Multiple Time Analyticity of a Quantum Statistical State Satisfying the KMS Boundary Condition

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Abstract
A multiple time expectation \( \varphi(AB_{t_1} \cdots B_n(t_n)) \) in a stationary state \( \varphi \) satisfying the KMS boundary condition is studied. It is found to be holomorphic in a simplicial tube domain \( 0 < \text{Im } t_1 < \text{Im } t_2 < \cdots < \text{Im } t_n < \beta \), continuous and bounded in the closure and the expectation of cyclic permutation of operators are obtained as its values on various distinguished boundaries of the domain.

§1. Introduction

The Gibbs ensemble in quantum statistical mechanics satisfies the Kubo-Martin-Schwinger (KMS) boundary condition and a general property of such a state has been discussed by several authors [1], [2], [3], [4]. In this paper we shall study the analyticity of \( \varphi(AB_{t_1}(t_1) \cdots B_n(t_n)) \) in \( t_1 \cdots t_n \). The main theorem is Theorem 3.1 and 3.3 of section 3.

In passing, it is shown by the analyticity method that the center of the representing algebra is time translation invariant. It is also pointed out that the KMS boundary condition holds for the weak closure, which will be used in [4].

§2. The KMS Boundary Condition and Analyticity

We shall discuss an analyticity tube domain for single time expectation function in this section. We also give a proof that the center of the representative algebra is elementwise time translation invariant.

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Let $\mathfrak{A}$ be a $C^*$ algebra, $\tau(t)$ be a one parameter group of automorphisms of $\mathfrak{A}$, continuous in $t$, and $\varphi$ be a state of $\mathfrak{A}$ invariant under $\tau(t)$:

$$\varphi(A) = \varphi(\tau(t)A), \quad A \in \mathfrak{A}. \tag{2.1}$$

**Definition 2.1.** $\varphi$ satisfies the KMS boundary condition if

$$\int \varphi(A\tau(t)B)\tilde{f}_\beta(t)dt = \int \varphi([\tau(t)B]A)f_\beta(t)dt, \tag{2.2}$$

$$f_\beta(t) = \int_{-\infty}^{\infty} f(p) e^{-i\beta t - sp} dp \tag{2.3}$$

for arbitrary two elements $A$ and $B$ of $\mathfrak{A}$ and for arbitrary function $f$ in the class $\mathcal{D}$.

**Lemma 2.2.** If $\varphi$ is $\tau(t)$ invariant and satisfies the KMS boundary condition, and $A$ and $B$ are elements of $\mathfrak{A}$, then there exists a function $F(\xi)$ of a complex variable $\xi$ such that

1. $F$ is continuous and bounded for $0 \leq \text{Im} \xi \leq \beta$.
2. $F$ is holomorphic for $0 < \text{Im} \xi < \beta$.
3. For real $t$,

$$F(t) = \varphi(A\tau(t)B), \quad F(t + i\beta) = \varphi([\tau(t)B]A). \tag{2.4}$$

**Proof.** We note that a representation $\pi_{\varphi}$ of $\mathfrak{A}$ on a Hilbert space $H_{\pi}$, a cyclic vector $\Omega_{\pi}$ and a continuous one parameter group of unitary operator $U_{\pi}(t)$ are uniquely determined by the relation

$$\varphi(A) = (\Omega_{\pi}, \pi_{\varphi}(A)\Omega_{\pi}) \tag{2.5}$$

$$U_{\pi}(t)\pi_{\varphi}(A)\Omega_{\pi} = \pi_{\varphi}(\tau(t)A)\Omega_{\pi}. \tag{2.6}$$

In particular,

$$\varphi(A\tau(t)B) = (\Omega_{\pi}, \pi_{\varphi}(A)U_{\pi}(t)\pi_{\varphi}(B)\Omega_{\pi}) \tag{2.7}$$

is a Fourier transform of a finite complex measure $\mu_0$:

$$\varphi(A\tau(t)B) = \int e^{i\beta t} d\mu_0(p). \tag{2.8}$$

Similarly

$$\varphi([\tau(t)B]A) = \int e^{i\beta t} d\mu_0(p). \tag{2.9}$$
A complex finite measure can be considered as a dual to the Banach space $C_0$ of bounded continuous functions vanishing at infinity, in which $\mathcal{D}$ is dense. Hence (2.2) implies

\begin{equation}
(2.10) \quad d\mu_0 = e^{\beta p} d\mu_\beta.
\end{equation}

Let $\chi$ be the characteristic function of $(0, \infty)$ and set

\begin{equation}
(2.11) \quad d\mu = \chi d\mu_0 + (1-\chi)d\mu_\beta.
\end{equation}

It is a finite complex measure. Let

\begin{equation}
(2.12) \quad g_\alpha(p) = e^{-\alpha p} \chi(p) + e^{\beta - \alpha} p (1-\chi(p)),
\end{equation}

which is a bounded continuous function if $0 \leq \alpha \leq \beta$. Therefore

\begin{equation}
(2.13) \quad d\mu_\alpha = g_\alpha d\mu
\end{equation}

is a finite complex measure and

\begin{equation}
(2.14) \quad F(t + i\alpha) \equiv \int e^{itp} d\mu_\alpha(p)
\end{equation}

is a bounded continuous function of $t$ and $\alpha$ for $-\infty < t < +\infty$, $0 \leq \alpha \leq \beta$. From (2.10), (2.11), (2.12), we see that (2.4) is satisfied. For $0 < \alpha < \beta$, $e^{itp} g_\alpha(p)$ satisfies the Cauchy-Riemann relation with respect to $t + i\alpha$ in the topology of $C_0$ and hence $F(t + i\alpha)$ is holomorphic in $t + i\alpha$ for $0 < \alpha < \beta$.

**Remark 2.3.** The existence of $F$ satisfying (1), (2), (3) is equivalent to the KMS boundary condition. This is known except that the boundedness of $F$ in the tube has not been treated in the literature.

**Lemma 2.4.** Let $\mathfrak{A}_1 = (\pi_\varphi(\mathfrak{A}'))''$, $\varphi_1(A) = (\Omega_\varphi, A\Omega_\varphi)$, $\tau_1(t)A = U_\varphi(t)AU_\varphi(t)^{-1}$ $(A \in \mathfrak{A}_1)$. Then $\varphi_1$ satisfies the KMS boundary condition with respect to $\mathfrak{A}_1$ and $\tau_1$, if $\varphi$ satisfies the same with respect to $\mathfrak{A}$ and $\tau$.

**Proof.** We prove (2.2) for $\varphi_1$, $A \in \pi_\varphi(\mathfrak{A})$ and $B \in \pi(\mathfrak{A})''$. A similar argument will then yield (2.2) for general $A$ in $\pi_\varphi(\mathfrak{A})''$. Since $B = 1$ obviously satisfies (2.2), we consider $B$ in the weak closure of $\pi_\varphi(\mathfrak{A})$. By the density theorem, it is enough to consider
$B$ in the weak closure of the unit ball of $\pi_\varphi(\mathfrak{A})$.

Let $T$ be such that $\int_{|t|>T} |f_\varphi(t)| \, dt < \varepsilon$. Since $U_\varphi(t)$ is continuous in $t$, we can find an open interval $I_t$ containing $t$ such that

$$||U_\varphi(t)^{-1}\pi_\varphi(A)\Omega_\varphi - U_\varphi(t')^{-1}\pi_\varphi(A)\Omega_\varphi|| < \varepsilon$$

for any $t' \in I_t$. A finite number of such $I_t \cdots I_{tm}$ cover the compact interval $[-T, T]$. Let $N$ be the weak neighbourhood of $B$ defined by

$$N = \{B' \mid (U_\varphi(t_j)^{-1}\pi_\varphi(A)\Omega_\varphi, (B - B')\Omega_\varphi) < \varepsilon, j = 1 \cdots n\}.$$ 

Then we have for $B' \in N$, $||B'|| \leq 1$,

$$\int \phi_1(A\tau(t)(B-B'))f_\varphi(t) \, dt \leq 2||A||\varepsilon + 3\varepsilon \int |f_\varphi(t)| \, dt.$$

We have a similar equation for the right hand side of (2.2). Since (2.2) holds for $B' \in \pi(\mathfrak{A})$, we have (2.2) for $B$ in the weak closure of the unit ball of $\pi(\mathfrak{A})$. Q.E.D.

**Corollary 2.5.** The element of the center of $\pi_\varphi(\mathfrak{A})''$ is invariant under $\tau_\varphi(t)$.

**Proof.** Since $U_\varphi(t)RU_\varphi(t)^{-1} = R$ holds for $R = \pi_\varphi(\mathfrak{A})$, it holds for $R = \pi_\varphi(\mathfrak{A})'$ and hence for $R = \pi_\varphi(\mathfrak{A})''$ and therefore for $R = \pi_\varphi(\mathfrak{A})'''$. Thus

$$\phi_1(A\tau_\varphi(t)B) = \phi_1([\tau_\varphi(t)B]A)$$

if $B$ is in the center of $\pi_\varphi(\mathfrak{A})''$. Lemma 2.2 implies the existence of a function $F(\xi)$ which is holomorphic for $0 < \text{Im} \, \xi < \beta$ and continuous for $0 \leq \text{Im} \, \xi \leq \beta$. (2.4) and (2.18) implies, due to the edge of wedge theorem, that $F(\xi)$ is an entire function with period $i\beta$. Since $F$ is bounded, it must be a constant. Since $\tau_\varphi(t)B$ is in the center of $\pi_\varphi(\mathfrak{A})''$, we see that $\phi_1(A_1[\tau_\varphi(t)B]A_2)$ is constant of $t$ and hence $\tau_\varphi(t)B = B$.

**Remark 2.6.** This corollary can be proved also from (2.10) directly. Namely, $\mu_0 = \mu_\varphi$ and (2.10) imply $\mu_0 = c \delta(p) \, dp$, from which it follows that $\phi(A\tau(t)B)$ is independent of $t$. 


This proof can be used to show that a state is invariant under \( \tau(t) \), if it satisfies the KMS boundary condition (This is pointed out by H. Miyata).

The equation (2.10) at \( p=0 \) implies that \( \Omega_p \) is a trace vector for \( E(\{0\}) \pi_p(\mathcal{A}) E(\{0\}) \). This is used in [4].

§ 3. Analyticity of Multiple Time Expectation Values

**Theorem 3.1.** Let \( \varphi \) be a \( \tau(t) \) invariant state of \( \mathcal{A} \) satisfying the KMS boundary condition. Let \( A, B_1, \cdots, B_n \) be arbitrary \( n+1 \) elements of \( \mathcal{A} \) \((n=1, 2, \cdots)\). There exists a function \( H(\zeta_1, \cdots, \zeta_n) \) of \( n \) complex variables such that

1. \( F \) is holomorphic for

\[
0 < \text{Im} \, \zeta_1 < \cdots < \text{Im} \, \zeta_n < \beta
\]

2. The boundary value of \( F \) for \( \text{Im} \, \zeta_1 = \cdots = \text{Im} \, \zeta_j = 0, \text{Im} \, \zeta_{j+1} = \cdots = \text{Im} \, \zeta_n = \beta \) in the distribution sense is the function

\[
\varphi[\tau(t_{j+1})B_{j+1}] \cdots [\tau(t_n)B_n] A[\tau(t_{j})B_j] \cdots [\tau(t_j)B_j]
\]

where \( j=0, \cdots, n \) and \( t_k = \text{Re} \, \zeta_k \).

**Proof.** Let us consider the Fourier transform of (3.2) in distribution sense:

\[
f_j(p_1, \cdots, p_n) = \int \varphi[\tau(t_{j+1})B_{j+1}] \cdots [\tau(t_n)B_n] \times \\
A[\tau(t_{j})B_j] \cdots [\tau(t_j)B_j]) e^{-i(p_1 t_1 + \cdots + p_n t_n)} \times \\
\text{d}t_1 \cdots \text{d}t_n / (2\pi)^n
\]

(2.2) implies

\[
f_{j+1}(p_1, \cdots, p_n) = e^{\beta p_{j+1}} f_j(p_1, \cdots, p_n).
\]

Let \( \chi_j \) be the characteristic function of the following region

\[
B_j = \{(p_1, \cdots, p_n); p_k + p_{k+1} + \cdots + p_j > 0, \ k=1, \cdots, j \}
\]

\[
p_{j+1} + \cdots + p_k < 0, \ k=j+1, \cdots, n \}
\]

where \( j=0, \cdots, n \). Let \( g \) be a nonnegative function in the class \( \mathcal{D} \) such that
(3.6) \[ \int g(p_1, \ldots, p_n)dp_1 \cdots dp_n = 1 \]

and \( \chi_j^g \) be the regularization of \( \chi_j \) by \( g \):

(3.7) \[ \chi_j^g(p_1, \ldots, p_n) = \int \chi_j(p_1 - r_1, \ldots, p_n - r_n)g(r_1, \ldots, r_n)dr_1 \cdots dr_n. \]

If we denote a vector with the first \( k \) components equal to 1 and the last \( (n-k) \) components equal to 0 by \( q_k \) (\( k=0, \ldots, n \)), then

\[ \bar{B}_j = \{ p; \max_k (p, q_k) = (p, q_j) \}. \]

From this we see that \( \bigcup_j \bar{B}_j \) is the entire space and \( \bar{B}_j \cap \bar{B}_k \) is in the plane orthogonal to \( q_j - q_k \) if \( j \neq k \), namely \( \dim \bar{B}_j \cap \bar{B}_k < n \). Hence

(3.8) \[ \sum_{j=0}^{n} \chi_j^g = 1. \]

Let us define the following distribution

(3.9) \[ H(p_1, \ldots, p_n; t_1 + i\alpha_1, \ldots, t_n + i\alpha_n) = \sum \chi_j^g(p_1, \ldots, p_n)h_j(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)f_j(p_1, \ldots, p_n) \exp i \sum_{j=1}^{n} t_j p_j \]

where

(3.10) \[ h_j(p_1 \ldots p_n; \alpha_1 \ldots \alpha_n) = \exp \{ \sum_{k=j+1}^{n} (\alpha_{k+1} - \alpha_k)(p_{j+1} + p_{j+2} + \cdots + p_k) \]

\[ - \sum_{k=1}^{j} (\alpha_k - \alpha_{k-1})(p_k + p_{k+1} + \cdots + p_j) \}

and \( \alpha_0 = 0, \alpha_{n+1} = \beta \). If

(3.11) \[ 0 < \alpha_1 < \cdots < \alpha_n < \beta, \]

then (3.10) implies that \( h_j \) decreases exponentially whenever \( (p, q_j - q_l) \) tends to \( +\infty \) for one \( l \). On the other hand the part of \( B_j \), in which \( (p, q_j - q_l) < R \) for all \( l \) and a fixed \( R > 0 \), is compact. Hence

\[ \exp i(\sum_{k=1}^{n} t_k p_k)h_j(p_1 \ldots p_n; \alpha_1 \ldots \alpha_n)\chi_j^g(p_1 \ldots p_n) \]

is in the class \( S \) and satisfies the Cauchy-Riemann relation with respect to each \( t_k + i\alpha_k \). We now define
which is holomorphic for $\zeta$ satisfying (3.1). Furthermore, the Fourier transform of the boundary value of $F$ for $\text{Im} \zeta_1 = \cdots = \text{Im} \zeta_j = 0$, $\text{Im} \zeta_{j+1} = \cdots = \text{Im} \zeta_n = \beta$ becomes

\begin{equation}
\sum \chi_h^\alpha(p_1 \cdots p_n)h_k(p_1 \cdots p_n; \alpha_1 \cdots \alpha_n)f_h(p_1 \cdots p_n)
\end{equation}

where $\alpha_1 = \cdots = \alpha_j = 0$, $\alpha_{j+1} = \cdots = \alpha_n = \beta$. From (5.10) we have

\begin{equation}
h_k(p_1 \cdots p_n; \alpha_1 \cdots \alpha_n) = \begin{cases} 
\exp \beta(p_{k+1} + \cdots + p_j) & \text{if } k < j \\
1 & \text{if } k = j \\
\exp -\beta(p_{j+1} + \cdots + p_k) & \text{if } k > j
\end{cases}
\end{equation}

Hence, from (3.4), we have

\begin{equation}
h_k(p_1 \cdots p_n; \alpha_1 \cdots \alpha_n)f_h(p_1 \cdots p_n) = f_j(p_1 \cdots p_n)
\end{equation}

for all $k$. By using (3.8), we see that the boundary value in question is (3.2).

**Remark 3.2.** (i) The above theorem and its proof are stated in a form which holds for Wightman fields. The next theorem uses the fact that $A$ and $B_1$ are bounded operators. (ii) In the discussion of the analyticity, it is more symmetric to consider

\begin{equation}
\varphi(A_1(t_1) \cdots A_n(t_n))
\end{equation}

on the space $\{(t_1 \cdots t_n) \mod (1, \cdots, 1)\}$. The step function $\chi_j$ can be written in terms of the edge vectors of the simplicial domain in question. For such a technique, see generalized $\theta$ function introduced in [5].

**Theorem 3.3.** The function $F$ in Theorem 3.1 is continuous and bounded in the closure of the simplicial tube domain (3.1).

**Proof.** We investigate each summand more closely. By definition (3.3), we have

\begin{equation}
\begin{cases} 
\int h_k(p_1 \cdots p_n) \exp i\{\sum(t_i+s_i)p_i\} dp_1 \cdots dp_n \\
= (Q_\varphi Q_{k+1}U_\varphi(s_{k+1})Q_{k+2} \cdots Q_nU_\varphi(-s_n)Q_0U_\varphi(s_1) \\
Q_\varphi(U_\varphi(s_1-s_0) \cdots U_\varphi(s_n-s_{n-1}))Q_{k+2}Q_0 \\
= \pi_\varphi[\tau(t_i)B_1], \quad Q_0 = \pi_\varphi[A].
\end{cases}
\end{equation}
On the other hand

\begin{equation}
(3.18) \quad \chi_k(p_1 \cdots p_n) = \prod_{i=1}^{k} \theta(p_i + \cdots + p_n) \prod_{j=k+1}^{n} \theta(-(p_{k+1} + \cdots + p_j))
\end{equation}

\begin{equation}
(3.19) \quad h_k(p_1 \cdots p_n; \alpha_1 \cdots \alpha_n)
\end{equation}

\begin{equation}
\begin{aligned}
= \prod_{i=1}^{k} e^{-\alpha_i} e^{-a_{i-1} \alpha_i} e^{a_i} p_i \cdots p_{k+1} \prod_{j=k+1}^{n} e^{a_j} e^{j-1} e^{a_{j+1}} p_{k+1} \cdots p_j
\end{aligned}
\end{equation}

where \( \theta \) is the characteristic function for positive reals. We note that

\begin{equation}
(3.20) \quad \sum t_j p_j = t_1 (p_1 + \cdots + p_n) + (t_2 - t_1)(p_2 + \cdots + p_n) + \cdots
\end{equation}

\begin{equation}
+ (t_{k-1} - t_{k-2})(p_{k-1} + p_{k-2}) - (t_{k-2} - t_{k-3})(p_{k-2} + p_{k-3}) - \cdots - (t_n - t_{n-1})(p_{n-1} + p_n) + t_n (p_{n-1} + \cdots + p_n).
\end{equation}

If we set

\begin{equation}
(3.21) \quad \phi(z_1 \cdots z_n) = \int h_k(p_1 \cdots p_n) \exp -i \sum_{i=1}^{n} z_i p_i d \phi_1 \cdots d \phi_n
\end{equation}

we have

\begin{equation}
\begin{aligned}
(3.22) \quad & \int h_k(p_1 \cdots p_n; \alpha_1 \cdots \alpha_n) \chi_k(p_1 \cdots p_n) \exp -i (\sum s_i \phi_i) d \phi_1 \cdots d \phi_n / (2\pi)^n

= \tilde{g}(z_1 \cdots z_n) \theta(z_1) \theta(z_2 - z_1) \cdots \theta(z_k - z_{k-1}) \theta(z_{k+1} - z_{k+1}) \cdots

\cdots \theta(-z_n)
\end{aligned}
\end{equation}

where

\begin{equation}
(3.23) \quad z_l = s_l - i\alpha_l, \quad l = 1 \cdots k
\end{equation}

\begin{equation}
(3.24) \quad z_l = s_l - i\alpha_l + i\beta, \quad l = k+1 \cdots n.
\end{equation}

Combining (3.17) and (3.22), we obtain the following expression for the integral of the \( k \)th term of (3.9).

\begin{equation}
(3.25) \quad \int \tilde{g}(z_1 \cdots z_n) d s_1 \cdots d s_n(\Omega_{\varphi}, Q_{k+1} X(z_{k+1} - z_{k+1}) Q_{k+2} \cdots

\cdots Q_{\varphi} X(-z_n) Q_{\varphi} X(z_1) Q_{\varphi} X(z_2 - z_1) \cdots X(z_k - z_{k-1}) Q_{\varphi} \Omega_{\varphi})
\end{equation}

where

\begin{equation}
(3.26) \quad X(z) = U_{\varphi}(\text{Re } z) \theta(z).
\end{equation}

For any testing function \( \tilde{g} \) in the class \( \mathcal{O} \), we have
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\[ g(t) = \int e^{-it\eta} \bar{g}(\eta) d\eta, \]

where \( \text{Im} \, \zeta \geq 0, \, \text{Re} \, \zeta = 0, \)

\[ U^+(\zeta \mid q) = \int_q e^{i\zeta \lambda} dE(\lambda), \]

\( E \) is the spectral projection of \( U_v(t) \).

The integral in (3.29) is ambiguous at the lower end but this ambiguity does not affect (3.27). If the lower end is \( q \pm 0, \) we denote \( U^\pm \). We define (3.29) as an average of \( U^+ \) and \( U^- \).

The expression (3.25) is then equal to

\[ \int g(p_1 \cdots p_n) \Omega \Omega' Q_{k+1} U^+(\zeta_{k+1} ; q_{k+1}) Q_{k+2} \]

where

\[ (3.31) \quad \zeta_l = i(\alpha_{l+1} - \alpha_l) \quad l = 0, \ldots, n \]

\[ (3.32) \quad \alpha_{n+1} = \beta, \quad \alpha_0 = 0 \]

\[ (3.33) \quad q_l = \begin{cases} p_l + \cdots + p_k & \text{if } l \leq k \\ -(p_{k+1} + \cdots + p_l) & \text{if } l > k \end{cases} \]

We now take the limit of sequence \( g = g^{(\nu)} \) such that \( \int g^{(\nu)} d\eta \cdots d\eta_n = 1, \) \( g^{(\nu)} \geq 0; \) \( g^{(\nu)} = 0 \) for \( \sum p_i^2 \geq (1/\nu). \) Let \( B(\sigma_1 \cdots \sigma_n) \) be the region in which \( \sigma_i q_i > 0 \) for all \( i, \) where \( \sigma_i = \pm 1. \) Assume that

\[ (3.35) \quad \mu_\nu(\sigma_1) \cdots (\sigma_n) = \lim_{\nu \to \infty} \int g^{(\nu)} d\eta \cdots d\eta_n. \]

Then the limit of (3.30) is

\[ \sum_{\sigma_1 \cdots \sigma_n} \mu_\nu(\sigma_1 \cdots \sigma_n) \Omega \Omega' Q_{k+1} U^+(\zeta_{k+1} ; 0) Q_{k+2} \]

\[ \cdots Q_n U^+(\zeta_n ; 0) Q_{n+1} U^+(\zeta_{n+1} ; 0) Q_{n+2} \cdots \]

where \( \mu \geq 0 \) and \( \sum \mu(\sigma_1 \cdots \sigma_n) = 1. \)

In obtaining (3.36) we have used the fact that \( U^+(\zeta \mid q) - U^+(\zeta \mid 0) \) strongly tends to zero as \( q \to 0 \) with \( \sigma q > 0, \) and that \( ||U^+(\zeta \mid q)|| \) is bounded uniformly in \( q \) in the neighbourhood of 0.
Since
\[ U stated) \int_{\sigma_0}^+ e^{i\lambda \xi} \, dE(\lambda) \]
is bounded and continuous for \( \text{Im } \xi > 0 \) (and holomorphic for \( \text{Im } \xi > 0 \)), (3.36) is bounded and continuous in \( \text{Im } \xi \geq 0 \).

**Corollary 3.4.** \( F \) is given by
\[ F(\xi_1, \ldots, \xi_n) = \sum \mu_k(\xi_1, \ldots, \xi_n) U_{\xi_1+\cdots+\xi_n} \]
and
\[ F^+_{\xi_1+\cdots+\xi_n} = \sum \mu_k(\xi_1, \ldots, \xi_n) \Omega \]
where
\[ \mu_k(\xi_1, \ldots, \xi_n) = \frac{1}{\pi} \int_{\sigma_0}^+ e^{i\lambda \xi} \, dE(\lambda) \]
and \( \mu_k(\xi_1, \ldots, \xi_n) \) is the volume of those parts of the ball \( \mathcal{B}(K) \) which is defined by \( \sigma_1 q_1 \leq 0, \ldots, q_n \leq 0 \) for \( l \leq k \), \( q_i^{(k)} = -p_k - \cdots - p_l \) for \( l > k \).

**Proof.** This follows from (3.36) where we take as \( g^{(\nu)} \) a function obtained by smoothly cutting off tails of
\[ C(\nu) \exp \nu \sum_{i=1}^n p_i^2. \]

**Remark 3.5.** If we insert a formal expression
\[ U^+_{\xi} = \frac{1}{2\pi i} \int_{\xi} U^+_{\xi}(t) \frac{dt}{t-\xi} \quad (\text{Im } \xi > 0) \]
into (3.38) and (3.39), we obtain an unsubtracted form of the Bergman Weil formula.

**Remark 3.6.** As a special case of \( n=1 \), we obtain
\[ F(\xi) = \langle \Omega_\psi, Q_0 U^+ (i\beta - \xi) Q_0 \Omega_\psi \rangle + \langle \Omega_\psi, Q_0 U^+ (\xi) Q_0 \Omega_\psi \rangle \]
where
\[ U^+ (\xi) = \int_{\sigma_0}^+ e^{i\lambda \xi} \, dE(\lambda) + \frac{1}{2} E(\{0\}). \]
By setting \( \xi = 0 \), we have
\begin{equation}
(\Omega_\varphi, Q_b U^+(i\beta)Q_a \Omega_\varphi) = (\Omega_\varphi, Q_a (1 - U^+(0)) Q_b \Omega_\varphi).
\end{equation}

\textbf{References}
