On Holonomic Systems for $\prod_{i=1}^{N} (f_i + \sqrt{-1}0)^{\lambda_i}$

By

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§ 0. The purpose of this paper is to give a description of the characteristic variety of the holonomic $\mathcal{D}$-Module of which a hyperfunction of the form $\Phi(x) = \prod_{j=1}^{d} \delta(\varphi_j) \prod_{i=1}^{N} (f_i(x) + \sqrt{-1}0)^{\lambda_i}$ is a solution. The existence of such a holonomic system was shown in [7] (See also [3].) The description of the characteristic variety in terms of $\varphi_j$'s and $f_i$'s was announced in Lemma 1 of [8]. The result immediately gives an information on the singularity spectrum of $\Phi(x)$. (See Theorem 18 below.) Although a little more precise result was announced in [8] (Lemma 2), we have recently found a gap in our original proof of Lemma 2 of [8]. Even though we have not yet succeeded in filling the gap, we still believe that our original claim should be true and we feel it worth while presenting here as a conjecture. We also discuss some examples in order to show how subtle and delicate the conjecture is. In any case, we should correct our article [8], so that Lemma 2 still remains a conjecture and hence the last six lines of Theorem of [8] should be deleted at the moment.\(***) Needless to say, Lemma 2 of [8] follows immediately from Lemma 1 of [8] if we replace $(\mathbb{R}^+)^N$ in line 16 of

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As a consequence of this correction, all adjective "real" appearing in [9] p. 142 line 17 through line 20 should be replaced by "complex". See for details (and some improvements) the full paper which is being prepared by H. P. Stapp and the authors. (See reference [9].)
§ 1. The situation we shall discuss is the following:

Let $M$ be a real analytic manifold and $X$ a complexification of $M$. Let $\varphi = (\varphi_1, \ldots, \varphi_d)$ and $f = (f_1, \ldots, f_N)$ be sets of real-valued real analytic functions defined on $M$ which extend to $X$. Let $Y$ be a subvariety of $X$ which satisfies the following conditions:

1. $Y$ is irreducible.
2. $Y \subset \{ x \in X ; \varphi_1(x) = \cdots = \varphi_d(x) = 0 \}$
3. There exists a proper complex analytic subset $Y_{\text{sing}}$ of $Y$ such that the following conditions are satisfied:
   3.a) $Y - Y_{\text{sing}}$ is a non-singular subvariety of codimension $d$.
   3.b) $d\varphi_1, \ldots, d\varphi_d$ are linearly independent at any point of $Y - Y_{\text{sing}}$.
4. $f_1|_Y \not\equiv 0$ ($l = 1, \ldots, N$)
5. $Y_{\text{sing}} \subset \{ x \in X ; \prod_{i=1}^N f_i(x) = 0 \}$

Under these assumptions we can easily show the following theorem by the desingularization theorem of Hironaka ([1], [5]).

**Theorem 1.** For $s = (s_1, \ldots, s_N) \in \mathbb{C}^N$ with $\Re s_i \gg 0$ and a compactly supported $C^\infty$-function $g(x)$ on $M$, the integral

$$
\int_{(s-Y_{\text{sing}})\cap M} g(x) \prod_{j=1}^d \delta(\varphi_j(x)) \prod_{i=1}^N (f_i(x) + \sqrt{-1})^{s_i} dx
$$

converges and it defines a distribution $\Phi(x; s) = \prod_{j=1}^d \delta(\varphi_j(x)) \times \prod_{i=1}^N (f_i(x) + \sqrt{-1})^{s_i}$ with holomorphic parameters $s$ when $\Re s_i \gg 0$. Furthermore this distribution can be extended as a meromorphic function in $s = (s_1, \ldots, s_N) \in \mathbb{C}^N$. More precisely, we can choose a $\gamma$-factor $\gamma(s)$ that makes $\gamma(s)\Phi(x; s)$ entire in $s$ and has the form

$$
\gamma(s) = \prod_{k=1}^N \Gamma(\sum_{i=1}^N \nu_{i,k}s_i + d_k),
$$

where $\nu_{i,k}$ are non-negative integers such that $(\nu_{1,k}, \ldots, \nu_{N,k}) \neq 0$ for any $k$ and $d_k$ is an integer.

**Remark.** If $Y_{\text{sing}} = \emptyset$, then we can choose $d_k$ to be a strictly posi-
tive integer.

Since the proof is essentially the same as that given in [4] and [2], we omit the details. See also Section 5 and Section 6, especially the proof of Lemma 14.

§ 2. In order to formulate our main results, we introduce the following notations:

(7) $W_{f,Y}$ is the closure of $\{(\sigma; x, \xi) \in C^N \times T^*X; x \in Y - Y_{\text{sing}}, f_l(x) \neq 0 \ (l = 1, \ldots, N) \text{ and } \xi = \sum_{j=1}^N c_j d \varphi_j(x) + \sum_{j=1}^N \sigma_j d \log f_l(x) \text{ for some } c = (c_1, \ldots, c_d) \in C^d \}$ in $C^N \times T^*X$.

Remark. It is known that $W_{f,Y}$ is an irreducible analytic set. (Cf. Whitney [11]). Clearly it is involutory and of dimension $n + N$. Note that $W_{f,Y}$ depends only on $f|_Y$ and $Y$ and not on $\varphi$.

(8) $W_0 = W_{f,Y} \cap \{\sigma_i = 0, l = 1, \ldots, N\}$, which we consider as a subset of $T^*X$.

(9) We denote by $C[s]$ the ring of polynomials in $s = (s_1, \ldots, s_N)$ and, for a set of $N$ indeterminates $t = (t_1, \ldots, t_N)$ with commutation relations

\[
\begin{cases}
[t_j, t_k] = 0 \\
[t_j, s_k] = \delta_{jk} t_j
\end{cases}
\]

we denote by $C[s, t]$ the non-commutative algebra generated by $s$ and $t$.

(10) $\mathcal{D}[s] = \mathcal{D} \otimes C[s]$ and $\mathcal{D}[s, t] = \mathcal{D} \otimes C[s, t]$. Here $\mathcal{D} = \mathcal{D}_X$ denotes the sheaf of linear differential operators of finite order defined on $X$.

For an integer $m$ and an Ideal $\mathcal{I}$ of $\mathcal{D}[s]$, we define $\mathcal{I}_m$ by

\[
\{P(s) = \sum_{\alpha = (\alpha_1, \ldots, \alpha_N)} P^{\alpha} s^\alpha \in \mathcal{I}; \text{ord } P \leq m - |\alpha|\}.
\]

For $P(s)$ in $\mathcal{I}_m$ we define the function $\tilde{\mathcal{I}}_m(P(s))$ of $(\sigma; x, \xi) \in C^* \times T^*X$ by

\[
\sum_{n=1}^1 \sigma_{m-n} (P_d) \sigma_1^{\alpha_1} \cdots \sigma_N^{\alpha_N},
\]

where $\sigma_f(Q)$ denotes the principal symbol of an operator $Q$ of order at most $j$. 
Let $\mathcal{I}$ be the Ideal of $P(s) \in \mathcal{D}[s]$ such that

\begin{equation}
(13) \quad P(s) \prod_{j=1}^{d} \delta(\varphi_j) \prod_{i=1}^{N} f_i^{t_i} = 0
\end{equation}

holds. This means that the equality (13) holds in $\mathcal{D} \prod_{j=1}^{d} \delta(\varphi_j)$ on $Y - Y_{\text{sing}}$ and for any complex numbers $s$. It is clear that $\gamma(s) \Phi(x; s)$ is annihilated by $P(s)$ in $\mathcal{I}$, where $\gamma(s)$ is the $\gamma$-factor introduced in Theorem 1.

We define $\mathcal{N}_{f,s}$ by $\mathcal{D}[s]/\mathcal{I}$. We shall denote by $u$ the differential operator $1$ modulo $\mathcal{I}$. Note that the characteristic variety $\text{SS}(\mathcal{N}_{f,s})$ of $\mathcal{N}_{f,s}$ contains $W_{f,Y}$, because $W_{f,Y}$ is contained in $\text{SS}(...)$ at generic points of $W_{f,Y}$.

If we define the multiplication of $t_i$ by

\begin{equation}
(14) \quad t_i: P(s) u \rightarrow P(s, \ldots, s_i+1, \ldots, s_N) f_i u,
\end{equation}

$\mathcal{N}_{f,s}$ naturally acquires a structure of $\mathcal{D}[s, t]$-Module.

We also define

\begin{equation}
(15) \quad C_{f,Y} = \cap_{\mathcal{I}} \{ (\sigma; x, \xi) \in C^X \times T^*X; \partial_{\sigma}(P(s)) (\sigma; x, \xi) = 0 \}
\end{equation}

for $P(s) \in \mathcal{I}_m$.

Then we have following

**Theorem 2.** $C_{f,Y} = W_{f,Y}$ holds.

In order to prove this theorem, we introduce auxiliary variables $w_i \in C (l=1, \ldots, N)$ and define $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_N)$ by $(w_1 f_1(x), \ldots, w_N f_1(x))$. We denote $C^X \times X$ (resp. $C^Y \times Y$) by $\bar{X}$ (resp. $\bar{Y}$). Then by defining the vector fields $\partial_{\bar{f}}(l=1, \ldots, N)$ by $w_i \frac{\partial}{\partial w_i}$, we immediately find that

\begin{equation}
(16) \quad \Theta_{\nabla} \bar{f}_k = \delta_{ik} \bar{f}_k \quad (i, k = 1, \ldots, N)
\end{equation}

and

\begin{equation}
(17) \quad \Theta_{\varphi_j} = 0 \quad (j = 1, \ldots, d, l = 1, \ldots, N)
\end{equation}

hold. Then we have

\begin{equation}
(18) \quad s_i^n \prod_{j=1}^{d} \delta(\varphi_j) \prod_{i=1}^{N} \bar{f}_i^{t_i} = \Theta_{\nabla} \prod_{j=1}^{d} \delta(\varphi_j) \prod_{i=1}^{N} \bar{f}_i^{t_i}
\end{equation}
for any positive integer $m$. Hence $\mathcal{J}_{f,v}$ turns out to be a coherent $\mathcal{D}_X$-Module generated by the section $u = \prod \delta (\varphi_j)^{\sum \bar{f}_j}$. Since the function $\varphi_1$ on $C^N \times T^X$ equals the principal symbol $\sigma (\Theta_1)$ of $\Theta_1$ on $W_{f,v}$, $W_{f,v}$ is imbedded into $T^X$ by the projection from $C^N \times T^X$ onto $T^X$. Therefore, we regard $W_{f,v}$ as a subset of $T^X$. Then we can prove the following lemmas.

**Lemma 3.** $SS(\mathcal{J}_{f,v}) = \{(w, x; \tau, \xi) \in T^*(C^N \times X); (w; x; \tau, \xi) \in C_{f,v}\}$.

**Lemma 4.** $SS(\mathcal{J}_{f,v}) = \{(w, x; \tau, \xi) \in T^*(C^N \times X); (w; x; \tau, \xi) \in W_{f,v}\}$.

It is clear that Theorem 2 follows from these two lemmas.

§ 3. We first prove Lemma 3.

**Proof of Lemma 3.**

We first show that the left hand side of the formula in Lemma 3 is included in the right hand side. Let $\bar{u}$ be the generator of $\mathcal{J}_{f,v}$. Let $P(s)$ be an element in $S_m$. Then we have

$$P\left( w_1 \frac{\partial}{\partial w_1}, \ldots, w_N \frac{\partial}{\partial w_N}\right) u = 0.$$  

Furthermore

$$\sigma_m\left( P\left( w_1 \frac{\partial}{\partial w_1}, \ldots, w_N \frac{\partial}{\partial w_N}\right)\right)(w, x; \tau, \xi)$$

$$= \partial^{m}(P(s))(w_1, \tau, \ldots, w_N, \tau, x, \xi)$$

holds. Hence the inclusion relation in question has been proved.

We now prove the opposite inclusion relation. Let $P(w, x, D_w, D_x)$ be an annihilator of $\bar{u}$. Let $m$ be the order of $P$. We decompose $P$ into the form

$$\sum_{a \in \mathbb{Z}^s} P_a,$$

where $P_a$ is of degree $\alpha_i$ with respect to $w_i$ (by counting $\partial / \partial w_i$ to be of degree $-1$). Then $P_a$ has the form
\[(20) \quad \left( \prod_{a_i > 0} w_i^{a_i} \right) P_a^0 \left( \prod_{a_i < 0} (\partial / \partial w_i)^{-a_i} \right),\]

where \( P_a^0 \) is of degree 0 in each \( w_i \). It is clear that \( P_a u = 0 \) and hence we find \( P_a^0 u = 0 \). Since \( P_a^0 \) has the form \( P_a^0 \left( w_1 \frac{\partial}{\partial w_1}, \ldots, w_N \frac{\partial}{\partial w_N}, x, D_x \right) \), we can define \( P_a^0 \in \mathcal{D}[x] \) by

\[(21) \quad \bar{P}_a^0 (s_1, \cdots, s_N, x, D_x).\]

Then we have

\[(22) \quad \sigma(P_a^0) (w, x, \tau, \xi) = \sigma(\bar{P}_a^0) (w_1 \tau_1, \cdots, w_N \tau_N, x, \xi).\]

On the other hand, it follows from the definition of \( P_a \) that

\[(23) \quad \sigma_m(P) = \sum_a (\prod_{a_i > 0} w_i^{a_i}) \sigma(P_a^0) (\prod_{a_i < 0} \tau_i^{-a_i}).\]

This proves the required inclusion relation. Q.E.D.

§ 4. Before giving the proof of Lemma 4, we prepare several auxiliary results. They have some interest in their own right.

**Proposition 5.** Let \( \mathcal{M} \) be a (non-zero) coherent \( \mathcal{E}_X \)-Module\(^a\), and let \( f_j \) and \( g_j \) \((j=1, \cdots, l)\) be endomorphisms of \( \mathcal{M} \). Suppose the following commutation relations hold:

\[(24) \quad \begin{align*}
[ f_j, f_k ] &= [ g_j, g_k ] = 0 \\
[ f_j, g_k ] &= \delta_{jk} \quad \text{for } j, k = 1, \cdots, l.
\end{align*}\]

Then we have

\[\text{codim Supp } \mathcal{M} + l \leq n = \dim X.\]

**Corollary 6.** Let \( \mathcal{M} \) be a coherent \( \mathcal{E}_X \)-Module and let \( s_j \) and \( t_j \) \((j=1, \cdots, l)\) be endomorphisms of \( \mathcal{M} \). Suppose the following conditions hold:

\[(25) \quad t_j : \mathcal{M} \to \mathcal{M} \text{ is injective } (j=1, \cdots, l)\]

\[(26) \quad \begin{align*}
[ t_j, t_k ] &= [ s_j, s_k ] = 0 \\
[ t_j, s_k ] &= \delta_{jk} t_j \quad (j, k = 1, \cdots, l).
\end{align*}\]

\(^a\) \( \mathcal{E}_X \) denotes the sheaf of micro-differential operators of finite order.
Then we have
\[ \text{codim Supp } \mathcal{M} + l \leq n = \dim X. \]

**Proof of Corollary 6.**

By the additive property of the multiplicity of coherent \(\mathcal{D}\)-Modules, the multiplicity of the cokernel of \(t_j: \mathcal{M} \to \mathcal{M}\) along an irreducible component of the characteristic variety of \(\mathcal{M}\) is the difference of the multiplicity of \(\mathcal{M}\) and the same one. Hence the characteristic variety of the cokernel does not contain any irreducible components of the characteristic variety of \(\mathcal{M}\). Therefore, we might assume from the first that \(t_j\) are isomorphisms. Then, \(f_j = t_j\) and \(g_j = t_j^{-1}\) satisfy the commutation relations (24) and this result immediately follows from Proposition 5.

**Proof of Proposition 5.**

If \(\mathcal{M}\) is holonomic, then \(E = \text{End}(\mathcal{M})\) is a finite-dimensional vector space and hence \(\text{tr } 1 = \text{tr } [f_j, g_j] = 0\), which is a contradiction if \(l \geq 1\). Thus the theorem holds in this case.

Let us assume that \(\mathcal{M}\) is not holonomic. Let \(V\) be the support of \(\mathcal{M}\). It is enough to prove the theorem at a generic point of \(V\). Therefore, by a quantized contact transformation, we may assume that \(V = \{(x, \xi); \xi_1 = \cdots = \xi_d = 0\}\) with \(d = \text{codim } V\). Set \(Y = \{x \in X; x_1 = \cdots = x_d = 0\}\) and \(\mathcal{N} = \mathcal{M}|_Y\). Then \(\text{Supp } \mathcal{N} = T^*Y\), and \(\mathcal{N}\) has the endomorphisms \(\tilde{f}_j\) and \(\tilde{g}_j\) induced from \(f_j\) and \(g_j\), respectively. The endomorphisms \(\tilde{f}_j\) and \(\tilde{g}_j\) satisfy also the same commutation relations (17). Hence, by replacing \(\mathcal{M}\) with \(\mathcal{N}\), we might assume from the first that \(\text{Supp } \mathcal{M} = T^*X\).

At a generic point of \(T^*X\), \(\mathcal{M}\) is a free \(\mathcal{E}_x\)-Module. Hence we can represent \(f_j\) and \(g_j\) by matrices of micro-differential operators. Thus the proposition immediately follows from the following

**Proposition 7.** Let \(P_1, \cdots, P_l, Q_1, \cdots, Q_l\) be \(N \times N\) matrices of micro-differential operators on \(C^n\), where \(N\) is an integer. If the relations
\[
\begin{align*}
[P_j, P_k] &= [Q_j, Q_k] = 0 \\
[P_j, Q_k] &= \delta_{jk} \\
&\quad \text{for } j, k = 1, \cdots, l
\end{align*}
\]
hold, then \( l \leq n \).

In order to prove this proposition, we prepare the following lemma.

**Lemma 8.** Let \( P = (P_{jk})_{j,k=1,\ldots,N} \) be an \( N \times N \) matrix of micro-differential operators of order \( \leq m \). Assume that the eigenvalues of the matrix \( \overline{P} = (\sigma_m(P_{jk}))_{j,k=1,\ldots,N} \) are mutually different and do not vanish. Then there is an invertible matrix \( \Lambda \) of micro-differential operators defined at a generic point of \( T^*X \), such that \( \Lambda PA^{-1} \) is a diagonal matrix.

**Proof.** Consider \( \lambda D_i^m - P \) as a micro-differential operator defined on \( C_\lambda \times C_t \times X \).

Then \( \det(\sigma(\lambda D_i^m - P)) = \det(\lambda \tau_i^m - \overline{P}) \). Hence, if one denotes by \( \rho_j \) the eigenvalues of \( \overline{P} \), then \( \lambda D_i^m - P \) is invertible provided \( \lambda \neq \rho_j \tau_i^m \).

We shall define \( A_j \) by

\[
A_j = \frac{1}{2\pi \sqrt{-1}} \oint \frac{D_i^m}{\lambda \tau_i^m - P} \, d\lambda,
\]

where the integral is a contour integral along a path around \( \rho_j \tau_i^m \). It is easy to see that \( A_j A_k = \delta_{jk} A_j \) and \( 1 = \sum_j A_j \) hold. Since \( A_j \) commutes with \( t \) and \( D_t \), \( A_j \) is a matrix of micro-differential operators on \( X \). Setting \( \mathcal{L} = \mathcal{E}^n \), we regard \( P \) and \( A_j \) as endomorphisms of \( \mathcal{L} \). Set \( \mathcal{L}_j = A_j \mathcal{L} \). Since \( P \) is invertible, \( \mathcal{L} = \bigoplus_j \mathcal{L}_j \). Since \( A_j \) is a matrix of micro-differential operators of order \( \leq m \) and \( \sigma_m(A_j) \) is a projector onto the eigenspace for \( \rho_j \), \( \mathcal{L}_j \) is not zero. Hence \( \mathcal{L}_j \) is with multiplicity 1. Thus \( \mathcal{L}_j \) is isomorphic to \( \mathcal{E} \) at a generic point with a base \( u_j \). If we take an invertible matrix \( \Lambda \) corresponding to the base \( u_1, \ldots, u_N \), then \( \Lambda PA^{-1} \) is a diagonal matrix. Q.E.D.

Let us return to the proof of Proposition 7.

Let \( \Lambda \) be an invertible constant matrix whose eigenvalues are mutually different. For a sufficiently large integer \( N \), we set

\[
\widetilde{P}_j = P_j + AD_i^\eta \quad \text{and} \quad \widetilde{Q}_j = Q_j \quad \text{for} \ j = 1, \ldots, l
\]

\[
\widetilde{P}_{l+1} = AD_i^\eta
\]
\[
\tilde{Q}_{t+1} = \frac{1}{N} A^{-1}tD_t^{-N} - \sum_{j=1}^{l} Q_j.
\]

We shall regard \( \tilde{P}_j \) and \( \tilde{Q}_j \) as micro-differential operators on \( \mathbb{C} \times X \). Then \( \tilde{P}_1 \) satisfies the condition in Lemma 8. Hence it is diagonalizable by an inner automorphism. Thus, replacing \( X \) with \( \mathbb{C} \times X \) and \( P_j \) and \( Q_j \) with \( \tilde{P}_j \) and \( \tilde{Q}_j \), respectively, we might assume from the first that

(28) \( P_1 \) is a diagonal matrix with diagonal components \( A_1, \ldots, A_N \).

(29) \( A_1, \ldots, A_N \) are micro-differential operators of order \( m \) and \( \sigma_m(A_j) \) are mutually different.

Here we note the following

**Sublemma 9.** Let \( R \) be an \( N \times N \) matrix of micro-differential operators such that \([P_1, R]\) is a diagonal matrix. Then \( R \) is also diagonal.

*Proof.* Let \( \{R_{jk}\} \) be components of \( R \). Then \( A_j R_{jk} = R_{jk} A_k \) for \( j \neq k \). If \( R_{jk} \neq 0 \), then \( \sigma(A_j) \sigma(R_{jk}) = \sigma(R_{jk}) \sigma(A_k) \). This is a contradiction. Q.E.D.

Now we resume proving Proposition 7. It follows from Sublemma 9 that all \( P_j \) and \( Q_j \) are diagonal matrices. Set \( m_j = \text{ord} P_j \) and \( n_j = \text{ord} Q_j \) \((j = 1, \ldots, l)\). Then \( m_j + n_j \geq 1 \). We shall prove the proposition by the induction on \( \sum_{j=1}^{l} (m_j + n_j - 1) \).

If \( \sum_{j=1}^{l} (m_j + n_j - 1) = 0 \), then \( m_j + n_j = 1 \) for every \( j \). Hence we obtain

\[
\begin{align*}
\{\sigma(P_j), \sigma(R_k)\} = \{\sigma(Q_j), \sigma(Q_k)\} &= 0 \\
\{\sigma(P_j), \sigma(Q_k)\} &= \delta_{jk} \quad \text{for } j, k = 1, \ldots, l.
\end{align*}
\]

Then, as is well-known in symplectic geometry, we have \( l \leq n \).

Next suppose that \( \sum_{j=1}^{l} (m_j + n_j - 1) > 0 \). Then either \( m_j \geq 1 \) or \( n_j \geq 1 \) holds for each \( j \). If \( n_j \geq 1 \), by replacing \( P_j \) and \( Q_j \) with \( -Q_j \) and \( P_j \), respectively, we might assume from the first that \( m_j \geq 1 \). Set \( \tilde{P}_j = P_j^{1/m_j} \) and \( \tilde{Q}_j = m_j P_j^{-1/m_j} Q_j \). Then it is easy to see that \( \tilde{P}_j \) and \( \tilde{Q}_j \) satisfy the same commutation relations as \( P_j \) and \( Q_j \). Furthermore we have
ord $P_j = 1$ and ord $\bar{P}_j = m_j + n_j$. Hence we may assume without loss of generality that ord $P_j = 1$ ($j = 1, \ldots, l$) and $n_1 \geq n_2 \geq \ldots \geq n_l \geq 0$. Set $f_j = \sigma(P_j)$ ($j = 1, \ldots, l$). Suppose first that $df_1, \ldots, df_l$ are linearly independent. Since $\{f_j, f_k\} = 0$ ($1 \leq j, k \leq l$), it is well-known in symplectic geometry that $l \leq n$ holds in this case. Therefore we may assume that there exists $r$ ($1 \leq r < l$) such that $df_1, \ldots, df_r$ are linearly independent and $df_{r+1} = 0 \mod df_1, \ldots, df_r$. This means that $f_{r+1}$ is a function of $f_1, \ldots, f_r$, namely, there exists a homogeneous function $\varphi(t_1, \ldots, t_r)$ of degree 1 such that $f_{r+1} = \varphi(f_1, \ldots, f_r)$. Denote $\frac{\partial}{\partial t^j} \varphi(t_1, \ldots, t_r)$ by $\varphi_j$. Set $\bar{P}_{r+1} = P_{r+1} - \varphi(P_1, \ldots, P_r)$, $\bar{Q}_j = Q_j$ ($j > r$) and $\bar{Q}_j = Q_j + \varphi_j(P_1, \ldots, P_r)Q_{r+1}$ ($j \leq r$). Then it is easy to see that

\[
\begin{align*}
[\bar{P}_j, \bar{P}_k] &= [\bar{Q}_j, \bar{Q}_k] = 0 \\
[\bar{P}_j, \bar{Q}_k] &= \delta_{jk}.
\end{align*}
\]

Furthermore we have ord $\bar{Q}_j \leq n_j$, ord $\bar{P}_j \leq 1$ and ord $\bar{P}_{r+1} < 0$. Hence the induction proceeds and we conclude $l \leq n$. This completes the proof of Proposition 7, hence also that of Proposition 5.

The second result we need for the proof of Lemma 4 is the following

**Proposition 10.** Let $f: X \to Y$ be a proper map of complex manifolds. Let $V$ be an involutory variety (possibly with singularities) of $T^*X$ whose dimension is at most $\dim X + l$ at any point. Then the dimension of $\mathfrak{V} \rho^{-1}(V)$ is at most $\dim Y + l$ at any point in $\mathfrak{V} \rho^{-1}(V)$. Here $\mathfrak{V}$ (resp. $\rho$) is the canonical projection from $X \times T^*Y$ to $T^*Y$ (resp. $T^*X$).

**Proof.** Set $W = \mathfrak{V} \rho^{-1}(V)$. The question being local on $W$, we can choose a point $p$ in $\rho^{-1}(V)$ such that $W$ is non-singular at $\mathfrak{V}(p)$, $\rho^{-1}(V)$ is non-singular at $p$ and the projection $\rho^{-1}(V) \to W$ is smooth at $p$. Moreover it is enough to show the statement in a neighborhood of $\mathfrak{V}(p)$. Let $\omega_X$ and $\omega_Y$ be the canonical 1-forms on $T^*X$ and $T^*Y$. Then $\mathfrak{V}^*(\omega_Y) = \rho^*(\omega_X)$.

Let $\varphi: V' \to V$ be a desingularization of $V$ ([1], [5]). Then there
exist a manifold $U$, $\psi: U \to \rho^{-1}(V)$ and $\phi: U \to V'$ such that $\varphi\phi = \rho\psi$ and that $\phi$ is generically surjective.

\[
\begin{array}{ccc}
V' & \xrightarrow{\varphi} & V \\
\downarrow \phi & & \downarrow \rho \\
U & \xrightarrow{\psi} & \rho^{-1}(V) \\
\downarrow & & \downarrow W \\
\end{array}
\]

(30)

At a generic point of $V$, we have $(d\omega_x|_V)^{l+1} = 0$, because $V$ is involutory. Hence $(d(\psi^*\phi^*\omega_x))^{l+1} = (\psi^*\rho^*d\omega_x)^{l+1} = (\psi^*\omega^*d\omega_x)^{l+1}$ vanishes. Since $\psi\omega$ is generically surjective, $(d\omega_y|_W)^{l+1} = 0$. This implies immediately that the dimension of $W$ is at most $\dim Y + l$ at any point in $W$.

Q.E.D.

We shall further prove some related results in contact geometry, which will be used later.

**Lemma 11.** Let $Y$ be a submanifold of $X$ (i.e., subvariety without singularities) and let $V$ be an involutory variety in $T^*X$. Assume that $V$ is non-characteristic with respect to $Y$, namely $\rho|_{T^*X}: Y \times X \to T^*Y$ is a finite map. Then $W = \rho(Y \times V)$ is an involutory subvariety of $T^*Y$.

**Proof.** We denote by $n$ (resp. $l$) $\dim X$ (resp. codim$_X Y$). We may assume without loss of generality that codim$_X Y = 1$. Then codim$_X (Y \times V) = l + 1$ and $\dim W = 2(n - 1) - (l - 1)$. As in the proof of Proposition 10, we have

\[
(d\omega_y|_W)^{n-l+1} = 0,
\]

since $(d\omega_x|_V)^{n-l+1} = 0$.

Let $a$ be a generic point in $W$. Then by a suitable contact transformation we may assume that

\[
W = \{(x, \xi) \in T^*Y; x_1 = \cdots = x_p = 0, \xi_1 = \cdots = \xi_q = 0\}
\]

in a neighborhood of $a$. Here
Then we have
\[(d\omega_r|_W)^{n-1-\max(p,q)} \neq 0.\]
Comparing (31) and (33), we conclude
\[n-l+1>n-1-\max(p,q).\]
Combining (32) and (34), we have \(p = 0\) or \(q = 0\). This is equivalent
to saying that \(W\) is involutory. Q.E.D.

**Proposition 12.** Let \(V\) be an involutory variety in \(T^*X\). Let \(f(x,\xi)\) be a holomorphic function which is homogeneous in \(\xi\). Assume that \(df\neq 0\) on \(\{f=0\}\). Assume furthermore that \(V_0=\{f=0\}\cap V\)
is invariant under the Hamiltonian vector field
\[H_f=\sum_{j}(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j})\]
associated with \(f\). Then \(V_0\) is an involutory variety.

**Proof.** By adding a dummy variable, we may assume without loss of
generality that \(df\wedge \omega_x\neq 0\) and \(f\) is homogeneous of degree 1 with
respect to \(\xi\). Hence by a suitable contact transformation we may assume
that \(f=\xi_1\). We may also assume that \(V\) is irreducible. If \(V\subset f^{-1}(0)\),
then there is nothing to prove. Therefore we may assume that \(V_0\) is a
hypersurface of \(V\).

In order to prove the proposition, it is enough to discuss at a
generic point of \(V_0\). Hence, we may assume that \(V_0\) and \(V-V_0\) are
non-singular and \((V-V_0, V_0)\) satisfies the condition of Whitney. Suppose
that \(V_0\) is not involutory. Then there is a point \(p\) in \(V_0\) such that
\(T_pV_0\) is not involutory. Let us take a sequence \(\{p_n\}\) in \(V-V_0\) con-
verging to \(p\) such that \(T_{p_n}V\) converges to a linear subspace \(\tau\).
The space \(\tau\) is involutory and \(T_pV_0\) is a hyperplane of \(\tau\). Next we
show that \(\tau\subset df^{-1}(0)\). If \(\tau\not\subset df^{-1}(0)\), then \(T_pV_0=\tau\cap df^{-1}(0)\).
Therefore \((T_pV_0)\perp=(\tau\cap df^{-1}(0))\perp=\tau\perp+CH_f\). Since \(\tau\perp\subset \tau\) and \(H_f\in T_pV_0\)
\(\subset \tau\) we have \(\tau\perp+CH_f\subset T_pV_0\), which contradicts the fact that \(T_pV_0\) is not
involutory. Thus we have seen \(\tau\) is contained in \(df^{-1}(0)\). Since we
have assumed $f = \xi^1$, this implies that $V$ is non-characteristic with respect to $Y = \{x_1 = 0\}$.

On the other hand, we may assume that $V_0$ has the form $\{(x, \xi); \xi_1 = \xi_2 = \cdots = \xi_s = 0, x_2 = \cdots = x_p = 0\}$ at a generic point of $V_0$, because $V_0$ is invariant by $H_{\xi^1}$. Then we have

$$\rho(X \times V) \supset \{(x_2, \cdots, x_n; \xi_2, \cdots, \xi_n) \in T^*Y; \xi_1 = \cdots = \xi_s = 0, x_2 = \cdots = x_p = 0\}. \tag{35}$$

Since $\text{codim}_{X} \rho(X \times V) = (\nu - 1) + (\mu - 1)$, the left hand side of (35) must coincide with the right hand side of (35).

On the other hand, the preceding lemma asserts that $\rho(X \times V)$ is involutory. This means that $\nu = 1$ or $\mu = 1$.

Thus we have proved that $V_0$ is involutory. Q.E.D.

§ 5. Now we embark on the proof of Lemma 4.

By the desingularization theorem of Hironaka ([1], [5]) we can find an analytic manifold $X'$, a projective map $F$ from $X'$ to $X$ and a proper subvariety $Z$ of $Y$ so that the following conditions (36) ~ (39) are satisfied.

$$Y_{\text{sing}} \subset Z \subset \{x \in Y; \prod_{i=1}^{N} f_i(x) = 0\} \tag{36}$$

(37) $Z' = F^{-1}(Z)$ is a normally crossing hypersurface and $F|_{X' - Z'}$ is an isomorphism from $X' - Z'$ onto $X - Z$.

(38) The proper transform $Y' = F^{-1}(Y - Z)$ is a non-singular subvariety of $X'$.

(39) At any point $a$ in $Y'$ we can find a local coordinate system $(v, y, z) = (v_1, \cdots, v_d, y_1, \cdots, y_m, z_1, \cdots, z_k)$ $(d + m + k = n)$ so that the following conditions are satisfied:

(39, a) $v_1(a) = \cdots = v_d(a) = y_1(a) = \cdots = y_m(a) = z_1(a) = \cdots = z_k(a) = 0$

(39, b) $Y' = \{(v, y, z); v_1 = \cdots = v_d = 0\}$

(39, c) $Z' \subset \{(v, y, z); \prod_{p=1}^{m} y_p \prod_{q=1}^{k} z_q = 0\}$
(39. d) $\varphi'_j = \varphi_j \circ F = \psi_j (v, y, z) v_j \prod_{p=1}^{m} y_p^{\rho_{j,p}} \prod_{q=1}^{k} z_q^{r_{j,q}}$

(39. e) $f'_i |_{Y'} = f_i \circ F |_{Y'} = \chi_i (y, z) \prod_{q=1}^{k} z_q^{r_{i,q}}$

where $\rho_{j,p}, \rho'_j, p$ and $r_{i,q}$ are non-negative integers, $(r_1, q, \cdots, r_k, q) \neq 0$ for each $q = 1, \cdots, k$ and $\psi_j$ and $\chi_i$ are non-vanishing holomorphic functions.

As a matter of fact, we have

(39. e') $Y' \cap Z' \subset \{(v, y, z); v = 0, \prod_{q=1}^{k} z_q = 0\},$

because $Z \subset f_1^{-1}(0) \cup \cdots \cup f_k^{-1}(0)$. We also find that $\rho'_j, p$ is actually zero, because $\{d\psi_1, \cdots, d\psi_d\}$ are linearly independent at any point in $Y' - Z'$.

We define $X'$ (resp. $Y'$) by $C^\infty \times X'$ (resp. $C^\infty \times Y'$) and extend $F$ to $\tilde{F}: \tilde{X}' \to \tilde{X}$ ($= C^\infty \times X$). We denote $(w_1 f_1', \cdots, w_N f_N')$ by $\tilde{f}' = (\tilde{f}_1', \cdots, \tilde{f}_N')$. Then we have the following

Sublemma 13. We have

(40) $\tilde{W}_{\tilde{f}, \tilde{y}} = \{ (w, v, y, z; \tau, \sigma, \eta, \zeta) \in T^* \tilde{X}'; \ v = 0 \}$

$\eta_\tau = \sum_{i=1}^{N} w_i \tau_i \frac{\partial}{\partial y_\tau} \log \chi_i, \ z_q \eta_q = \sum_{i=1}^{N} r_{i,q} w_i \tau_i, \ \text{where}$

$\tilde{y}_q = \zeta_q - \sum_{i=1}^{N} w_i \tau_i \frac{\partial}{\partial z_q} \log \chi_i$

in a neighborhood $\tilde{a} = (0, a) \in \tilde{Y}'$.

Proof. Since the right hand side of (40) is a closed subset, it suffices to show that the following set $\tilde{W}_{\tilde{f}, \tilde{y}}$ is contained in the right hand side and that any point in the right hand side is reached by a sequence of points in $\tilde{W}_{\tilde{f}, \tilde{y}}$.

(41) $\tilde{W}_{\tilde{f}, \tilde{y}} = \{ (w, v, y, z; \tau, \sigma, \eta, \zeta) \in T^* \tilde{X}'; \ v = 0 \}$

$\tilde{f}_i' \neq 0 \ (l = 1, \cdots, N), \ (\eta, \zeta) = \sum_{i=1}^{N} w_i \tau_i \grad_{(i,a)} \log \tilde{f}_i' \ (l = 1, \cdots, N) \}$.

Recall that $W_{f, y}$ is the closure of $\tilde{W}_{f, y}$ by the definition. Clearly $\tilde{W}_{\tilde{f}, \tilde{y}}$ coincides with the right hand side of (40) in a neighborhood of
the points where \( \prod_{q=1}^{k} z_q \neq 0 \). It is also clear that \( \tilde{W}_{\mathcal{J}, \mathcal{V}} \) is contained in the right hand side of (40). Hence it remains to be proved only that any point \( b = (w, v, y, z; t, \sigma, \eta, \zeta) \) in the right hand side of (40) where \( \prod_{q=1}^{k} z_q = 0 \) can be reached by a sequence of points in \( \tilde{W}_{\mathcal{J}, \mathcal{V}} \). Assume that \( z_1 = \cdots = z_k = 0 \) and \( z_{k+1} \neq 0, \ldots, z_k \neq 0 \) hold at \( b \). Here we may assume without loss of generality that \( \tilde{r}_q \neq 0 \) \( (q = 1, \ldots, \ell) \) at \( b \). Since \( (r_{1,q}, \ldots, r_{N,q}) \neq 0 \) for any \( q \), we can choose sequences \( w_1^{(m)} \to w_1 \) and \( \tau_{i}^{(m)} \to \tau_i \) \( (i = 1, \ldots, N) \) so that

\[
\sum_{l=1}^{N} r_{1,q} w_{l}^{(m)} \tau_{i}^{(m)} \neq 0 \quad (q = 1, \ldots, \ell).
\]

We also define sequences \( \tilde{r}_q^{(m)} \) and \( z_q^{(m)} \) by the following:

\[
\begin{align*}
\tilde{r}_q^{(m)} &= \tilde{r}_q \quad (q = 1, \ldots, \ell) \\
\tilde{r}_q^{(m)} &= \frac{1}{\tilde{r}_q} \sum_{l=1}^{N} r_{1,q} w_{l}^{(m)} \tau_{i}^{(m)} \quad (q = 1, \ell, \ldots, k) \\
z_q^{(m)} &= \frac{1}{\tilde{r}_q^{(m)}} \sum_{l=1}^{N} r_{1,q} w_{l}^{(m)} \tau_{i}^{(m)} \quad (q = 1, \ell, \ldots, i) \\
z_q^{(m)} &= z_q \quad (q = i + 1, \ell, \ldots, k)
\end{align*}
\]

Then it is clear that \( (w_1^{(m)}, v, y, z^{(m)}; \tau^{(m)}, \sigma, \eta, \zeta^{(m)} + \sum_{l=1}^{N} w_{l}^{(m)} \tau_{i}^{(m)} \frac{\partial}{\partial z_q} \log z_i) \) is a required limiting sequence which converges to \( b \). Q.E.D.

By using this result we shall prove

\[
W_{\mathcal{J}, \mathcal{V}} \times (G^{N} \times Z) \text{ is an involutory variety.}
\]

It follows from (39.c') that \( Y' \cap Z' \) is a union of hypersurfaces \( \{(z, y, z); v = 0, z_0 = 0\} \) of \( Y' \). Hence it suffices to prove that \( W_{\mathcal{J}, \mathcal{V}} \cap \{z_1 = 0\} \) is involutory. Since \( W_{\mathcal{J}, \mathcal{V}} \) is involutory and \( W_{\mathcal{J}, \mathcal{V}} \cap \{z_1 = 0\} \) is invariant by \( H_{z_1} = -\partial/\partial z_1 \), this set is involutory by Proposition 12.

Now let us consider \( \mathcal{D}_{\mathcal{J}} \)-Module \( \mathcal{N}_{\mathcal{J}, \mathcal{V}} \). Then by the aid of Sublemma 13 we find the following

**Lemma 14.** \( \text{SS} (\mathcal{N}_{\mathcal{J}, \mathcal{V}}) = W_{\mathcal{J}, \mathcal{V}} \) holds.

**Proof of Lemma 14.** Since \( \text{SS} (\mathcal{N}_{\mathcal{J}, \mathcal{V}}) \) clearly contains \( W_{\mathcal{J}, \mathcal{V}} \), it
suffices to prove the opposite inclusion relation. Let \( u' \) be the generator
\[
\prod_{j=1}^{d} \delta(\phi_j) \prod_{i=1}^{N} f_i^{x_i} = \prod_{i=1}^{N} (x_i(y, z) w_i \prod_{q=1}^{k} z_q^{x_q}) \prod_{j=1}^{d} \delta(\psi_j(v, y, z) v_j \prod_{q=1}^{k} z_q^{x_q}) = \]
\[
= \prod_{j=1}^{d} \delta(v_j) \left( \prod_{i=1}^{N} x_i(y, z)^{x_i} w_i^{x_i} \right) \left( \prod_{j=1}^{d} \psi_j(0, y, z)^{-1} \right) \left( \prod_{q=1}^{k} z_q^{x_q} \prod_{j=1}^{d} \delta(\rho_j) \right)
\]

Therefore, \( u' \) satisfies the differential equations
\[
\begin{align*}
\frac{\partial}{\partial y_p} - \sum_{i=1}^{N} \left( \frac{\partial}{\partial y_q} \log x_i \right) w_i \frac{\partial}{\partial w_i} \left( \prod_{j=1}^{d} \psi_j(0, y, z) \right) u' &= 0 \quad (p = 1, \ldots, m) \\
\frac{\partial}{\partial z_q} - \sum_{i=1}^{N} \left( \frac{\partial}{\partial z_q} \log x_i \right) w_i \frac{\partial}{\partial w_i} - \sum_{i=1}^{N} r_{i,q} w_i \frac{\partial}{\partial w_i} + \sum_{j=1}^{d} \rho_j q \right) \times \\
\times \prod_{j=1}^{d} \psi_j(0, y, z) u' &= 0 \quad (q = 1, \ldots, k).
\end{align*}
\]

Since the characteristic variety of \( \mathcal{N}_{j_1, \varphi} \) is contained in the common zeros of the principal symbols of the differential operators used in (45), it is contained in \( W_{j_1, \varphi} \). This completes the proof of the lemma.

Q.E.D.

Now we resume the proof of Lemma 4. Since \( \tilde{F} \) is a projective map, \( \mathcal{D}_F \)-Module \( \mathcal{N}' = \int_{\tilde{F}} \mathcal{N}_{j_1, \varphi} \) is well-defined and coherent and its characteristic variety is contained in \( w\rho^{-1}(W_{j_1, \varphi}) \) (Theorem 4.2 of [6]). Let \( v \) be the section of \( \mathcal{N}' \) corresponding to \( 1_{\tilde{F}} \times_{\tilde{F}} \prod_{j=1}^{d} f_i^{x_i} \prod_{j=1}^{d} \delta(\phi_j) \) and let \( \mathcal{N}'' \) be the sub \( \mathcal{D}_F \)-Module of \( \mathcal{N} \) generated by \( v \). Then we can define a natural \( \mathcal{D}_F \)-linear surjective homomorphism from \( \mathcal{N}'' \) to \( \mathcal{N}_{j_1, \varphi} \) by assigning \( \prod_{j=1}^{d} \delta(\phi_j) \prod_{i=1}^{N} f_i^{x_i} \) to \( v \). Since the characteristic variety of \( \mathcal{N}'' \) is contained in \( w\rho^{-1}(W_{j_1, \varphi}) \), we have
\[
(46) \quad \text{SS}(\mathcal{N}_{j_1, \varphi}) \subset w\rho^{-1}(W_{j_1, \varphi}).
\]

On the other hand, we have
\[
\begin{align*}
w\rho^{-1}(W_{j_1, \varphi}) &= w\rho^{-1}(W_{j_1, \varphi} \times (\tilde{X}' - C_N \times Z')) \\
&\cup w\rho^{-1}(W_{j_1, \varphi} \times (C_N \times Z')) \\
&\subset W_{j_1, \varphi} \cup w\rho^{-1}(W_{j_1, \varphi} \times (C_N \times Z)),
\end{align*}
\]
because $\tilde{F}$ is an isomorphism outside $C^N \times Z$. Clearly
\begin{equation}
(47) \quad \dim W_{f, r} \times C^N \times Z \leq 2N + n - 1
\end{equation}
Therefore Proposition 10 combined with (44) and (47) entails that
\begin{equation}
(48) \quad \dim \mathcal{W}^{-1}(W_{f, r} \times (C^N \times Z)) \leq 2N + n - 1.
\end{equation}
On the other hand, Corollary 6 claims that
\begin{equation}
(49) \quad \dim \text{SS}(\mathcal{N}_{f, \phi}) \geq N + (n + N) = 2N + n
\end{equation}
holds at any point $p$ of $\text{SS}(\mathcal{N}_{f, \phi})$. Thus we finally conclude from (48) and (49) that
\begin{equation}
(50) \quad \text{SS}(\mathcal{N}_{f, \phi}) = W_{f, r}.
\end{equation}
This completes the proof of Lemma 4, and, hence at the same time, that of Theorem 2.

§ 6. In order to study the holonomic character of the distribution $\Phi(x; s)$ we further need the following geometric result.

**Proposition 15.** The set $W_\theta$ is Lagrangian.

*Proof.* We shall first show that $W_\theta$ is isotropic. Again by the desingularization theorem of Hironaka, we can find a complex manifold $\tilde{W}$, a proper analytic subset and a proper surjective map $G$ from $\tilde{W}$ onto $W_{f, r}$. Let $F$ be the projection from $C^N \times T^*X$ onto $T^*X$. Let $\omega_x = \sum \xi_j dx_j$ be the canonical 1-form on $T^*X$. Here we may assume the following:
\begin{equation}
(51) \quad \tilde{W} \text{ is a closure of } G^{-1}(\{x; \prod_{i=1}^N f_i(x) \neq 0\}).
\end{equation}
\begin{equation}
(52) \quad G^{-1}(W_\theta) \text{ is a hypersurface of } \tilde{W}.
\end{equation}
The question being local on $W_\theta$, it is enough to show that $\omega_x|_{W_\theta} = 0$ at a generic point $p$ of $W_\theta$. We may assume that $W_\theta$ is non-singular at $p$ and that there exists a point $p'$ in $G^{-1}(p)$ such that $G^{-1}(W_\theta)$ is non-singular at $p'$ and $G^{-1}(W_\theta) \to W_\theta$ is smooth at $p'$. Let $\phi$ be a defining function of $G^{-1}(W_\theta)$. Then, we have $f_i \circ G = \theta_i \phi^{\xi_i}$ and $\partial_t \circ G = \zeta_t \phi^{k_t}$,
where $\theta_i$ and $\chi_i$ are holomorphic functions on $W$ defined in a neighborhood of $p$ such that their restrictions onto $G^{-1}(W_0)$ do not vanish identically. Here we have $k_i \geq 1$. Then, we have

$$(FG)^* \omega_x = \sum_{i=1}^{N} (\sigma_i G) d(\log f_i \circ G)$$

outside

$$\{x; \prod_{i=1}^{N} f_i(x) \neq 0\}.$$ 

Therefore we have

$$(FG)^* \omega_x = \sum_{i=1}^{N} \nu_i \chi_i \psi^{k_i - 1} d\psi + \sum_{i=1}^{N} \chi_i \psi^{k_i} d \log \theta_i$$

in a neighborhood of $p'$. Thus $(FG)^* \omega_x|_{W_0}$ vanishes at generic points and hence vanishes in a neighborhood of $p'$. Since $G^{-1}(W_0) \to W_0$ is smooth at $p'$, $\omega_x|_{W_0}$ vanishes at $p$. Therefore $W_0$ is isotropic, and hence $\dim W_0 \leq n$.

On the other hand, $W_0$ is defined as the common zeros of $N$ functions on $W_{f,r}$, and hence we have $\dim W_0 \geq n$. Thus $\dim W_0 = n$, and this implies that $W_0$ is Lagrangian. Q.E.D.

Here we note the following interesting property of $W_0$ (Theorem 16), even though we do not need it in our subsequent discussions of this paper.

In stating the theorem we use the following notation:

For a set of complex numbers $a = (a_1, \ldots, a_N)$, $W(a)$ denotes the closure of $\{(\tau, x, \xi) \in C \times T^* X; x \in Y - Y_{\text{sing}}, \prod_{i=1}^{N} f_i(x) \neq 0 \text{ and } \xi = \sum_{j=1}^{N} c_j d\varphi_j(x) + \tau \sum_{i=1}^{N} a_i \tau \sigma_i d \log f_i(x) \text{ for some complex numbers } c_j \text{ and } \tau \}$ and $W_0(a)$ denotes the intersection of $W(a)$ and $\tau^{-1}(0)$. We identify $W_0(a)$ with the subset of $T^* X$.

**Theorem 16.** The set $W_0(a)$ coincides with $W_0$, if $\sum_{i=1}^{N} a_i \nu_i \neq 0$ holds for $\nu_i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with $(\nu_i, \ldots, \nu_N) \neq (0, \ldots, 0)$.

**Proof.** At least one of $a_i$ does not vanish, say $a_i$. We can normalize $a_i$
to be 1. Let $W'$ be the subvariety of $W_{f,r}$ defined by $\sigma_i = a_i \sigma_i$ ($l=1, \ldots, N$). Let $E$ be the subvariety of $W_{f,r}$ defined by $\prod_{i=1}^{N} f_i(x) = 0$. It is enough to show $\bar{W}' - E \supset W_0$.

We shall prove it by the reduction to absurdity.

If this were not true, there should be a point $p$ in $W_0$ and a neighborhood $U$ of $p$ such that $U \cap E \supset U \cap W'$. Again by the desingularization theorem of Hironaka, we can find a complex manifold $\tilde{W}$ and a surjective proper map $G: \tilde{W} \to W_{f,r}$ which satisfy the following conditions.

(53) $\tilde{W}$ is the closure of $G^{-1}(W_{f,r} - E)$.

(54) $G^{-1}(W')$ is a hypersurface of $\tilde{W}$.

Let $\psi$ be the defining function of $G^{-1}(W')$. Then, at a generic point of $G^{-1}(U \cap W')$, we have

(55) $(\sigma_i - a_i \sigma_i) \circ G = \kappa_i \psi^{k_i}$ ($l = 2, \ldots, N$)

(56) $f_i \circ G = \theta_i \psi^{\nu_i}$ ($l = 1, \ldots, N$),

where $\kappa_i$ and $\theta_i$ are non-vanishing and $k_i \geq 1$ and $\nu_i \geq 0$. Since we have supposed $\prod_{i=1}^{N} f_i(x) = 0$ on $W'$ in a neighborhood of $p$, we have also $\sum_{i=1}^{N} \nu_i \geq 1$. Hence we have there

(57) $\omega = \sum_{i=1}^{N} (\sigma_i \circ G) \frac{d}{d \log(f_i \circ G)}$

$= (\sigma_i \circ G) \left( \sum_{i=1}^{N} a_i \frac{d}{d \log(f_i \circ G)} \right) + \sum_{i=2}^{N} ((\sigma_i - a_i \sigma_i) \circ G) \frac{d}{d \log(f_i \circ G)}$

$= \left( \sum_{i=1}^{N} a_i \nu_i \right) \frac{(\sigma_i \circ G)}{\psi} d\phi + \text{holomorphic form}.$

Since $\omega$ is a holomorphic form, this is a contradiction if $\sum_{i=1}^{N} a_i \nu_i \neq 0$.

Q.E.D.

After proving these preparatory results, it is now easy to study the holonomic character of $\gamma(\lambda) \Phi(x; \lambda)$. Here $\gamma(\lambda)$ denotes the $\gamma$-factor introduced in Theorem 1. First we define the coherent $\mathcal{D}_X$-Module $\mathcal{N}_{-\lambda}$ by

(58) $\mathcal{N}_{-\lambda} = \mathcal{F}_{f_{-\lambda}} / \left( \sum_{i=1}^{N} (s_i - \lambda_i), \mathcal{F}_{f_{-\lambda}} \right)$
for \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{C}^N \). It immediately follows from the definition of \( \mathcal{N}_\lambda \) that the distribution \( \gamma(\lambda) \Phi(x; \lambda) \) satisfies the system of linear differential equations \( \mathcal{N}_\lambda \).

The sheaf \( \mathcal{D}_x[s] \) is contained in \( \mathcal{D}_{x \times \mathbb{C}^N} \) if we regard \( s_j \) as the coordinate functions on \( \mathbb{C}^N \). In the sequel we shall denote by \( \mathcal{N}_{f,s} \) the \( \mathcal{D}_{x \times \mathbb{C}^N} \)-Module

\[
\mathcal{D}_{x \times \mathbb{C}^N} \otimes_{\mathcal{D}_x[s]} \mathcal{N}_{f,s} = \mathcal{D}_{x \times \mathbb{C}^N} / \mathcal{D}_{x \times \mathbb{C}^N} \mathcal{I}.
\]

Then we have the following

**Theorem 17.** (a) The characteristic variety of \( \mathcal{N}_\lambda \) is contained in \( W_0 \) (hence \( \mathcal{N}_\lambda \) is a holonomic system).

(b) The characteristic variety of \( \mathcal{N}_{f,s} \) is contained in

\[
\{(s, x; \xi, \sigma) \in T^*(X \times \mathbb{C}^N); (x, \xi) \in W_0\}.
\]

**Proof.** Let \( a(x, \xi) \) be a holomorphic function on \( T^*X \) vanishing on \( W_0 \) and homogeneous of degree \( r \) in \( \xi \). Then, by Hilbert's Nullstellensatz, there exist an integer \( m \) and holomorphic functions \( g_i(\sigma, x, \xi) \) which are homogeneous of degree \( rm - 1 \) in \( (\sigma, \xi) \) such that

\[
a(x, \xi)^m = \sum_{i=1}^{N} \sigma_i g_i(\sigma, x, \xi) \quad \text{on} \quad W_{f,y}.
\]

By Theorem 2 we can find an integer \( m' \) and \( P(s) \in \mathfrak{J}_{mm'} \) such that

\[
\partial_{mm'}(P(s)) = (a(x, \xi)^m - \sum_{i=1}^{N} \sigma_i g_i(\sigma, x, \xi))^m'.
\]

Therefore, if we regard \( P(s) \) as a differential operator on \( \mathbb{C}^N \times X \) (or if we substitute \( \lambda \) into \( s \)), then the principal symbol of \( P(s) \) is \( a(x, \xi)^{mm'} \). This proves the desired results.

Q.E.D.

§ 7. Concerning the singularity spectrum of \( \gamma(s) \Phi(x; s) \) (regarded as a distribution in \( (x, s) \) which depends holomorphically on \( s \)), Theorem 17 (b) immediately gives the following Theorem 18. (See Theorem 2.1.1 of [10] Chap. III. § 2.1. Note also that a distribution on \( M \) with holomorphic parameters \( s \in \mathbb{C}^N \) is a distribution on \( M \times \mathbb{C}^N \) which satisfies the Cauchy-Riemann equations in \( s \).)
Theorem 18. The singularity spectrum of $\gamma(s)\Phi(x; s)$ is confined to
\[
\{(x, s; \sqrt{-1}(\langle x, dx \rangle + 2 \text{Re} \langle \tau, ds \rangle)\infty) \in \sqrt{-1} S^* (M \times C^n) ; \\
\tau = 0, (x, \xi) \in W_0\}.
\]

However, the conclusion of this theorem is not the best possible one of the sort. One of the typical examples which manifest this fact is given by considering the case where
\[
\sum_{i=1}^{N} \alpha_i \text{grad}_x f_i(x) \quad (\alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1)
\]
is contained in a proper convex cone. In order to take into account such phenomena, we introduced the following set $W_0'(+)\) in [8]. Unfortunately, our proof for the claim announced in Lemma 2 of [8] turns out to be incomplete.\(^{(e)}\) Still we believe that the claim itself should be true. Hence we feel it worth while presenting here as a conjecture.

Conjecture. The singularity spectrum of $\gamma(\lambda)\Phi(x; x)$ (considered as a distribution in $x$) is contained in the following set $W_0'(+)\), if $X_{\text{sing}} = \phi$.

$W_0'(+) = \{(x, \sqrt{-1}\xi)\infty) \in \sqrt{-1} S^* M ; \text{ there exist a sequence } x_m \in X\text{ such that } \varphi_j(x_m) = 0 (j = 1, \cdots, d) \text{ and that converges to } x \text{ with } \prod_{i=1}^{N} f_i(x) = 0 \text{ and sequences } \alpha_m = (\alpha_1^{(m)}, \cdots, \alpha_N^{(m)}) \in \mathbb{R}^+ \times \mathbb{R}^N \text{ and } \beta_m = (\beta_1^{(m)}, \cdots, \beta_N^{(m)}) \in C^d \text{ such that}
\]
\[
\alpha_i^{(m)} f_i(x_m) \to 0 \quad (l = 1, \cdots, N)
\]
\[
\sum_{i=1}^{N} \alpha_i^{(m)} \text{grad}_x f_i(x_m) + \sum_{j=1}^{d} \beta_j^{(m)} \text{grad}_x f_j(x_m) \to \eta_i.
\]

The rest of this paper is devoted to the discussion on some examples which support this conjecture and indicate that it should be very difficult to improve it much. In the study of the examples given below, we concentrate our attention on delicate points and leave the complete argument to the reader.

\(^{(e)}\) One rather trivial case covered by our original argument is the case where $N=1$. 
Example 1. Let \( \mathcal{O} \) be \((tx + \sqrt{-1} 0)^i(ty + \sqrt{-1} 0)^*(\lambda, \mu \gg 0)\). Let 
\((t, x, y; \sqrt{-1}(\tau, \xi, \eta) \infty)\) denote a point in \(\sqrt{-1}S^*\mathbb{R}^4\). At the origin, 
the prescription given by Theorem 18 allows the following set \(I\) as a 
possible singularity set over the origin for \(\mathcal{O}\).

\[
I = \{(0; \sqrt{-1}(\tau, \xi, \eta) \infty); (\tau, \xi, \eta) \neq 0\}.
\]

On the other hand, the prescription given by the conjecture allows only 
the part \(I^+\) of \(I\) where \(\xi \eta \geq 0\).

Actually, we find

\[
\text{S.S.}\mathcal{O} \subset \{(t, x, y; \sqrt{-1}(\tau, \xi, \eta) \infty) \in \sqrt{-1}S^*\mathbb{R}^4; \xi \eta \geq 0\}.
\]

In order to see this, we first decompose \(\mathcal{O}\) as follows:

\[
\mathcal{O} = \mathcal{O}_+ + \mathcal{O}_-, \quad \text{where} \quad \mathcal{O}_+ = (tx + \sqrt{-1} 0)^i(ty + \sqrt{-1} 0)^*Y(t)
\]
\[
\text{and} \quad \mathcal{O}_- = (tx + \sqrt{-1} 0)^i(ty + \sqrt{-1} 0)^*Y(-t),
\]

Here \(Y(t)\) is the Heaviside function. Since \(\mathcal{O}_+\) (resp. \(\mathcal{O}_-\)) is a hyper-
function which is holomorphic in \(\{(x, y) \in C^1; \text{Im } x > 0, \text{Im } y > 0\}\) (resp. 
\(\{(x, y) \in C^1; \text{Im } x < 0, \text{Im } y < 0\}\)), we find by [10] Chap. I., §3.2 
(p. 308) that

\[
\text{S.S.}\mathcal{O}_+ \subset \{\xi \eta \geq 0\}.
\]

Hence we find (62). Thus in this case, the more delicate prescription 
based on the conjecture turns out to be a correct one.

Example 2. Let \(\mathcal{O}\) be \((tx + x^2 + \sqrt{-1} 0)^i(ty + z^2 + \sqrt{-1} 0)^*(\lambda, \mu \gg 1)\). Let 
\((t, x, y, z; \sqrt{-1}(\tau, \xi, \eta, \zeta) \infty)\) denote a point in \(\sqrt{-1}S^*\mathbb{R}^4\). Again, 
at the origin the prescription based on the conjecture gives rise to an 
intriguing condition \(\xi \eta \geq 0\), which is not derived from the result stated 
in Theorem 18. In this case, again the prescription based on the conjec-
ture is correct. To see this, we again decompose \(\mathcal{O}\) into the sum 
\(\mathcal{O}Y(t) + \mathcal{O}Y(-t)\). Then the same argument as in Example 1 succeeds.

From the viewpoint of applications to the problems in physics (e.g., 
[9], [12]), it would be more desirable if we could choose \(x_m\) to be 
real in the definition of \(W^e(+)\). The following Example 3 might give
us a hope for such an improvement. Such a hope, however, is nullified by the subsequent Example 4. See also a very interesting paper [12] for related topics.

**Example 3.** Let \( \Phi \) be \( |ts|^l (xt^2 + ys^2)^\mu, (\lambda, \mu \gg 0). \)

\[ (\star) \]

Let \((x, y, t, s; \sqrt{-1}(\xi, \eta, \tau, \sigma) \infty) \) denote a point in \( \sqrt{-1} S^* R^l \). Then after some calculation we have

\[
(65) \quad \int e^{t(x^2 + y^2 + t^2)} \Phi \, dt \, dx \, dy \, ds = \int e^{(x^2 + y^2 + t^2)} c (\mu) |ts|^l (\xi^2 - \eta^2) (\xi + i0)^{-\mu-1} (t^2)^{-\mu-1} dt \, ds,
\]

where \( c (\mu) = 2\pi i \exp (\mu \pi i / 2) I (\mu + 1) \). Then S.S. \( \Phi \) is contained in the set \( (\xi \eta \leq 0) \) due to the factor \( \partial (\xi^2 - \eta^2) \). The prescription given by the conjecture does not give this result, while this constraint naturally appears if we assume \( x_m \) to be real in the definition of \( W^*_0 (+) \).

**Example 4.** Let \( \Phi \) be \((x^3 - y^3)^l, (\lambda \not\in N, \lambda \gg 0) \).

\[ (\star\star) \]

Let \((x, y; \sqrt{-1}(\xi, \eta) \infty) \) denote a point in \( \sqrt{-1} S^* R^l \). If we use the recipe given by the conjecture, then there is no constraint on the cotangential component of the possible singularity spectrum of \( \Phi \) at the origin. On the other hand, if we suppose that \( x_m \) should be chosen to be real in the description of \( W^*_0 (+) \), then we have there an additional constraint \( \xi \eta \leq 0 \). However, as we shall see below, a point \((0, 0; \sqrt{-1}(\xi, \eta) \infty) \) with \( \xi \eta \geq 0 \) really appears in the singularity spectrum of \( \Phi \).\( (***) \)

This implies that we cannot keep \( x_m \) to be real.

In order to see this, it suffices to show that the following Radon transform \( R (\xi, \eta) \) of \( \Phi \) does not vanish identically for \( (\xi, \eta) \in R^l \) with \( \xi \eta \geq 0. \)

\[ (\star\star\star) \]

(\( * \)) Note that \( |f(x)|^l \) and \( f(x)^l \) are expressed as a linear combination of \( (f(x) + \sqrt{-1} 0)^l \) and \( (f(x) - \sqrt{-1} 0)^l \) for generic \( \lambda \).

(\( ** \)) As a matter of fact, the argument below implies that S.S. \( \Phi \) contains any point \((0, 0; \sqrt{-1}(\xi, \eta) \infty) \) with \( (\xi, \eta) \not= (0, 0) \).

(\( *** \)) Since \((x^3 - y^3)^l \) is homogeneous in \((x, y) \), we can use the Radon transform of \( \Phi \) instead of its Fourier transform. Note also that there is no contribution to the set \( \{\xi \eta \geq 0\} \) from the points different from the origin. This is the reason why we do not need to use a cut-off function with respect to \((x, y)\)-variables.
Here we may assume without loss of generality that \( \xi + \eta = 1 \). Then what we should show is that the resulting integral \( I(\eta) \) does not vanish identically. In order to see this we calculate \( I(\eta) \) by shifting the path of integration \( \gamma_0 = (-\infty, \infty) \) to \( \gamma_1 \), described in the figure below.

Such a change of the path of integration is legitimate if \( \lambda > 0 \), as the integral along the dotted circle tends to zero as its radius tends to zero. In view of the \( + \sqrt{-1} \) in the integrand of \( I(\eta) \), we may regard \( \eta \) to be a complex number running over the domain \( \{ \text{Im } \eta < 0 \} \). If we move \( \eta \) so that it lies on the segment joining \( \alpha \) and \( \beta \), \( I(\eta) \) acquires the form

\[
-\sqrt{-1} (1 - e^{\pi i \lambda}) e^{-\pi i \lambda} \int_{\lambda \sqrt{3}}^{\infty} \left( x - \frac{1}{2\sqrt{3}} \right)^{\lambda} \left( x + \frac{1}{2\sqrt{3}} \right)^{\lambda} \times \\
\left( x - \frac{1}{\sqrt{-1}} \left( \eta - \frac{1}{2} \right) \right)^{-3\lambda - 1} dx.
\]

Clearly the integral

\[
\int_{\lambda \sqrt{3}}^{\infty} \left( x - \frac{1}{2\sqrt{3}} \right)^{\lambda} \left( x + \frac{1}{2\sqrt{3}} \right)^{\lambda} \left( x - \frac{1}{\sqrt{-1}} \left( \eta - \frac{1}{2} \right) \right)^{-3\lambda - 1} dx
\]
converges for \( \lambda \gg 0 \). Furthermore, its integrand is non-negative if \( \eta = \frac{1}{2} + \sqrt{-1} \zeta \) with \( \zeta < \frac{1}{2\sqrt{3}} \). Therefore \( I(\eta) \) is different from zero for \( \eta = \frac{1}{2} + \sqrt{-1} \zeta \) with \( \zeta < \frac{1}{2\sqrt{3}} \). This implies that \( I(\eta) \) cannot be identically zero for real \( \eta \). This completes the proof of the assertion that a point \((0, 0; \sqrt{-1}(\xi, \eta) \infty)\) with \( \xi \eta > 0 \) really appears in the singularity spectrum of \( \Phi \).

References


Added in proof: The statement in page 9, line 4 is erroneous, because \( \tilde{P}_i^\dagger \)'s and \( \tilde{Q}_j^\dagger \)'s do not satisfy the commutation relation. Hence the proof of Proposition 7 is not correct. Although the proof of Theorem 2 depends on Proposition 7, if we replace \( \mathcal{M}_{i,r} \) with \( \mathcal{M}_{i,t,\alpha} \), it does not depend on Proposition 7. Here \( \alpha = (\alpha_1, \cdots, \alpha_i) \), \( \beta = (\beta_1, \cdots, \beta_i) \) \( \in \mathbb{C}^i \) and \( \mathcal{M}_{i,t,\alpha} \) is obtained from \( \mathcal{M}_{i,r,\alpha} \) by letting \( s_j \) subject to the relation \( s_j = \alpha_\beta = \alpha_j \beta_j \) with one indeterminate \( s \). Therefore the proof of Theorem 18 is complete as it stands. The detailed corrections will be submitted to this journal. See also our paper “On the characteristic variety of a holonomic system with regular singularities,” which will appear in Adv. in Math. It gives a complete proof for a generalization of Theorem 18.