Theory of Connexes. II

By
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Introduction

Here we have a display of the famous game named Hex, where two players White and Black occupy the vertices in the rhombus and who obtains a path between his initially posed pieces wins. It is remarkable that this game always gives a single winner. Regarding the board as the upper half of the sphere, we notice the following statement:

![Figure 1](image)

Suppose there be a simplicial decomposition of the sphere invariant by the antipodal mapping. If two players occupy whole the dipoles of vertices, then there exists strictly one player who obtains in his territory a connected set of vertices invariant under the antipodal action.

Our purpose in this paper is to prove the above statement in more general situation. We have already proved in [3] the converse of the relevant statement, namely, a graph with an action of $\mathbb{Z}_2$ is essentially spherical besides certain exceptions if it admits the unique winner property.

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§ 1. Preliminary

We fix a set $\Pi$ of two players $\top$ and $\bot$ and the involution $\wedge$ of $\Pi$ namely, $\top = \bot$ and $\bot = \top$. For any finite set $X$, we call a mapping $b$ from $X$ to $\Pi$ as a division on $X$. We consider a compact real 2-dimensional manifold $M$ with an action of a finite group $G$. We consider also a $G$-invariant simplicial decomposition $K = (K^0, K^1, K^2)$ of $M$.

For $i = 0$ or 1, we say two $i$-simplices to be adjacent if they are distinct and are contained in the boundary of an $i+1$-simplex. The connectivity of a subset of $K^i$ is considered with respect to this adjacency. We assume that the action of $G$ is faithful on $K^0$ and that any complete subset of $K^0$ is contained in the boundary of 2-simplex.

For a subset $A$ of $K^0$, we denote by $[A]$ the subset of $M$ defined as follows:

$$[A] = \{ x \in M \mid x \text{ is a point of a simplex whose vertices are all in } A \}.$$

Let $X$ be a subset of $M$. Then we denote by $\bar{X}$ the closure of $X$ and define a subgroup $S(X)$ of $G$ as

$$S(X) = \{ g \in G \mid X \text{ is } g\text{-invariant} \}.$$

**Lemma 1.** Let $A$ be a subset of $K^0$. Then $S([A])$ coincides with $S(A)$. Let $B$ be a connected component of $A$. Then $[B]$ is a connected component of $[A]$.

The proof of this lemma is easy and is omitted.

Let $b$ be a $G$-invariant division on $K^0$. Then we denote by $B$ the set of connected components of the open set $M - \bigcup_{\pi \in H} [b^{-1}(\pi)]$. We fix the division $b$ in the rest of this section. We assume that $b$ is not constant.

**Lemma 2.** Let $\pi$ be a player, $C$ a connected component of $b^{-1}(\pi)$ and $B_C$ a subset of $B$ defined as follows:

$$B_C = \{ \sigma \in B \mid \sigma \cap b^{-1}(\pi) \subseteq C \}.$$

Assume there be given an element $\sigma_0$ of $B_C$. Then

$$S(C) = \{ g \in G \mid g\sigma_0 \in B_C \}.$$

Especially, $S(C)$ contains $S(\sigma_0)$.

This lemma follows immediately the above lemma and the facts that $b$ is $G$-invariant and that $C$ is a connected component.

**Lemma 3.** An element of $B$ is orientable.
Proof. Let \( \mathcal{A} \) be an element of \( \mathcal{B} \). Then for any 1-simplex in \( \mathcal{A} \), there exist exactly two 1-simplices adjacent to it. Touring along the 1-simplices of \( \mathcal{A} \), we obtain an orientation with the 0-simplices occupied by \( \top \) on the right side.

Figure 2

Lemma 4. Let \( \mathcal{A} \) be an element of \( \mathcal{B} \) and \( S_0(\mathcal{A}) \) the set consisting of the elements of \( S(\mathcal{A}) \) which preserve an orientation of \( \mathcal{A} \). Then \( S_0(\mathcal{A}) \) is a cyclic subgroup of \( S(\mathcal{A}) \) of index 1 or 2. If this index is 2, then any element of \( S(\mathcal{A}) \) stabilizes exactly two elements of \( K^1 \cup K^2 \) contained in \( \mathcal{A} \).

Proof. Let \( \Gamma \) be the graph whose vertices are the 1-simplices contained in \( \mathcal{A} \) and the adjacency be defined before. Then there is a natural homomorphism from \( S(\mathcal{A}) \) to the automorphism group of \( \Gamma \), which is injective because \( G \) is faithful on \( K^0 \). Now our lemma is clear.

Lemma 5. Let \( \mathcal{A} \) be an element of \( \mathcal{B} \). Then for each player \( \pi \), \( \mathcal{A}^{-1}(\pi) \cap \mathcal{A} \) is connected and is invariant under \( S(\mathcal{A}) \).

This lemma is easily verified and its proof is omitted.

Let \( \pi \) be a player and \( C \) a connected component of \( \mathcal{A}^{-1}(\pi) \). We define a family \( \mathcal{N}_C \) of connected components of \( \mathcal{A}^{-1}(\mathcal{A}) \) as follows:

\[
\mathcal{N}_C = \{ E \mid E \text{ is a connected component of } \mathcal{A}^{-1}(\pi) \text{ such that } C \cup E \text{ is connected} \}.
\]

For \( E \in \mathcal{N}_C \) we define a subset of \( G \) as follows:

\[
\begin{pmatrix} C \\ E \end{pmatrix} = \{ g \in G \mid gE \in \mathcal{N}_C \}.
\]

We define also a new division \( \mathcal{A}_C \) on \( K^0 \) as follows:

\[
\mathcal{A}_C^{-1}(\pi) = \mathcal{A}^{-1}(\pi) - GC.
\]

Lemma 6. Let the assumptions be as above. Assume moreover that \( \mathcal{A} \) is not constant. Then the stabilizer of the connected component \( C' \) of \( \mathcal{A}_C^{-1}(\mathcal{A}) \) containing \( C \) is given as follows:
\[ S(C') = \left\langle \begin{pmatrix} C \\ E \end{pmatrix} \middle| E \in \mathcal{N}_C \right\rangle. \]

If \(|\mathcal{N}_C| = 1\), moreover, then \(S(C') = S(E)\), where \(E\) is the element of \(\mathcal{N}_C\).

**Proof.** It is evident that \(S(C')\) contains the relevant group. Let \(\gamma\) be an element of \(S(C')\). Then there exists a series \(\gamma_1 C, g_1 E_1, \gamma_2 C, g_2 E_2, \ldots, \gamma_{n-1} C, g_{n-1} E_{n-1}, \gamma_n C\) for a positive integer \(n\) where \(E_i \in \mathcal{N}_C\), \(g_i \in G\), \(\gamma_i \in G\), \(\gamma_1^{-1} \gamma_n = \gamma\) and the union of each neighbouring two is connected. If \(n = 1\), then \(\gamma_1^{-1} \gamma_n \in S(C)\). If \(n \geq 2\), then for \(1 \leq i \leq n - 1\)

\[ \gamma_i^{-1} g_i \text{ and } \gamma_{i+1}^{-1} g_i \in \begin{pmatrix} C \\ E_i \end{pmatrix}. \]

Therefore \(\gamma = \gamma_1^{-1} \gamma_n\) is an element of the relevant group. The latter part is evident.

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**§ 2. Linear Groups on the Unit Sphere (1)**

From now on we assume \(M\) as the unit sphere in \(\mathbb{R}^3\). For a positive integer \(n\) we define \(3 \times 3\)-matrices \(g_-(n)\) and \(g_+\) as follows:

\[
g_-(n) = \begin{pmatrix} \cos \frac{\pi}{n} & \sin \frac{\pi}{n} & 0 \\ -\sin \frac{\pi}{n} & \cos \frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]
In this section we assume $G$ as one of the linear groups $G_-(n) = \langle g_-(n) \rangle$ and $G_+(n) = \langle g_-(n)^2, g_+ \rangle$ with the usual action on $M$. In case $G = G_+(n)$, we assume $n \geq 2$. We fix a $G$-invariant simplicial decomposition $K = (K^0, K^1, K^2)$ of $M$.

**Theorem.** Let the assumptions be as above. Let $\mathcal{b}$ be a $G$-invariant division on $K^0$. We regard a player $\pi$ as a winner if $\mathcal{b}^{-1}(\pi)$ has a $G$-invariant connected component. Then there exists a unique winner.

**Proof.** This statement is obvious if $\mathcal{b}$ is a constant mapping. Suppose it to be false and let $\mathcal{b}$ be a counter example minimal with respect to the number $|\mathcal{B}|$. We choose a pair $(\mathcal{C}, [C])$ of an element $\mathcal{C}$ of $\mathcal{B}$ and a connected component $[C]$ of $M - \mathcal{C}$ such that $[C]$ is minimal. Then, by the Jordan curve theorem, $\mathcal{C}$ is the only element of $\mathcal{B}$ whose closure intersects with $C$. Lemma 2 tells us $S(C) = S(\mathcal{C})$. If $G = G_+(n)$ for $n \geq 2$, then

$$S(C) = S(\mathcal{C}) = S(\mathcal{C}) \setminus g_-(n),$$

and if $G = G_-(n)$, then by Lemma 4

$$S(C) = S(\mathcal{C}) \setminus g_-(n).$$

In any way, we have $S(C) \neq G$.

Now we consider a division $\mathcal{b}_C$ with respect to the player $\pi = \mathcal{b}(C)$. Then $\mathcal{b}_C^{-1}(\pi) \subset b^{-1}(\pi)$. We have seen above that there is no $G$-invariant connected component of $b^{-1}(\pi)$ besides the ones of $\mathcal{b}_C^{-1}(\pi)$. On the other hand, by Lemma 6, every $G$-invariant connected component $\mathcal{b}_C^{-1}(\mathcal{C})$ remains a connected component even if it is restricted to $b^{-1}(\mathcal{C})$. This contradicts the minimality of $\mathcal{b}$.
§3. Linear Groups on the Unit Sphere (2)

In the last section we have studied the action of $G_-(n)$ and $G_+(n)$ on the unit sphere. We know that the finite linear group of degree 3 is conjugate in $\text{SL}(3, \mathbb{R})$ to a subgroup of $\langle g_-(n), g_+ \rangle$ for a positive integer $n$ or of polyhedral groups. Then it is still possible that the simplicial decomposition in the last section admits an action of larger groups. We give here an example.

Let $G$ be the group generated by the reflections on $xy$, $yz$- and $zx$-planes, which contains $G_-(1)$. Let $K = (K^0, K^1, K^2)$ be a $G$-invariant simplicial decomposition of $M$. Let $\mathfrak{b}$ be a $G$-invariant division on $K^0$. Then one of $\mathfrak{b}^{-1}(\top)$ and $\mathfrak{b}^{-1}(\bot)$ has a $G_-(1)$ invariant connected component by our theorem, which is $G$-invariant. This causes the following proposition.

**Proposition.** Let $K = (K^0, K^1, K^2)$ be a simplicial decomposition of a triangle and $\mathfrak{b}$ a division on $K^0$. Then exactly one of $\mathfrak{b}^{-1}(\top)$ and $\mathfrak{b}^{-1}(\bot)$ contains a connected components which intersects each edge of the initial triangle.

If the initial triangle is on a plane and any 1-simplex is parallel to an edge of the previous triangle, then this example is equivalent to what Komiya [1] calls trinitrix, which was announced to the author by his friend Mr. Tsujino.

![Figure 5](image.png)
Bibliography


