Construction of a Continuous $SL(3, \mathbb{R})$ Action on 4-Sphere

Dedicated to Professor Nobuo Shimada on his 60th birthday

Fuichi UCHIDA*

§ 0. Introduction

Let $\Phi_0 : SO(3) \times M_3(\mathbb{R}) \to M_3(\mathbb{R})$ denote the $SO(3)$ action on the vector space $M_3(\mathbb{R})$ of all real matrices of degree 3, defined by $\Phi_0(A, X) = AXA^{-1}$ for $A \in SO(3)$ and $X \in M_3(\mathbb{R})$. Put $(X, Y) = \text{trace } XY$ for $X, Y \in M_3(\mathbb{R})$. Then $(X, Y)$ is an $SO(3)$ invariant inner product of $M_3(\mathbb{R})$. Denote by $V$ and $S(V)$ the linear subspace of $M_3(\mathbb{R})$ consisting of symmetric matrices of trace 0 and its unit sphere, respectively. Then $V$ and $S(V)$ are $SO(3)$ invariant.

Let $\Phi : SO(3) \times S(V) \to S(V)$ denote the restricted action of $\Phi_0$. This is an orthogonal $SO(3)$ action on the 4-sphere $S(V)$. In this note, we shall show that the $SO(3)$ action $\Phi$ on $S(V)$ is extended to a continuous $SL(3, \mathbb{R})$ action $\Psi$ on $S(V)$, but the action $\Psi$ is not $C^1$-differentiable. It is still open whether the $SO(3)$ action $\Phi$ can be extended to a $C^1$-differentiable $SL(3, \mathbb{R})$ action or not.

The problem is motivated by the following (cf. [1]). We studied real analytic $SL(n, \mathbb{R})$ actions on spheres, and it was important to consider the restricted $SO(n)$ actions. Moreover, we gave an orthogonal $SO(4)$ action on 6-sphere which was not extendable to any continuous $SL(4, \mathbb{R})$ action.

§ 1. An Action of $GL(2, \mathbb{R})$ on 2-Disk

1.1. Denote by $D$ the set of complex numbers with modulus $\leq 1$. We regard $D$ as a closed unit 2-disk. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $GL(2, \mathbb{R})$, and put

$$\alpha = (a + d + (b - c)i)/2, \quad \beta = (a - d - (b + c)i)/2.$$
Then \( \det A = |\alpha|^2 - |\beta|^2 \) and

\[
TAT^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{for} \quad T = \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}.
\]

Define a map \( \psi_1 : GL(2, \mathbb{R}) \times D \to D \) by

\[
\psi_1(A, w) = \begin{cases} 
\frac{(\alpha w + \beta)}{(\bar{\beta} w + \bar{\alpha})} & \text{if } \det A > 0 \\
\frac{(\beta \bar{w} + \alpha)}{(\bar{\alpha} \bar{w} + \beta)} & \text{if } \det A < 0.
\end{cases}
\]

The map \( \psi_1 \) is well-defined, because

\[
|\bar{\beta} w + \bar{\alpha}| \geq |\bar{\alpha}| - |\bar{\beta} w| \geq |\alpha| - |\beta| > 0 \quad \text{for } |w| \leq 1, \det A > 0,
\]

\[
|\bar{\alpha} \bar{w} + \beta| \geq |\beta| - |\bar{\alpha} \bar{w}| \geq |\beta| - |\alpha| > 0 \quad \text{for } |w| \leq 1, \det A < 0
\]

and

\[
|\alpha + \beta \bar{w}|^2 - |\alpha w + \beta|^2 = (|\alpha|^2 - |\beta|^2)(1 - |w|^2)
\]

for any complex numbers \( \alpha, \beta, w \). Moreover, we see that the map \( \psi_1 \) is a continuous action of \( GL(2, \mathbb{R}) \) on \( D \) and \( \psi_1(A, 1) = 1 \) if and only if \( A \) is of the form \( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \), by a routine work.

Here we describe a distinct property of the action \( \psi_1 \). Define \( M_1(x - iy) = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \) for real numbers \( x, y \). Then

\[
M_1(\psi_1(A, w)) = AM_1(w)A^{-1} \quad \text{for } w \in D, A \in O(2).
\]

Finally we notice the following fact. Consider a correspondence \( w \to z \) of complex numbers defined by

\[
z = i(1 + w)/(1 - w), \quad w = (z - i)/(z + i).
\]

The correspondence induces a homeomorphism of the interior \( \mathring{D} \) onto the upper half plane \( \mathcal{H} \), and the action \( \psi_1 \) corresponds to an action \( \psi_2 \) of \( GL(2, \mathbb{R}) \) on \( \mathcal{H} \). We see that the action \( \psi_2 \) is well-known, in fact,

\[
\psi_2(A, z) = \begin{cases} 
(az + b)/(cz + d) & \text{if } \det A > 0, \\
(a\bar{z} + b)/(c\bar{z} + d) & \text{if } \det A < 0,
\end{cases}
\]

where \( z \in \mathcal{H} \) and \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).
§ 2. An Action of $SL(3, \mathbb{R})$ on 4-Sphere

2.1. Let $N(3)$ and $T(3)$ denote the closed subgroups of $SL(3, \mathbb{R})$ consisting of matrices of the forms

\[
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{pmatrix}
\]

respectively. Let $\pi: N(3) \to GL(2, \mathbb{R})$ be a projection defined by

\[
\pi \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

Put

\[
M(x-iy) = \frac{1}{\sqrt{8}} \begin{pmatrix}
2m & 0 & 0 \\
0 & x-m & y \\
0 & y & -x-m
\end{pmatrix}, \quad m = \sqrt{(4-x^2-y^2)/3}
\]

for real numbers $x, y$ such that $x^2 + y^2 \leq 1$. Then we have an injection $M: \mathcal{D} \to S(V)$. Define a map $\psi: N(3) \times M(\mathcal{D}) \to M(\mathcal{D})$ by

\[
\psi(A, M(w)) = M(\psi_1(\pi(A), w)) \quad \text{for} \quad w \in \mathcal{D}, A \in N(3).
\]

We see that the map $\psi$ is a continuous action of $N(3)$ on $M(\mathcal{D})$. Then

\[(a) \quad \psi(A, M(1)) = M(1) \quad \text{if and only if} \quad A \in T(3).
\]

By the property $(\ast)$ for $\psi_1$, we see that

\[(b) \quad \psi(A, M(w)) = AM(w)A^{-1} \quad \text{for} \quad w \in \mathcal{D}, A \in SO(3) \cap N(3).
\]

In addition, for each $w \in \mathcal{D}$, there is an element $A \in SO(3) \cap N(3)$ such that

\[(c) \quad M(w) = AM(\mid w \mid)A^{-1}.
\]

2.2. Denote by $S_+(V)$ (resp. $S_-(V)$) the set of $X \in S(V)$ such that $\det X \geq 0$ (resp. $\det X \leq 0$). If $X \in S_+(V)$ (resp. $X \in S_-(V)$), then $X = AM(x)A^{-1}$ (resp. $X = -AM(x)A^{-1}$) for some $A \in SO(3)$ and a unique real number $x$ such that $0 \leq x \leq 1$. Notice that $\det X = 0$ if and only if $x = 1$; in addition, $AM(x)A^{-1} = M(x)$ if and only if

\[(d) \quad A \in SO(3) \cap T(3) \quad \text{for} \quad 0 < x \leq 1,
\]

\[A \in SO(3) \cap N(3) \quad \text{for} \quad x = 0.
\]
Let \( P \in SL(3, \mathbb{R}) \) and \( A \in SO(3) \). We can express

1. \( PA = A_1N_1 \) and \( (PA)^{-1} = A_2N_2 \)

for some \( A_p \in SO(3) \) and \( N_p \in N(3) \). Put

\[
 PA \Delta AM(x)A = A_1\psi(N_1, M(x))A_1^{-1}, \quad PP(-AM(x)A^{-1}) = -A_2\psi(N_2, M(x))A_2^{-1}.
\]

If \( AM(x)A^{-1} = A'M(x)A'^{-1} \), then \( A' = AK \) for some \( K \in SO(3) \cap N(3) \) by (d); hence \( PA' = A_1(N_1K) \) and \( (PA')^{-1} = A_2(N_2K) \) where \( N_pK \in N(3) \). Therefore we see that the definition (ii) does not depend on the choice of \( A \), by the condition (b).

Next we show that the definition (ii) does not depend on the expression (i) by the condition (b). Suppose

\[
 PA = A_1N_1 = A_1'N_1' \quad \text{and} \quad (PA)^{-1} = A_2N_2 = A_2'N_2'
\]

for \( A_p \in SO(3), \ N_p \in N(3) \). Then \( A_p = A_pB_p \) and \( N_p = B_p^{-1}N_p \) for some \( B_p \in SO(3) \cap N(3) \). Hence

\[
 A_p\psi(N_p, M(x))A'^{-1} = A_pB_p\psi(N_p', M(x))B_p^{-1}A'^{-1} = A_p\psi(B_pN_p', M(x))A'^{-1} = A_p\psi(N_p, M(x))A'^{-1}.
\]

Consequently we can define continuous mappings

\[
 \Psi_+: SL(3, \mathbb{R}) \times S_+(V) \longrightarrow S_+(V), \quad \Psi_-: SL(3, \mathbb{R}) \times S_-(V) \longrightarrow S_-(V)
\]

by \( \Psi_+(P, X) = PAX \) (resp. \( \Psi_-(P, X) = PFX \)) for \( P \in SL(3, \mathbb{R}) \) and \( X \in S_+(V) \) (resp. \( X \in S_-(V) \)).

2.3. Next we show that \( \Psi_+ \) (resp. \( \Psi_- \)) is an action of \( SL(3, \mathbb{R}) \) on \( S_+(V) \) (resp. \( S_-(V) \)). Let \( P, Q \in SL(3, \mathbb{R}) \) and \( A \in SO(3) \). Express

\[
 PA = A_1N_1, \quad QA = A_1'N_1'; \quad (PA)^{-1} = A_2N_2, \quad (QA_2)^{-1} = A_2'N_2'
\]

for some \( A_p, A_p' \in SO(3) \) and \( N_p, N_p' \in N(3) \). Then

\[
 QPA = A_1'N_1'N_1 \quad \text{and} \quad (QPA)^{-1} = A_2'N_2'N_2.
\]

By the conditions (b) and (c), we see that

\[
 \psi(N_p, M(x)) = B_pM(x)pB_p^{-1} = \psi(B_p, M(x)p)
\]

for some \( B_p \in SO(3) \cap N(3) \) and a real number \( x_p \) such that \( 0 \leq x_p \leq 1 \). Since

\[
 QA_1B_1 = A_1'(N_1B_1) \quad \text{and} \quad (QA_2B_2)^{-1} = A_2'(N_2B_2),
\]

we see that
Thus we obtain
\[ \mathcal{Q}(\Delta \psi(N, M(x))) = \mathcal{Q}(\Delta \psi(N_1 B_1 M(x) B_1^{-1} A_1^{-1})) = \mathcal{Q}(\Delta (A_1 B_1 M(x) B_1^{-1} A_1^{-1})) \]
\[ = A_1^t \psi(N_1 B_1, M(x)) B_1^{-1} A_1^t = QP \Delta A M(x) A^{-1}, \]
\[ QP(-A M(x) A^{-1}) = QP(-A_2 M(x) A_2^{-1}) = QP(-A_2 B_2 M(x_2) B_2^{-1} A_2^{-1}) \]
\[ = -A_2^t \psi(N_2 B_2, M(x_2)) A_2^{-1} = A_2^t \psi(N_2 B_2, M(x)) A_2^{-1} = QP(-A M(x) A^{-1}). \]

Thus we obtain \( \mathcal{Q}(\Delta X) = \mathcal{Q} \Delta X \) for \( X \in S_+ \), and \( \mathcal{Q}(\Delta X) = \mathcal{Q} \Delta X \) for \( X \in S_-(V) \), respectively; hence \( \Psi_+ \) and \( \Psi_- \) are actions.

2.4. Here we show that the actions \( \Psi_+ \) and \( \Psi_- \) coincide on the intersection \( S_+(V) \cap S_-(V) \). Let \( X \in S_+(V) \cap S_-(V) \). Then

\[ X = A M(1) A^{-1} = -A S M(1) S^{-1} A^{-1} \]

for some \( A \in SO(3) \), where \( S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in SO(3) \). Let \( P \in SL(3, R) \). We can express

\[ (P S)^{-1} = A_1 N_1 \]

for some \( A_1 \in SO(3) \) and \( N_1 \in T(3) \). Then

\[ P A = A_1' N_1^{-1} S^{-1} = (A_1 S^{-1}) (S' N_1^{-1} S^{-1}) \]

where \( A_1 S^{-1} \in SO(3) \) and \( S' N_1^{-1} S^{-1} \in T(3) \). Therefore, we see that by the condition (a),

\[ P \Delta A M(1) A^{-1} = A_1 S^{-1} \psi(S' N_1^{-1} S^{-1}, M(1)) S A_1^{-1} = A_1 S^{-1} M(1) S A_1^{-1}, \]
\[ P \Delta (-A S M(1) S^{-1} A^{-1}) = -A_1 \psi(N_1, M(1)) A_1^{-1} = -A_1 M(1) A_1^{-1}. \]

Hence we see that the actions \( \Psi_+ \) and \( \Psi_- \) coincide on \( S_+(V) \cap S_-(V) \). Thus we obtain a continuous action \( \Psi \) of \( SL(3, R) \) on \( S(V) \) whose restriction on \( S_+(V) \) (resp. \( S_-(V) \)) is the action \( \Psi_+ \) (resp. \( \Psi_- \)).

By the definition of \( \Psi \), we see that

\[ \Psi(P, X) = P X P^{-1} = \Phi(P, X) \]

for each \( P \in SO(3) \) and \( X \in S(V) \). Hence the action \( \Psi \) is a desired continuous action of \( SL(3, R) \) on \( S(V) \).

§ 3. Non-Differentiability of \( \Psi \)

Denote by \( S_d(V) \) the set consisting of the diagonal matrices of \( S(V) \). Then \( S_d(V) \) is a one-dimensional \( C^\infty \)-submanifold of \( S(V) \). Put \( G_t = \text{diag}(e^{-2t}, e^t, e^t) \)
for each real number \( t \). The correspondence \( X \to \Psi(G_t, X) \) defines a homeomorphism \( h_t \) of \( S_d(V) \) onto itself. We shall show that the homeomorphism \( h_t \) is not \( C^1 \)-differentiable for each \( t \neq 0 \). Put

\[
D(\theta) = (1/\sqrt{6}) \text{diag} (\cos \theta + \sqrt{3} \sin \theta, \cos \theta - \sqrt{3} \sin \theta, -2 \cos \theta)
\]

for each real number \( \theta \). The correspondence \( \theta \to D(\theta) \) defines a \( C^\infty \)-differentiable submersion of \( \mathbb{R} \) onto \( S_d(V) \). The point \( D(\pi/6) = M(1) \) is a fixed point of the homeomorphism \( h_t \) for each real number \( t \). Define a function \( f(t, \theta) \) by

\[
h_t(D(\theta)) = \text{diag} (-, -, f(t, \theta))
\]

for each real numbers \( t, \theta \). We show that \( f(t, \theta) \) is not \( C^1 \)-differentiable at \( \theta = \pi/6 \) for each \( t \neq 0 \).

Suppose first \( \pi/6 \leq \theta \leq \pi/3 \). Then

\[
D(\theta) = M(\sqrt{3} \cos \theta - \sin \theta)
\]

and hence

\[
h_t(D(\theta)) = M(\psi_1(\text{diag} (e^t, e^t), \sqrt{3} \cos \theta - \sin \theta)) = D(\theta).
\]

Therefore \( f(t, \theta) = (-2/\sqrt{6}) \cos \theta \); hence

\[
\lim_{\theta \to \pi/6^+} \frac{\partial}{\partial \theta} f(t, \theta) = 1/\sqrt{6}.
\]

Suppose next \( 0 \leq \theta \leq \pi/6 \). Then

\[
D(\theta) = -SM(2 \sin \theta)S^{-1} \quad \text{for} \quad S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\]

and hence

\[
h_t(D(\theta)) = -SM(\psi_1(\text{diag} (e^t, e^{-2t}), 2 \sin \theta)S^{-1} = -SM(x(t, \theta))S^{-1},
\]

where

\[
x(t, \theta) = \frac{2(e^t + e^{-2t}) \sin \theta + (e^t - e^{-2t})}{2(e^t - e^{-2t}) \sin \theta + (e^t + e^{-2t})}
\]

and \( f(t, \theta) = -\sqrt{(4-x(t, \theta)^2)/6} \). Therefore we obtain

\[
\lim_{\theta \to \pi/6^-} \frac{\partial}{\partial \theta} f(t, \theta) = e^{3t}/\sqrt{6}.
\]

Consequently, we see that \( f(t, \theta) \) is not \( C^1 \)-differentiable at \( \theta = \pi/6 \) for each \( t \neq 0 \), and hence the action \( \Psi \) of \( SL(3, \mathbb{R}) \) on \( S(V) \) is not \( C^1 \)-differentiable.
Reference
