text{COH})\), while the example preceding Lemma 4.8 shows that this fails for a general \(S \in \mathcal{L}(L^2_a)\). This example appeared in [4], where it is also shown that there is an infinite dimensional orthogonal projection \(P\) such that \(B_0 P(z) \to 0\) when \(|z| \to 1\). We do not know the answer even for a general Toeplitz operator with bounded symbol.

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References


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