Quantum Toroidal Algebras and Their Vertex Representations

By

Yoshihisa SAITO*

Abstract

We construct the vertex representations of the quantum toroidal algebras $U_q(\mathfrak{g}_{n+1,tor})$. In the classical case the vertex representations are not irreducible. However in the quantum case they are irreducible.

§1. Introduction

The classical toroidal algebras have been studied by many authors [MEY], [S], [Y], etc. Here "classical" means $q = 1$. The definition of the quantum toroidal algebras is given in [GKV]. They gave a geometric realization of the quantum toroidal algebras without any results on their representation theory. Recently Varagnolo and Vasserot [VV] proved Schur-type duality between representations of the quantum toroidal algebras and the double affine Hecke algebra introduced by Cherednik [C]. This is an analogue of the duality between the quantum affine algebras and the affine Hecke algebras given by Chari and Pressly [CP]. In [VV] only the representations of "trivial central charge" was studied. It is known that there are two subalgebras $U_q^{(1)}(\mathfrak{g}_{n+1})$ and $U_q^{(2)}(\mathfrak{sl}_{n+1})$ of $U_q(\mathfrak{g}_{n+1,tor})$ such that there are surjective algebra homomorphisms $U_q(\mathfrak{g}_{n+1}) \rightarrow U_q^{(i)}(\mathfrak{sl}_{n+1})$ for $i = 1, 2$. In this paper we say that $M$ has a level $(0,0)$ instead of the trivial central charge. The first 0 means that $M$ has a level 0 as a $U_q^{(1)}(\mathfrak{g}_{n+1})$-module and the second 0 means that $M$ has a level 0 as a $U_q^{(2)}(\mathfrak{sl}_{n+1})$-module. This notation is an analogue of level 0 representations of the affine quantum algebras.

In this paper we try to consider an analogue of "integrable representations"
in the toroidal case. Let us recall the integrability of quantum Kac–Moody modules. Let $U_q(g)$ be a quantum Kac–Moody algebra and $V$ a $U_q(g)$-module. We say $V$ is integrable if $V$ has a weight space decomposition and locally nilpotent actions of the Chevalley generators of $U_q(g)$. Therefore the definition of “integrability” needs Chevalley-type generators. The toroidal algebras is defined through Drinfeld type of generators and its Chevalley-type generators are not known. Therefore we are not able to define the integrability at this moment. However, in the affine case, Frenkel-Jing [FJ] realized the integrable representations with level 1 by the vertex representations. Thus if there are “vertex representations” of quantum toroidal algebras, they must be interesting example of the integrable representations still not defined. In the $q = 1$ case, vertex representations of the toroidal algebras have been already considered by Moody–Eswara Rao–Yokonuma [MEY]. In this paper we construct the $q$-analog of the representations defined by them for $g = \mathfrak{sl}_{n+1}$ with level $(1,0)$ and $(1,1)$. Therefore we give a new class of the representations of the quantum toroidal algebras. In the $q = 1$ case, the Fock modules are not irreducible over the Heisenberg algebra and the vertex representations are not irreducible over the toroidal algebra. In the quantum case, it is not the case: the Fock modules are irreducible and also the vertex representations with level $(1,0)$ are irreducible.

The algebra $U_q(g_{tor})$ has infinitely many generators satisfying infinitely many relations (See §2). It is preferable that $U_q(g_{tor})$ is written by finitely many generators with finitely many relations. According to [GKV] there are finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ but the relations among these generators are highly non-trivial. In this paper we give an explicit form of finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ and closed relations of them (See §4). They coincide with the generators by Vasserot [V].

§2. Definition of Quantum Toroidal Algebras

2.1. Notations. Let $g$ be a complex semisimple Lie algebra of type $A_n$ and $\hat{g}$ an affine Kac-Moody Lie algebra of type $A_n^{(1)}$. We denote their Cartan subalgebras by $\mathfrak{h}$ and $\hat{\mathfrak{h}}$ respectively. We denote by $\alpha_1, ..., \alpha_n$ the simple roots of $\mathfrak{g}$, by $\tilde{\alpha}_1, ..., \tilde{\alpha}_n$ the simple coroots of $\mathfrak{g}$, by $\Lambda_1, ..., \Lambda_n$ the fundamental weights of $\mathfrak{g}$, by $\alpha_0, ..., \alpha_n$ the simple roots of $\hat{\mathfrak{g}}$, by $h_0, ..., h_n$ the simple coroots of $\hat{\mathfrak{g}}$ and $\Lambda_0, ..., \Lambda_n$ the fundamental weights of $\hat{\mathfrak{g}}$. Let $Q = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha$, be the root lattice of $\mathfrak{g}$, $P = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha$, the weight lattice of $\mathfrak{g}$, $Q = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha$, the root lattice of $\hat{\mathfrak{g}}$ and $P = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha$, the weight lattice of $\hat{\mathfrak{g}}$. Here $\delta$ is the null root.

We denote the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$ (resp. $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$) by $\langle , \rangle$. The invariant bilinear form on $P$ is given by $\langle \alpha_i, \alpha_j \rangle = -\delta_{ij} - 2\delta_{ij} - \delta_{ij+1}$ and $\langle \delta, \delta \rangle = 0$. The projection form $P$ to $\bar{P}$ is given by $\bar{\Lambda}_i = \Lambda_i - \Lambda_0$ and $\bar{\delta} = 0$. 
2.2. We will give the definition of the quantum toroidal algebra $U_q(\mathfrak{g}_{tor})$.

**Definition 2.2.1.** Let $M = (m_{ij})_{0 \leq i, j \leq n}$ be a skew-symmetric $(n+1) \times (n+1)$ matrix with integral coefficients and let $\kappa$ be an element of $\mathbb{Q}(q)^*$. $U_q(\mathfrak{g}_{tor})$ is an associated algebra over $\mathbb{Q}(q)$ with generators:

$$E_{1,k}, F_{1,k}, H_{i,l}, K_{t}^+, q^{\frac{1}{2}t}, q^{\pm d_i}, q^{\pm d_2},$$

for $k \in \mathbb{Z}$, $l \in \mathbb{Z}\setminus\{0\}$ and $i = 0, 1, ..., n$.

We introduce $K_{t,k}^+$ as the Fourier components of the following generating functions:

$$K_{t}^+(z) = \sum_{k \geq 0} K_{t,k}z^{-k} = K_{t}^+ \exp \left( (q-q^{-1}) \sum H_{i,k}z^{-k} \right),$$

$$K_{t}^-(z) = \sum_{k \leq 0} K_{t,k}z^{-k} = K_{t}^- \exp \left( -(q-q^{-1}) \sum H_{i,-k}z^{k} \right).$$

The defining relations of $U_q(\mathfrak{g}_{tor})$ are then written as follows:

1. $q^{\frac{1}{2}t}$ are central,
2. $K_{t}^+ K_{t}^- = K_{t}^- K_{t}^+ = 1$,
3. $[K_{t}^+, K_{t}^-] = 0$,
4. $[K_{t}^+, H_{i,l}] = 0$,
5. $[H_{i,k}, H_{j,l}] = \delta_{k+l,0} \frac{1}{k} \left[ k \mathbf{h}_1, \mathbf{c} \right] q^{\frac{k}{2} - \frac{k}{2} - \frac{k}{2}} q^{-\frac{k}{2}} \kappa^{-k \alpha_0}$,
6. $[q^{\pm d_i}, K_{t}^+] = 0$,
7. $q^{d_i} H_{j,l} q^{-d_i} = q^l H_{j,l}$,
8. $[q^{\pm d_i}, H_{i,l}] = 0$,
9. $q^{d_i} E_{i,k} q^{-d_i} = q^k E_{i,k}$,
10. $q^{d_i} F_{i,k} q^{-d_i} = q^k F_{i,k}$,
11. $q^{d_2} E_{j,k} q^{-d_2} = q^2 E_{j,k}$,
12. $q^{d_2} F_{j,k} q^{-d_2} = q^2 F_{j,k}$,
13. $K_{t}^+ E_{j,k} = q \left( k \alpha_0 \right) E_{j,k}$,
14. $K_{t}^+ F_{j,k} = q^{-k \alpha_0} F_{j,k}$,
15. $[H_{i,k}, E_{i,l}] = \frac{1}{k} \left[ k \mathbf{h}_1, \mathbf{c} \right] q^{-\frac{k}{2} + \frac{k}{2} - \frac{k}{2}} \kappa^{-k \alpha_0} E_{j,k+l}$. 

Quanum Toroidal Algebras
\[
[H_{i,k}, F_{j,l}] = -\frac{1}{k} [k \langle h_{i}, \alpha_{l} \rangle] q^{\frac{1}{2} k \langle \alpha_{l} \rangle} K^{-k \alpha_{l} \rho} F_{j,k+i}.
\]

(2.2.13)

\[
\kappa^{\mu \nu} E_{i,k} F_{j,l} - q^{-\langle h_{i} \rangle \langle h_{l} \rangle} \kappa^{\mu \nu} E_{i,k} F_{j,l} = q^{-\langle h_{i} \rangle \langle h_{l} \rangle} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k}.
\]

(2.2.14)

\[
[E_{i,k}, F_{j,l}] = \delta_{i,j} \frac{1}{q-q^{-1}} q^{\frac{1}{2} k \langle h_{i} \rangle} K_{k+i}^+ - q^{\frac{1}{2} (i-k) \langle h_{i} \rangle} K_{k+i}^-.
\]

(2.2.15)

\[
\sum_{\sigma \in S_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} E_{i,k_{r_{1}}} \cdots E_{i,k_{r_{m}}} E_{j,l} E_{l,k_{r_{m+1}}} \cdots E_{l,k_{r_{1}}} = 0,
\]

\[
\sum_{\sigma \in S_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} F_{i,k_{r_{1}}} \cdots F_{i,k_{r_{m}}} F_{j,l} F_{l,k_{r_{m+1}}} \cdots F_{l,k_{r_{1}}} = 0,
\]

for \( i \neq j \).

where \( m = 1 - \langle h_{i}, \alpha_{l} \rangle \).

In these relations we denote \([k] = \frac{q^{k} - q^{-k}}{q - q^{-1}}, \quad [n]! = \prod_{k=1}^{n} [k], \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!}.\)

2.3. Let \( U'_{q} (g_{\text{tor}}) \) be the subalgebra of \( U_{q} (g_{\text{tor}}) \) generated by \( E_{i,k}, F_{i,k}, K_{i}^+, H_{i,l} \).

Let \( U''_{q} (g_{\text{tor}}) \) (resp. \( U'^{q}_{q} (g_{\text{tor}}) \)) be the subalgebra generated by \( U'_{q} (g_{\text{tor}}) \) and \( q^{\pm i} \) (resp. \( q^{\pm i} \)). Let \( U_{q}^{(1)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) be the subalgebra generated by \( E_{i}, F_{i}, K_{i}^+, H_{i,l} \).

Let \( U_{q}^{(2)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) be the subalgebra generated by \( E_{i}, F_{i}, K_{i}^+, H_{i,l} \) for \( 0 \leq i \leq n \) and \( U_{q}^{(2)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) the subalgebra generated by \( U_{q}^{(1)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) and \( q^{\pm i} \).

By the definition it is clear that there are surjective homomorphisms \( U_{q} (\hat{\mathfrak{g}}_{n+1}) \to U_{q}^{(2)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) and \( U_{q} (\hat{\mathfrak{g}}_{n+1}) \to U_{q}^{(2)'(i)} (\hat{\mathfrak{g}}_{n+1}) \).

The following are straightforward.

Lemma 2.3.1. For \( 1 \leq i \leq n \) let \( \overline{E}_{i,k} = E_{i,k} K_{i}^\Sigma_{i \alpha_{j}}^{\Sigma_{j} \alpha_{i}} \), \( \overline{F}_{i,k} = F_{i,k} K_{i}^\Sigma_{i \alpha_{j}}^{\Sigma_{j} \alpha_{i}} \), \( \overline{H}_{i,l} = H_{i,l} K_{i}^\Sigma_{i \alpha_{j}}^{\Sigma_{j} \alpha_{i}} \). Then the relations between \( \overline{E}_{i,k} \overline{F}_{i,k}, \overline{H}_{i,l} \) and \( K_{i}^\Sigma_{i \alpha_{j}}^{\Sigma_{j} \alpha_{i}} \) are precisely the relations of Drinfeld generators of \( U_{q} (\hat{\mathfrak{g}}_{n+1}) \). That is, there are surjective homomorphisms \( U_{q} (\hat{\mathfrak{g}}_{n+1}) \to U_{q}^{(1)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) and \( U_{q} (\hat{\mathfrak{g}}_{n+1}) \to U_{q}^{(2)'(i)} (\hat{\mathfrak{g}}_{n+1}) \).

Lemma 2.3.2. Let \( K_{i}^\Sigma_{i \alpha_{j}}^{\Sigma_{j} \alpha_{i}} = \prod_{l=0}^{\infty} K_{i} \). Then \( K_{i} \) are central elements of \( U_{q} (g_{\text{tor}}) \).

Note that \( q^{\pm i} \) is the central elements of \( U_{q}^{(1)'(i)} (\hat{\mathfrak{g}}_{n+1}) \) and \( K_{i} \) the central
§3. Vertex Representations

3.1. Heisenberg algebras. In this section we shall give the vertex representations of $U_q(\mathfrak{g}_{\text{tor}})$. We assume $c=1$.

Consider a $\mathbb{Q}(q)$-algebra $S_n$ generated by $H_{i,l}$ ($0 \leq i \leq n$, $l \in \mathbb{Z} \setminus \{0\}$) satisfying:

\begin{equation}
[H_{i,k}, H_{j,l}] = \frac{1}{k} [k \langle h_i, \alpha_j \rangle] q^k - q^{-k} \delta_{k,0} \delta_{i,0}.
\end{equation}

We call $S_n$ the Heisenberg algebra.

Let $S_n^\ast$ (resp. $S_n$) be the subalgebra of $S_n$ generated by $H_{i,l}$ ($0 \leq i \leq n$, $l > 0$) (resp. $0 \leq i \leq n$, $l < 0$).

We introduce the Fock space

\[ \mathcal{F}_n = S_n v_0 \]

with the defining relations:

\begin{equation}
H_{i,l} v_0 = 0, \quad \text{for } i > 0,
\end{equation}

\begin{equation}
q^\frac{1}{k} v_0 = q^\frac{1}{k} v_0.
\end{equation}

Note that $\mathcal{F}_n$ is a free $S_n^\ast$-module of rank 1.

Let $\mathbb{F}$ be a field of characteristic zero and let $a$ be an associative $\mathbb{F}$-algebra generated by $x_p, y_p (p \in \mathbb{Z}_{>0}), z$ and its inverse $z^{-1}$ with the following relations:

\[ [x_p, z] = [y_p, z] = 0, \]

\[ [x_p, x_r] = [y_p, y_r] = 0, \]

\[ [x_p, y_r] = \delta_{pr} z. \]

Let $a^\ast$ (resp. $a^-$) be the subalgebra of $a$ generated by $x_p$ (resp. $y_p$). We set $b = a^\ast \otimes \mathbb{F} [z, z^{-1}]$. This is a maximal abelian subalgebra of $a$. Fix a nonzero scalar $\lambda \in \mathbb{F}^\ast$. Let $\mathcal{F}_\lambda$ be the one-dimensional space $\mathbb{F}$ viewed as a $b$-module by:

\[ z \cdot 1 = ? \lambda, \quad a^\ast \cdot 1 = 0. \]

Let $F(\lambda)$ be the induced $a$-module

\[ F(\lambda) = \text{Ind}_b^a \mathcal{F}_\lambda = a \otimes \mathcal{F}_\lambda. \]

By the defining relations of $a$ we obtain an $\mathbb{F}$-linear isomorphism

\[ F(\lambda) \cong a^-_. \]
Since \( a^- \) is abelian we may regard it as the algebra of polynomials in the variables \( y_1, y_2, \ldots \). Then we see that \( \varepsilon \) acts on \( a^- \cong \mathbb{F}[y_1, y_2, \ldots] \) by the multiplication of \( \lambda \), \( x_p \) acts by \( \lambda \delta_{xy}^p \). By this realization we immediately have the following lemma.

**Lemma 3.1.1.** \( F(\lambda) \) is an irreducible \( a^- \)-module.

Fix an skew-symmetric \((n+1) \times (n+1)\)-matrix with integral coefficients \( M = (m_{ij})_{0 \leq i,j \leq n} \). We say that \( \kappa \in \mathbb{Q}(q)^* \) is generic with respect to \( M \) if for any \( k \in \mathbb{Z}_{>0} \) the matrix \( \langle [k \langle h_i, \alpha_j \rangle] \kappa^{-km_{ij}} \rangle \) is invertible.

Note that if \( n = 1 \) any \( \kappa \) is not generic with respect to any \( M \). Since the matrix \( \langle [k \langle h_i, \alpha_j \rangle] \rangle_{0 \leq i,j \leq n} \) is invertible for \( n > 1 \), there exists a generic \( \kappa \) for \( n > 1 \).

**Lemma 3.1.2.** (1) For a fixed \( M \), we assume that \( \kappa \in \mathbb{Q}(q)^* \) is generic with respect to \( M \) (in particular \( n > 1 \)). Then \( \mathcal{F}_n \) is an irreducible \( S_n \)-module.

(2) \( \mathcal{F}_1 \) is not irreducible.

**Proof.** (1) Set \( G(k) = \langle (g(k))_{ij} \rangle \langle [k \langle h_i, \alpha_j \rangle] \kappa^{-km_{ij}} \rangle \). Since \( \kappa \) is generic with respect to \( M \) there exists its inverse \( G(k)^{-1} = (g(k))_{ij}^{-1} \) for any \( k \). Note that by the definition \( \sum_{0 \leq s \leq n} g(k)^{ts} g(k)^{st} = \delta_{tt} \).

We set

\[
\tilde{H}_{ik} = \begin{cases} 
\sum_{0 \leq s \leq n} k^{[s]} g(k)^{ts} H_{ik}, & \text{for } k > 0, \\
H_{ik}, & \text{for } k < 0.
\end{cases}
\]

Then we have

\[
[H_{ik}, \tilde{H}_{ij}] = [H_{ik}, \tilde{H}_{ij}] = 0,
\]

for \( k,l > 0 \). Since all \( G(k) \) are regular, \( \tilde{H}_{ik} (0 \leq i \leq n, k > 0) \) generate \( S_n^\ast \).

We shall use Lemma 3.1.1. Put \( \mathcal{F} = \mathbb{Q}(q), a = S_n, a^\pm = S_n^\ast, \lambda = 1, x_p = \tilde{H}_{ik} (k > 0), y_r = \tilde{H}_{i1} (l < 0) \) where \( p = (k-1)(n+1)+i+1 \) and \( r = (-l-1)(n+1)+j+1 \).

Then it is clear that \( F(1) = \mathcal{F}_n \). By Lemma 3.1.1 we conclude that \( \mathcal{F}_n \) is an irreducible \( S_n \)-module.

(2) It is easy to see that \( \kappa^{-km_{ij}} H_{0-i} + H_{1-k} \) is a central element of \( S_1 \) for each \( k \in \mathbb{Z}_{>0} \). Therefore \( \mathcal{F}_1 \) has infinitely many singular vectors. □

### 3.2. Construction of level \((1,0)\) modules.

In this subsection we assume \( \kappa = 1 \). Note that \( 1 \in \mathbb{Q}(q) \) is generic with respect to any \( M \) with \( n > 1 \).
Note that one can rewrite $\bar{P} = \bigoplus_{r=2}^{n} \mathbb{Z} \bar{\alpha}_{r} \bigoplus \mathbb{Z} \bar{\alpha}_{n}$. We introduce a twisted version of the group algebra $Q(q) [\bar{P}]$ by $\mathbb{Z}/2\mathbb{Z}$. We denote it by $Q(q) (\bar{P})$. This is the $Q(q)$-algebra generated by symbols $e^{\bar{\alpha}_{2}}, e^{\bar{\alpha}_{3}}, \ldots, e^{\bar{\alpha}_{n}}, e^{\bar{\alpha}_{n}}$ which satisfy the following relations:

\begin{align}
\tag{3.2.1} e^{\bar{\alpha}_{i}} e^{\bar{\alpha}_{j}} &= (-1) \langle \bar{\alpha}_{i}, \bar{\alpha}_{j} \rangle e^{\bar{\alpha}_{i}} e^{\bar{\alpha}_{j}}, \\
\tag{3.2.2} e^{\bar{\alpha}_{i}} e^{\bar{\alpha}_{n}} &= (-1) \delta_{i} e^{\bar{\alpha}_{i}} e^{\bar{\alpha}_{n}}.
\end{align}

For $\bar{\alpha} = \sum_{i=1}^{n} m_{i} \bar{\alpha}_{i} + m_{n+1} \bar{\alpha}_{n}$ we denote $e^\bar{\alpha} = (e^{\bar{\alpha}_{2}})^{m_{2}} (e^{\bar{\alpha}_{3}})^{m_{3}} \ldots (e^{\bar{\alpha}_{n}})^{m_{n}} (e^{\bar{\alpha}_{n}})^{m_{n+1}}$. For example $e^{\bar{\alpha}_{1}} = e^{-2\bar{\alpha}_{2}} e^{-3\bar{\alpha}_{3}} \ldots e^{-n \bar{\alpha}_{n}} e^{\bar{\alpha}_{n+1}} \bar{\alpha}_{n} e^{\bar{\alpha}_{1}} = e^{-\bar{\alpha}_{1}} e^{-2\bar{\alpha}_{2}} e^{-3\bar{\alpha}_{3}} \ldots e^{-(n-1) \bar{\alpha}_{n}} e^{(n+1) \bar{\alpha}_{n}}$ where $\bar{\alpha}_{i}$ is the $i$-th fundamental weight. We denote $\bar{\alpha}_{0} = - \sum_{i=1}^{n} \bar{\alpha}_{i}$ and $\bar{h}_{0} = - \sum_{i=1}^{n} \bar{h}_{i}$.

Note that $\langle \bar{h}_{i}, \bar{\alpha}_{j} \rangle = \langle \bar{\alpha}_{i}, \bar{\alpha}_{j} \rangle$ for $0 \leq i, j \leq n$.

We denote by $Q(q) (\bar{Q})$ the subalgebra of $Q(q) (\bar{P})$ generated by $e^{\bar{\alpha}_{i}} (1 \leq i \leq n)$.

Set

$$W(p)_{n} = F_{n} \otimes Q(q) (\bar{Q}) e^{\bar{\alpha}_{p}}$$

for $1 \leq p \leq n$

and

$$W(0)_{n} = F_{n} \otimes Q(q) (\bar{Q}).$$

We define the operators $H_{i,j}, e^{\bar{\alpha}_{i}} (\bar{\alpha} \in \bar{Q})$, $\partial_{\bar{\alpha}_{i}}$ on $W(p)_{n}$ for $i = 0, 1, \ldots, n$ as follows:

for $v \otimes e^{\bar{\beta}} = H_{i,1} v \otimes e^{\bar{\beta}}$, $H_{i,k} v \otimes e^{\bar{\beta}} \in W(p)_{n}$,

$$H_{i,j} (v \otimes e^{\bar{\beta}}) = (H_{i,j} v) \otimes e^{\bar{\beta}},$$

$$e^{\bar{\alpha}_{i}} (v \otimes e^{\bar{\beta}}) = v \otimes e^{\bar{\beta}} e^{\bar{\alpha}_{i}},$$

$$\partial_{\bar{\alpha}_{i}} (v \otimes e^{\bar{\beta}}) = \langle \bar{h}_{i}, \bar{\beta} \rangle v \otimes e^{\bar{\beta}}.$$

$$d (v \otimes e^{\bar{\beta}}) = \left( - \sum_{s=1}^{n} k_{s} (\bar{\beta}, \bar{\beta}) + \frac{(\bar{\beta}, \bar{\beta})}{2} \right) v \otimes e^{\bar{\beta}}.$$

We have the following lemma.

**Lemma 3.2.1.** As operators on $W(p)_{n}$,

$$e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{i}} = (-1) \langle \bar{h}_{i}, \bar{\alpha}_{p} \rangle e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{i}},$$

$$e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{p}} = q^{\langle \bar{h}_{p}, \bar{\alpha}_{p} \rangle} e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{p}},$$

$$e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{p}} = z^{\langle \bar{h}_{p}, \bar{\alpha}_{p} \rangle} e^{\bar{\alpha}_{p}} e^{\bar{\alpha}_{p}},$$

for $0 \leq i, j \leq n$. 
We introduce the following generating functions:

\[ E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \]
\[ F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}. \]

**Proposition 3.2.2.** Let \( c = 1 \) and \( \kappa = 1 \). Then for each \( p \) and \( n \), the following action gives a \( U_q^\prime (g_{tor}) \)-module structure on \( W(p)_n \):

\[ q^{\frac{1}{2}i} \mapsto q^{\frac{1}{2}i}, \]
\[ q^{ci} \mapsto q^{ci}. \]

\[ E_i(z) \mapsto \exp \left( \sum_{k \geq 1} \frac{H_{i,-k}}{[k]} (q^{-1/2}z)^k \right) \exp \left( \sum_{k \geq 1} \frac{H_{i,k}}{[k]} (q^{1/2}z)^{-k} \right) e^{\bar{\alpha}_i z^m_{i,-1}}. \]

\[ F_i(z) \mapsto \exp \left( \sum_{k \geq 1} -\frac{H_{i,-k}}{[k]} (q^{1/2}z)^k \right) \exp \left( \sum_{k \geq 1} \frac{H_{i,k}}{[k]} (q^{-1/2}z)^{-k} \right) e^{-\bar{\alpha}_i z^m_{i,-1}}. \]

\[ K_i^+ (z) \mapsto \exp \left( (q^{-1} - q) \sum_{k \geq 1} H_{i,-k} z^{-k} \right) q^{\delta_i}, \]
\[ K_i^- (z) \mapsto \exp \left( -(q^{-1} - q) \sum_{k \geq 1} H_{i,k} z^{k} \right) q^{-\delta_i} \]

for \( 0 \leq i \leq n \).

The proof will be given in Appendix.

We have immediately the following lemma.

**Lemma 3.2.3.** The \( U_q^\prime (g_{tor}) \)-module \( W(p)_n \) is cyclic:

\[ W(p)_n = U_q^\prime (g_{tor}) (v_0 \otimes e^{\delta_i}). \]

**Theorem 3.2.4.** If \( n > 1 \) then \( W(p)_n \) is irreducible for any \( p \).

**Proof.** Since \( \mathcal{F}_n \) is irreducible with respect to the action of \( S_n \), it is enough to show that for any non-zero \( v = v_0 \otimes \sum_{a \in \mathbb{Q} \otimes e^{\delta_i}} a \in \mathbb{Q} \otimes e^{\delta_i} \) there exists \( X \in U_q^\prime (g_{tor}) \) such that \( Xv = v_0 \otimes e^{\delta_i} \). Let \( S_n \) be the subalgebra of \( S_n \) generated by \( H_{i,l} \) \((1 \leq i \leq n, l \in \mathbb{Z} \setminus \{0\})\) and \( q^{\frac{1}{2}c} \). Let \( \mathcal{F}_n \) be the \( S_n \)-submodule of \( \mathcal{F}_n \) generated by \( v_0 \otimes e^{\delta_i} \), and let \( \overline{W(p)}_n = \mathcal{F}_n \otimes \mathbb{Q} \otimes (q) \otimes e^{\delta_i} \). As already known \( \overline{W(p)}_n \) is an irreducible \( U_q^\prime (\mathfrak{sl}_{n+1}) \)-module. It is obvious that \( v \in \overline{W(p)}_n \). Therefore there exists \( X \in U_q^\prime (\mathfrak{sl}_{n+1}) \subset U_q^\prime (g_{tor}) \) such that \( Xv = v_0 \otimes e^{\delta_i} \). \( \square \)

**Remark 3.2.5.** Since \( \mathcal{F}_1 \) is not irreducible as an \( S_1 \)-module \( W(p)_1 \) is not
irreducible.

Remark 3.2.6. Since $U_q^{(1)}(\widehat{sl}_{n+1})$ and $U_q^{(2)}(\widehat{sl}_{n+1})$ are subalgebras of $U_q^{(g_{tor})}$ we can regard $W(p)_n$ as a $U_q^{(1)}(\widehat{sl}_{n+1})$-module or as a $U_q^{(2)}(\widehat{sl}_{n+1})$-module. As a $U_q^{(1)}(\widehat{sl}_{n+1})$-module, $W(p)_n$ is a level 1 module. On the other hand it is a level 0 module as a $U_q^{(2)}(\widehat{sl}_{n+1})$-module.

3.3. On the structure of level (1,0) modules. In this subsection we will study the level 1 $U_q(\widehat{sl}_{n+1})$-module structure of $W(p)_n$.

Let $M$ be a $U_q(\widehat{sl}_{n+1})$-module. We can regard $M$ as a $U_q(\widehat{sl}_{n+1})$-module via a surjective homomorphism $U_q(\widehat{sl}_{n+1})\to U_q^{(1)}(\widehat{sl}_{n+1})$. As a $U_q(\widehat{sl}_{n+1})$-module, we denote the character of $M$ by $\chi_M$.

Let $L(\Lambda_p)$ be the irreducible highest weight $U_q(\widehat{sl}_{n+1})$-module with highest weight $\Lambda_p$. Note that the following identity holds:

$$\chi_{L(\Lambda_p)} = e^{\Lambda_p} \frac{\sum_{\alpha \in \Phi^+} e^{\alpha-a_{i,j}^{(1)}} e^{-\frac{1}{2}(\langle \alpha | a_{i,j}^{(1)} \rangle) \delta}}{\phi(e^{-\delta})^n}.$$  

Here $\phi(x) = \prod_{k \geq 1} (1-x^k)$.

We denote $\delta$ by the null root of $U_q(\widehat{sl}_{n+1})$.

By the definition of $W(p)_n$ and (3.2.1) it is immediate to see the following proposition.

Proposition 3.3.1. As a $U_q^{(1)}(\widehat{sl}_{n+1})$-module, we have

$$\chi_{W(p)_n} = e^{\Lambda_p} \frac{\sum_{\alpha \in \Phi^+} e^{\alpha-a_{i,j}^{(1)}} e^{-\frac{1}{2}(\langle \alpha | a_{i,j}^{(1)} \rangle) \delta}}{\phi(e^{-\delta})^{n+1}} = \chi_{L(\Lambda_p)} \frac{\phi(e^{-\delta})}{\phi(e^{-\delta})^n}.$$  

Lemma 3.3.2. For each $t \in \mathbb{Z}\setminus\{0\}$ there exist $H_t = \sum_{i=0}^n a_{i,t} H_i$ ($a_{i,t} \in \mathbb{Q}(q)$) such that

$$[\hat{H}_t, H_{i,k}] = 0$$

for any $1 \leq j \leq n$ and $k \in \mathbb{Z}\setminus\{0\}$. Moreover such $H_t$ is unique up to scalar.

Proof. Note that $\kappa = 1$. The rank of $n \times (n+1)$-matrix $([t \langle h_i, \alpha_j \rangle])_{1 \leq i \leq n, 0 \leq j \leq n}$ is equal to $n$. The lemma follows form this fact immediately. \qed

By the definition of $\hat{H}_t$ we have

$$[\hat{H}_t, \hat{H}_t] = \delta_{k+1,0} \gamma_t.$$  

(3.3.2)
where $\gamma_k \in \mathbb{Q}(q)$. We fix a normalization of $\widehat{H}_k$ by putting $\gamma_k = 1$ for all $k$.

Let $\widehat{S}_n$ be the subalgebra of $S_n$ generated by $\widehat{H}_k$. By the definition, $\widehat{S}_n$ acts on $W(p)^n$.

The following two lemmas are easy to see.

**Lemma 3.3.3.** For $l > 0$, $\widehat{H}_l (v_0 \otimes e^l) = 0$.

**Lemma 3.3.4.** The action of $U_q(\widehat{S}_{n+1})$ on $W(p)^n$ commutes with the action of $\widehat{S}_n$.

Let $\widehat{S}_n$ be the subalgebra of $\widehat{S}_n$ generated by $\widehat{H}_l$ ($l < 0$).

**Proposition 3.3.5.** As $U_q(\widehat{S}_{n+1})$-module

$$W(p)^n \cong L(A_p) \otimes^\mathrm{res}.$$

**Proof.** Set $\deg(\widehat{H}_k) = k$. Let $M_k = M_k (\widehat{H}_{-1}, \widehat{H}_{-2}, \ldots)$ be a monomial of degree $k$ in variables $\widehat{H}_{-1}, \widehat{H}_{-2}, \ldots$. Then by the above two lemmas, $M_k v_0 \otimes e^l$ is a singular vector of $U_q(\widehat{S}_{n+1})$-module $W(p)^n$. Let $W_{M_k}$ be the $U_q(\widehat{S}_{n+1})$-submodule which is generated by $M_k v_0 \otimes e^l$. Then by the definition of the action of $U_q(\widehat{S}_{n+1})$ on $W(p)^n$, we have

$$W_{M_k} \cong L(A_p - k \delta) \otimes L(A_p).$$

The vectors $(M_k v_0 \otimes e^l)$ are linearly independent. The number of the monomials of degree $k$ is equal to the $k$-th partition number $p(k)$. Therefore there is a $U_q(\widehat{S}_{n+1})$-submodule $W$ of $W(p)^n$ which is isomorphic to $\bigoplus_{k \geq 0} L(A_p - k \delta)^{\otimes p(k)}$. By Proposition 3.3.1 it coincides with $W(p)^n$. This completes proof.\[\Box\]

By Lemma 3.3.4 and the proof of Proposition 3.3.5, the following corollary follows immediately.

**Corollary 3.3.6.** As $U_q(\widehat{S}_{n+1}) \otimes \widehat{S}_n$-module $W(p)^n$ is isomorphic to $L(A_p) \otimes \widehat{S}_n$.

### 3.4. Construction of level $(1,1)$ modules

We introduce a twisted version of the group algebra $\mathbb{Q}(q)[Q]$ by $\mathbb{Z} \backslash \mathbb{Z}$. We denoted it by $\mathbb{Q}(q)\{Q\}$. This is the $\mathbb{Q}(q)$-algebra generated by symbols $e^a$, $e^{a_1}$, $e^{a_2}$, $\ldots$, $e^{a_n}$ which satisfy the following relations:

$$e^{a_i} e^{a_j} = (-1)^{\langle h, a_i \rangle_{e^{a_i}}} e^{a_j} e^{a_i}, \quad (3.4.1)$$

Similarly to §3.2, we denote $e^\alpha = (e^{a_0})^{m_0} (e^{a_1})^{m_1} \ldots (e^{a_n})^{m_n}$ for $\alpha = \sum_{i=0}^n m_i a_i \in \mathbb{Q}$. 

Let
\[ V(p)_n = \mathcal{F}_n \otimes \mathbb{Q}(q)(Q)e^{Ap}. \]
Here we regard \( e^{Ap} \) only a symbol indexed by \( p \).

We define the operators \( H_{i,t}(0 \leq i \leq n, t \neq 0), e^\alpha (\alpha \in Q), \partial \alpha, \) and \( z^{H_{i,s}}(0 \leq i \leq n) \) on \( V(p)_n \) as follows:

for \( v \otimes e^{\beta}e^{Ap} = H_{1i,-k_1}...H_{1n,-k_n}v \otimes e^{\beta}e^{Ap} \in V(p)_n \) \( (\beta = \sum_{k=1}^{n}m_k \alpha_k \in Q) \),
\[ H_{i,t}(v \otimes e^{\beta}e^{Ap}) = (H_{i,t}v) \otimes e^{\beta}e^{Ap}, \]
\[ e^\alpha (v \otimes e^{\beta}e^{Ap}) = v \otimes (e^\alpha e^{\beta})e^{Ap}, \]
\[ \partial \alpha, (v \otimes e^{\beta}e^{Ap}) = (h_i, \beta + \Lambda \alpha) v \otimes e^{\beta}e^{Ap}, \]
\[ z^{H_{i,s}}(v \otimes e^{\beta}e^{Ap}) = z^{(h_i, \beta + \Lambda \alpha)k^s_0} \sum_{t=0}^{n} \sum_{m_\alpha} m_v \otimes e^{\beta}e^{Ap}. \]

The following lemma is easy.

**Lemma 3.4.1.** As operators on \( V(p)_n \),
\[ e^\alpha e^\alpha = (-1)^{\langle \alpha, \alpha \rangle} e^\alpha e^\alpha, \]
\[ q^{-\alpha} e^\alpha = q^{\langle \alpha, \omega \rangle} e^\alpha q^{\omega}, \]
\[ z^{H_{i,s}} e^\alpha = z^{\langle \alpha, \omega \rangle} z^{\sum_{t=0}^{n} \sum_{m_\alpha} m_v \otimes e^{\beta}e^{Ap}.} \]

**Proposition 3.4.2.** Assume \( c = 1 \) then for each \( p \) and \( n \), the following action gives a \( U_q (\mathfrak{g}_{tor}) \) module structure on \( V(p)_n \):
\[ q^{1} \rightarrow q^{1}, \]
\[ q^{d_1} \rightarrow q^{d_1}, \]
\[ q^{d_2} \rightarrow q^{d_2}, \]
\[ E_t(z) \rightarrow \exp \left( \sum_{k \geq 1} H_{t,k} \frac{1}{k^s} (q^{1/2}z)^k \right) \exp \left( \sum_{k \geq 1} H_{t,k} \frac{1}{k^s} (q^{1/2}z)^{-k} \right) e^{\alpha_2 H_{1,0} + 1}, \]
\[ F_t(z) \rightarrow \exp \left( \sum_{k \geq 1} H_{t,k} \frac{1}{k^s} (q^{-1/2}z)^k \right) \exp \left( \sum_{k \geq 1} H_{t,k} \frac{1}{k^s} (q^{-1/2}z)^{-k} \right) e^{-\alpha_2 H_{1,0} + 1}, \]
\[ K_t^+(z) \rightarrow \exp \left( (q - q^{-1}) \sum_{k \geq 1} H_{t,k} \frac{1}{k^s} \right) q^{\omega}. \]
\[ K_i^\ominus (z) \mapsto \exp \left(-\left(q^{-1}\sum_{k\geq 1} H_i \cdot k z^k\right)q^{-\partial_0}\right) \]

for \(0 \leq i \leq n\).

The proof will be given in Appendix.

It is easy to see the following lemma.

**Lemma 3.4.3.** \(V(p)\) is a cyclic \(U_q(\frak{g}_{tor})\)-module: \(V(p)_n = U_q(\frak{g}_{tor}) (\nu_0 \otimes e^{A_J})\).

**Lemma 3.4.4.** \(V(p)\) has level 1 as a \(U_q^{(1)}(\frak{sl}_{n+1})\)-module and as a \(U_q^{(2)}(\frak{sl}_{n+1})\)-module.

**Proof.** It is clear that \(V(p)_n\) is a level 1 \(U_q^{(1)}(\frak{sl}_{n+1})\)-module. The center of \(U_q^{(2)}(\frak{sl}_{n+1})\) is \(\prod_{i=0}^{n} K_i\). By the definition it acts as the scalar \(q\) on \(V(p)_n\). \(\square\)

**§4. On \(U_q(\frak{sl}_{2,tor})\)**

4.1. In this section we assume that \(\frak{g} = \frak{sl}_2\). We shall try to find finitely many generators of \(U_q(\frak{sl}_{2,tor})\).

Let

\[ E_i = E_{i,0}, \quad F_i = F_{i,0}, \quad q^\pm h_i = K_i^\pm, \text{ for } i = 0,1, \]

\[ E_{-1} = F_{0,1}, K_0, \quad F_{-1} = K_0 E_{0,-1}, \quad q^\pm h_{-1} = q^{\pm c} K_0^\pm. \]

**Proposition 4.1.1.** \(U_q(\frak{sl}_{2,tor})\) is generated by \(E_i, F_i, q^\pm h_i (i = -1,0,1), q^\pm 1/2, q^\pm c, q^\pm d_1, q^\pm d_2\).

**Proof.** Let \(\mathcal{A}\) be the subalgebra of \(U_q(\frak{sl}_{2,tor})\) generated by \(E_i, F_i, q^\pm h_i (i = -1,0,1), q^\pm 1/2, q^\pm c, q^\pm d_1, q^\pm d_2\). By the definition we have \(E_{0,-1} = q^{-h_0} F_{-1}\) and \(F_{0,1} = E_{-1} q^{h_0}\). Since

\[ [E_{0,0} F_{0,1}] = \frac{1}{q-q^{-1}} q^{-c} K_0^+ \]

\[ = q^{-1/2} q^{h_0} H_{0,1} \]

and

\[ [E_{0,-1} F_{0,0}] = -\frac{1}{q-q^{-1}} q^{1/2} K_{0,-1} \]

\[ = q^{1/2} q^{-h_0} H_{0,-1}. \]

We deduce \(H_{0,1}\) and \(H_{0,-1} \in \mathcal{A}\). We recall (2.2.7)
By these formulas we have $E_{i,k}, F_{i,k} \in \mathcal{A}$ for $i=0,1, k \in \mathbb{Z}$ inductively.

On the other hand we know

$$[E_{0,1}, F_{0,1}] = \frac{1}{q - q^{-1}} K_{0,0}^2$$

$$= \frac{1}{2} (q - q^{-1}) H_{0,1}^2 + H_{0,2}.$$

Therefore we get $H_{0,2} \in \mathcal{A}$. Similarly we have $H_{i,l} \in \mathcal{A}$ for any $i, l$.

This completes the proof. □

**Lemma 4.1.2.** The following relations hold in $U_q(\mathfrak{sl}_2)_{tor}$:

\begin{align*}
(4.1.1) & \quad [q^{+h_i}, q^{-h_i}] = 0, \\
(4.1.2) & \quad [q^{d_l}, q^{d_l}] = 0, \\
(4.1.3) & \quad q^{d_l} E_i q^{-d_l} = q^{\delta_{l-1}} E_i, \\
(4.1.4) & \quad q^{d_l} F_i q^{-d_l} = q^{-\delta_{l-1}} F_i, \\
(4.1.5) & \quad q^{h_l} E_i q^{-h_l} = q^{\delta_{l-1}} E_i, \\
(4.1.6) & \quad q^{h_l} F_i q^{-h_l} = q^{-\delta_{l-1}} F_i.
\end{align*}

where

$$\begin{pmatrix}
(a_{ij})_{-1 \leq i, j \leq 1} = \begin{pmatrix}
2 & -2 & 2 \\
-2 & 2 & -2 \\
2 & -2 & 2
\end{pmatrix}.
\end{pmatrix}$$

\begin{align*}
(4.1.7) & \quad E_{-1}^2 F_1 - q^{-2} [3] E_{-1} F_1 E_{-1} + q^{-4} [3] E_{-1} F_1 E_{-1}^2 - q^{-6} F_1 E_{-1}^2 = 0, \\
(4.1.8) & \quad E_1 F_{-1} - q^2 [3] E_1 F_{-1} E_1 + q^4 [3] E_1 F_{-1} E_1^2 - q^6 F_{-1} E_1^2 = 0, \\
(4.1.9) & \quad F_{-1}^2 E_1 - q^{-2} [3] F_{-1}^2 E_1 E_{-1} + q^{-4} [3] F_{-1} E_1 E_{-1}^2 - q^{-6} E_1 E_{-1}^2 = 0,
\end{align*}
(4.1.10) \[ F_i E_{-1} - q^2 [3] F_i E_{-1} F_i + q^4 [3] F_i E_{-1} F_i^2 - q^6 E_{-1} F_i^3 = 0, \]

(4.1.11) \[ E_i^2 E_{-1} - [3] E_i^2 E_{-1} + [3] E_i E_{-1} - E_i F_i^2 = 0, \quad \text{for } |i - j| = 1, \]

(4.1.12) \[ F_i^2 F_{-1} - [3] F_i^2 F_{-1} F_i + [3] F_i F_{-1} F_i^2 - F_i F_{-1} F_i^3 = 0, \quad \text{for } |i - j| = 1, \]

(4.1.13) \[ E_{-1} E_i - q^2 E_i E_{-1} = 0, \]

(4.1.14) \[ F_{-1} F_i - q^2 F_i F_{-1} = 0. \]

**Proof.** By the definition of \( \mathcal{U}(\mathfrak{g}_{tor}) \) and \( E_i, F_i \) and \( q^h \) it is easy to check these relations. \( \square \)

Let \( \mathcal{U} \) be an associative algebra over \( \mathbb{Q}(q) \) generated by \( E_i, F_i, q^h \) (\( i = -1,0,1 \)), \( q^{\pm h_1}, q^{\pm d_1}, q^{\pm d_2} \) with relations (4.1.1) – (4.1.14). Then we have,

**Corollary 4.1.3.** There is a canonical surjective algebra homomorphism \( \Psi: \mathcal{U} \to \mathcal{U}(\mathfrak{sl}_{2,tor}) \).

**Remark 4.1.4.** \( \Psi \) has a highly nontrivial kernel. It is important to determine it. For example the following formulas holds in \( \mathcal{U}(\mathfrak{sl}_{2,tor}) \):

\[ \kappa^{m_0} E_{0,0} E_{1,-1} - q^{-2} \kappa^{m_0} E_{1,-1} E_{0,0} = q^{-2} E_{0,-1} E_{1,0} - E_{1,0} E_{0,-1}, \]

\[ E_{0,-1} = q^{-h_0} F_{-1} \]

and

\[ E_{1,-1} = \frac{\kappa^{-m_0}}{[-2]} \left[ F_{-1} F_0 - q^{-2} F_0 F_{-1}, E_1 \right]. \]

Therefore we have

\[ X = \frac{1}{[-2]} E_0 \left[ F_{-1} F_0 - q^{-2} F_0 F_{-1}, E_1 \right] - \frac{q^{-2}}{[-2]} \left[ F_{-1} F_0 - q^{-2} F_0 F_{-1}, E_1 \right] E_0 \]

\[ - q^{-2} F_{-1} E_1 - E_1 q^{-h_0} F_{-1} \]

\[ = 0 \]

in \( \mathcal{U}(\mathfrak{sl}_{2,tor}) \). Thus \( X \in \text{Ker } \Psi \). But, as an element of \( \mathcal{U} \), \( X \) is not equal to 0.

4.2. Let

\[ E_0^* = F_{1,1} K_1^1, \quad F_0^* = K_{1,1} E_{-1,1}, \quad q^{\pm h_0} = q^{\pm c} K_1^1. \]

**Proposition 4.2.1.** The subalgebra generated by \( E_i, F_i \) and \( q^h \) for \( i = 0,1 \), \( E_0^*, F_0^*, q^{h_0}, q^{\pm c}, q^{\pm d_1}, q^{\pm d_2} \), is equal to \( \mathcal{U}(\mathfrak{sl}_{2,tor}) \). That is, they are generators of
$U_q(\mathfrak{sl}_{2,\text{tor}})$. Moreover these generators satisfy relations similar to the ones in Lemma 4.1.2.

Proof. This proposition is proved in the same way as Proposition 4.1.1 and Lemma 4.1.2. □

We have immediately the following lemma.

**Lemma 4.2.2.** Let $U_q^{(1)}$ be the subalgebras generated by $E_i, F_i, q^{h_i}$ for $i = 1, 0^*$ and $q^{\pm(d_1+d_2)}$, $U_q^{(2)}$ the subalgebras generated by $E_i, F_i, q^{h_i}$ for $i = 0, 1$ and $q^{\pm(d_1+d_2)}$, $U_q^{(3)}$ the subalgebras generated by $E_i, F_i, q^{\pm h_i}$ for $i = 0^*, -1$ and $q^{\pm(d_1+d_2)}$, and $U_q^{(4)}$ the subalgebras generated by $E_i, F_i, q^{\pm h_i}$ for $i = 0, -1$ and $q^{\pm(d_1+d_2)}$. Then $U_q^{(i)} (i = 1, 2, 3, 4)$ are isomorphic to $U_q(\mathfrak{sl}_2)$.

Those four algebras are schematically visualized by Fig. 1.

![Diagram](image_url)

Fig. 1.

Let $U_q(\mathfrak{sl}_2) (i = -1, 0, 1, 0^*)$ be the subalgebra of $U_q(\mathfrak{sl}_{2,\text{tor}})$ generated by $E_i, F_i, q^{h_i}$. All $U_q(\mathfrak{sl}_{2,\text{tor}}) (i \in \{-1, 0, 1, 0^*\})$ are isomorphic to $U_q(\mathfrak{sl}_2)$. The upper left circle in Fig. 1 means $U_q(\mathfrak{sl}_2) (0)$, the upper right one means $U_q(\mathfrak{sl}_2) (1)$, the lower left one means $U_q(\mathfrak{sl}_2) (-1)$, and the lower right one means $U_q(\mathfrak{sl}_2) (0^*)$. The diagram

\[ \circ^i \iff \circ^j \quad (i, j \in \{-1, 0, 1, 0^*\}) \]

means the algebra generated by $U_q(\mathfrak{sl}_2) (i)$ and $U_q(\mathfrak{sl}_2) (j)$ is isomorphic to $U_q(\mathfrak{sl}_2)$. For example $\circ^0 \iff \circ^1$ means the algebra generated by $U_q(\mathfrak{sl}_2) (0)$ and $U_q(\mathfrak{sl}_2) (1)$ which we call $U_q^{(2)}$ is isomorphic to $U_q(\mathfrak{sl}_2)$. The meaning of the diagram

\[ \circ^1 \iff \circ^j \]

is as follows: In the algebra generated by $U_q(\mathfrak{sl}_2) (i)$ and $U_q(\mathfrak{sl}_2) (j)$, the following relations hold

\[ q^{h_i}E_i q^{-h_i} = q^2 E_i, \quad q^{h_j}E_j q^{-h_j} = q^2 E_j, \quad (4.2.1) \]
Appendix A.

A.1. Proof of Proposition 3.2.2 and 3.4.2. For the proof we rewrite the defining relation \( U_q(\mathfrak{g}_{tor}) \) generating function level.

\[(A.1.1)\] \( q^{\pm \frac{1}{2}S} \) are central.

\[(A.1.2)\] \( K^+_t K^-_t = K^-_t K^+_t = 1 \).

\[(A.1.3)\] \( K^+_t(z) K^-_t(w) = K^-_t(w) K^+_t(z) \).

\[(A.1.4)\] \( \theta_{-\langle \alpha, \alpha \rangle} \left( q^{-\frac{1}{2}S} \kappa^{-m_\alpha} \frac{Z}{w} \right) K^-_t(z) K^+_t(w) = \theta_{-\langle \alpha, \alpha \rangle} \left( q^{\frac{1}{2}S} \kappa^{-m_\alpha} \frac{Z}{w} \right) K^+_t(w) K^-_t(z) \).

\[(A.1.5)\] \( q^{d_1} K^+_t(z) q^{-d_1} = K^+_t(q^{-1}z) \).

\[(A.1.6)\] \( [q^{d_2}, K^+_t(z)] = 0 \).

\[(A.1.7)\] \( q^{d_1} E_j(z) q^{-d_1} = E_j(q^{-1}z) \).

\[(A.1.8)\] \( q^{d_2} E_j(z) q^{-d_1} = q^{d_2} E_j(z) \).

\[(A.1.9)\] \( q^{d_2} F_j(z) q^{-d_2} = q^{-d_2} F_j(z) \).

\[(A.1.10)\] \( K^+_t(z) E_j(w) = \theta_{-\langle \alpha, \alpha \rangle} \left( q^{-\frac{1}{2}S} \kappa^{-m_\alpha} \frac{W}{Z} \right) E_j(w) K^+_t(z) \).

\[(A.1.11)\] \( K^-_t(z) E_j(w) = \theta_{\langle \alpha, \alpha \rangle} \left( q^{\frac{1}{2}S} \kappa^{m_\alpha} \frac{W}{Z} \right) E_j(w) K^-_t(z) \).

\[(A.1.12)\] \( K^+_t(z) F_j(w) = \theta_{\langle \alpha, \alpha \rangle} \left( q^{\frac{1}{2}S} \kappa^{-m_\alpha} \frac{W}{Z} \right) F_j(w) K^+_t(z) \).
\[ K_t^{-}(z) F_j(w) = \theta_{-\langle h, \alpha \rangle}\left(q^{\frac{1}{2} m_0 \alpha} \frac{z}{w}\right) F_j(w) K_t^{-}(z) \]

(A.1.10)

\[ [E_i(z), F_j(w)] = \delta_{i,j} \frac{1}{q - q^{-1}} \left\{ \delta \left(q^z w \right) K_t^+(q^z w) - \delta \left(q^{-z} w \right) K_t^-(q^{-z} w) \right\} \]

(A.1.11)

\[ (k^{m \alpha}_u - q^{-\langle h, \alpha \rangle} w) E_i(z) E_j(w) = (q^{\langle h, \alpha \rangle} k^{m \alpha}_u - w) E_i(z) E_j(w) \]

\[ (k^{m \alpha}_u - q^{-\langle h, \alpha \rangle} w) F_i(z) F_j(w) = (q^{\langle h, \alpha \rangle} k^{m \alpha}_u - w) F_i(z) F_j(w) \]

(A.1.12)

\[ \sum_{\sigma \in S_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix}_q E_i(\sigma_{(1)}) \cdots E_i(\sigma_{(r)}) E_j(\sigma_{(r+1)}) \cdots E_i(\sigma_{(m)}) = 0 \]

\[ \sum_{\sigma \in S_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix}_q F_i(\sigma_{(1)}) \cdots F_i(\sigma_{(r)}) F_j(\sigma_{(r+1)}) \cdots F_i(\sigma_{(m)}) = 0 \]

where \( i \neq j \) and \( m = 1 - \langle h, \alpha \rangle \).

In these formulas we denote \( \theta_m(z) = \frac{z^m - 1}{z - q^m} \) for \( m \in \mathbb{Z} \), \( \delta(z) = \sum_{k \in \mathbb{Z}} z^k \).

If Proposition 3.4.2 holds, then, from Lemma 3.2.1 and 3.4.1, we have Proposition 3.2.2 by putting \( h_i \mapsto h_i, \alpha_i \mapsto \overline{\alpha}_i, \kappa \mapsto 1 \) and \( z^{\mathcal{H}} \mapsto z^{\overline{\mathcal{H}}} \). Therefore it is enough to show Proposition 3.4.2.

The relations (A.1.1), (A.1.2) and (A.1.3) are trivial. (A.1.4) is just the commutation relations of Heisenberg algebra \( \mathcal{H}_n \). Therefore, by the definition of \( V(p)_n \), it is clear that (A.1.4) holds. The relations (A.1.5), (A.1.6) immediately follows from the definition of \( d_1 \) and \( d_2 \).

Let us show (A.1.7) and (A.1.8). Take \( v \otimes e^{\beta} e^{A_p} \in V(p)_n \) where \( \beta = \sum_{k=0}^{n} m_k \alpha_k \in Q \). Then we have

\[ q^{d_1 e^{\pm \alpha_2 \pm H_{sp}} - 1} q^{-d_1} (v \otimes e^{\beta} e^{A_p}) \]

\[ = q^{\frac{1}{2} \sum \langle h, \beta + A_p \rangle + 1} \frac{1}{k} \sum \frac{1}{v \otimes e^{\pm \alpha_2} e^{A_p}} \]

\[ = q^{\frac{1}{2} \sum \langle h, \beta + A_p \rangle + 1} \frac{1}{k} \sum \frac{1}{v \otimes e^{\pm \alpha_2} e^{A_p}} \]

\[ = e^{\pm \alpha_2} (q^{-1/2}) (v \otimes e^{\beta} e^{A_p}) \]

Therefore we have

\[ q^{d_1 E_i(z) q^{-d_1}} = q^{d_1} \exp \left( \sum_{k \geq 1} \frac{H_i}{k} q^{-k} \frac{1}{2} z^k \right) \exp \left( - \sum_{k \geq 1} \frac{H_i}{k} q^{-k} \frac{1}{2} z^k \right) \]

\[ = \exp \left( \sum_{k \geq 1} \frac{H_i}{k} q^{-k} \frac{1}{2} (q^{-1/2})^k \right) \exp \left( - \sum_{k \geq 1} \frac{H_i}{k} q^{-k} \frac{1}{2} (q^{-1/2})^k \right) \]
Similarly we have

\[ q^{d_2}F_j(z)q^{-d_1} = F_j(q^{-1}z). \]

It is clear that

\( \text{(A.1.13)} \quad q^{d_2} \alpha_{r} \alpha_{l} H_{i,g+1} q^{-d_2} ( \psi \otimes e^g e^A ) = q^{d_2} \alpha_{r} \alpha_{l} H_{i,g+1} ( \psi \otimes e^g e^A ). \)

From (A.1.13) and the fact that \( q^{d_2} \) commutes with \( H_{i,k} \), we have (A.1.8). We shall show (A.1.9). We denote

\[ E_i^+ (z) = \exp \left( \sum_{k \geq 1} \frac{H_{i,k}}{k} q^{-\frac{1}{2} k} z^k \right), \]

\[ E_i^- (z) = \exp \left( - \sum_{k \geq 1} \frac{H_{i,k}}{k} q^{-\frac{1}{2} k} z^{-k} \right). \]

\[ F_i^+ (z) = \exp \left( - \sum_{k \geq 1} \frac{H_{i,k}}{k} q^{\frac{1}{2} k} z^k \right), \]

\[ F_i^- (z) = \exp \left( \sum_{k \geq 1} \frac{H_{i,k}}{k} q^{\frac{1}{2} k} w^{-k} \right). \]

Let us prove

\[ K_i^+ (z) E_j (w) = (q^{-\frac{1}{2} k m^{-1} w} z) E_j (w) K_i^+ (z). \]

We have

\[ \left( q^{-q^{-1}} \sum_{k \geq 1} H_{i,k} z^{-k}, \sum_{i \geq 1} \frac{H_{j,l}}{l} q^{-\frac{1}{2} l} w^l \right) = \sum_{k,l} (q^{-q^{-1}} \frac{1}{[l]} [H_{i,k}, H_{j,-l}] z^{-k} q^{-\frac{1}{2} l} w^l \]

\[ = \sum_{k} (q^{-q^{-1}} k \langle h, \alpha \rangle q^{-k} \frac{w}{z} )^k \]

\[ = \sum_{k} \frac{1}{k} (q^{k} (-\langle h, \alpha \rangle - \frac{1}{2} ) - q^{k} (-\langle h, \alpha \rangle - \frac{1}{2} ) ) k^{-k} \frac{w}{z} \]

\[ = \log \frac{1 - q^{-\langle h, \alpha \rangle - \frac{1}{2} k} \frac{w}{z}}{1 - q^{-\langle h, \alpha \rangle - \frac{1}{2} k} \frac{w}{z}} \]

and

\[ \left( q^{-q^{-1}} \sum_{k \geq 1} H_{i,k} z^{-k}, - \sum_{l \geq 1} \frac{H_{j,l}}{l} q^{-\frac{1}{2} l} w^{-l} \right) = 0. \]

We recall Campbell–Hausdorff formula: let \( A \) and \( B \) be noncommutative operators and \( C = [ A, B ] \). If \( [ C, A ] = [ C, B ] = 0 \) then we have \( e^A e^B = e^{C} e^B e^A \).

By Campbell–Hausdorff formula we get
On the other hand, by Lemma 3.4.1 we have
\[ q^\frac{a}{a_i} w^{H_{i,0}} = q^{\langle h_i, \alpha_j \rangle} e^{\alpha_j w^H_{i,0}} q^\frac{a}{a_i}. \]

Thus we get
\[
K_t^+(z) E_f(w) = \exp \left( (q - q^{-1}) \sum H_{1,k} z^{-k} \right) E_t^+(w) E_t^-(w) q^\frac{a}{a_i} w^{H_{i,0}} q^\frac{a}{a_i}
\]

The other formulas in (A.1.9) can be checked by similar arguments.

Let us show (A.1.10). We have
\[
\left[ - \sum_{k \geq 1} \frac{H_{1,k}}{k} q^{-\frac{1}{2} k} z^{-k}, - \sum_{k \geq 1} \frac{H_{j,i-k}}{k} q^{\frac{1}{2} k} w^k \right]
\]

\[
= \sum_{k \geq 1} \frac{1}{k [k]} [k \langle h_i, \alpha_j \rangle] k^{-m_i} \left( \frac{w}{z} \right)^k
\]

\[
= \log \frac{1}{(1 - q^w z) (1 - q^{-1} w z)}, \quad \langle h_i, \alpha_j \rangle = 2 \quad (i=j),
\]

\[
= \log \left( 1 - q^{m_i} \frac{w}{z} \right) \left( 1 - q^{-1} \frac{m_i w}{z} \right), \quad \langle h_i, \alpha_j \rangle = -2,
\]

\[
\log \left( 1 - \frac{w}{z} \right), \quad \langle h_i, \alpha_j \rangle = -1,
\]

\[
0, \quad \langle h_i, \alpha_j \rangle = 0.
\]

For example we will show in the case of \( \langle h_i, \alpha_j \rangle = -2 \). By Campbell-Hausdorff formula we get
\[
E_t^+(z) F_t^+(w) = F_t^+(w) E_t^+(z)
\]

and
\[ E_i^+(z) F_j^+(w) = \left(1 - q \kappa^{-m_i} w \right) \left(1 - q^{-1} \kappa^{-m_j} w \right) F_j^+(w) E_i^+(z). \]

On the other hand, by Lemma 3.4.1, we have

\[ z^{H_{i,j}} e^{-\alpha_i} = z^{2 \kappa^{-m_i} - \alpha_i} z^{H_{i,j}}. \]

Therefore we get

\[ E_i(z) F_j(w) = E_i(z) E_i(z) e^{\alpha_i z} e^{H_{i,j} + H_{i,j}} e^{-\alpha_i w} - \alpha_j F_j(w) e^{-\alpha_j w} - H_{i,j} \]

\[ = (z \kappa^{\frac{1}{2} m_i} - q \kappa^{-\frac{1}{2} m_j}) (z \kappa^{\frac{1}{2} m_j} - q^{-1} \kappa^{-\frac{1}{2} m_i}) E_i^+(z) F_j^-(w) E_i^+(z) F_j^-(w) \]

\[ \times e^{\alpha_i z} e^{-\alpha_j} z^{H_{i,j} + H_{i,j}} - H_{i,j}. \]

By a similar argument we have

\[ F_j(w) E_i(z) = (w \kappa^{-\frac{1}{2} m_j} - q z \kappa^{\frac{1}{2} m_i}) (w \kappa^{\frac{1}{2} m_i} - q^{-1} z \kappa^{\frac{1}{2} m_j}) E_i^+(z) F_j^-(w) E_i^+(z) F_j^-(w) \]

\[ \times e^{\alpha_i z} e^{-\alpha_j} z^{H_{i,j} + H_{i,j}} - H_{i,j}. \]

Therefore we get

\[ [E_i(z), F_j(w)] = 0. \]

Similarly one can check the other formulas.

We will show (A.1.11). We have

\[ \left[ - \sum_{k \geq 1} H_{i,k} q^{-\frac{1}{2} k} z^{-k} - k \sum_{k \geq 1} H_{i,k} q^{-\frac{1}{2} k} w^{-k} \right] \]

\[ = \sum_{k \geq 1} \frac{1}{k} \left[ k \langle h_i, \alpha_i \rangle \right] z^{-\frac{1}{2} m_i} w^k \]

\[ = \begin{cases} 
\log \left(1 - \frac{w}{z}\right) \left(1 - q^{-2} \frac{w}{z}\right), & i = j, \\
\log \left(1 - \kappa^{-m_i} w \right) \left(1 - q^{-2} \kappa^{-m_i} w \right), & \langle h_i, \alpha_i \rangle = -2, \\
\log \left(1 - q^{-1} \kappa^{-m_i} w \right) \left(1 - q^{-2} \kappa^{-m_i} w \right), & \langle h_i, \alpha_i \rangle = -1, \\
0, & \langle h_i, \alpha_i \rangle = 0. 
\end{cases} \]

For example let us show in the case of \( \langle h_i, \alpha_i \rangle = -2 \). If \( \langle h_i, \alpha_i \rangle \neq -2 \) one can show the formula by a similar argument. By Lemma 3.4.1 we get

\[ (z \kappa^{m_i} - q^{\langle h_i, \alpha_i \rangle} w) E_i(z) E_i(w) \]
QUANTUM TOROIDAL ALGEBRAS 175

\[(z \kappa^{m_u} - q^2 w) T^+_i(z) T^+_i(z) e^{\alpha_i z^{H_{1,0}} + 1} T^+_i(z) T^+_i(w) e^{\alpha_i w^{H_{1,0} + 1}} \]

\[= \frac{z \kappa^{m_u} - q^2 w}{z^2 \kappa^{m_u} \left(1 - \kappa^{-m_u} w \right) \left(1 - q^2 \kappa^{-m_u} w \right)} T^+_i(z) T^+_i(w) T^+_i(z) T^+_i(w) \times e^{\alpha_i z^{H_{1,0} + 1} w^{H_{1,0} + 1}} \]

\[= \frac{1}{(z - \kappa^{-m_u} w)} T^+_i(z) T^+_i(w) T^+_i(z) T^+_i(w) e^{\alpha_i z^{H_{1,0} + 1} w^{H_{1,0} + 1}}. \]

On the other hand we have

\[\langle q^{\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w \rangle T^+_i(z) T^+_i(z) = \langle q^{\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w \rangle T^+_i(z) T^+_i(w) \]

\[= \frac{z \kappa^{m_u} - q^2 \kappa^{m_{u2}} - w}{w^2 \kappa^{m_u} \left(1 - \kappa^{-m_u} z \right) \left(1 - q^2 \kappa^{-m_u} z \right)} T^+_i(z) T^+_i(w) T^+_i(z) T^+_i(w) \times e^{\alpha_i z^{H_{1,0} + 1} z^{H_{1,0} + 1}} \]

\[= \frac{1}{(z - \kappa^{-m_u} w)} T^+_i(z) T^+_i(w) T^+_i(z) T^+_i(w) e^{\alpha_i z^{H_{1,0} + 1} z^{H_{1,0} + 1}}. \]

Thus we have \[(z \kappa^{m_u} - q^{\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w) T^+_i(z) T^+_i(z) = \langle q^{\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w \rangle T^+_i(z) T^+_i(w) \]

The formula \[(z \kappa^{m_u} - q^{-\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w) F^+_i(z) F^+_i(w) = \langle q^{-\langle h, \alpha_i \rangle} \kappa^{m_{u2}} - w \rangle F^+_i(z) F^+_i(w) \]

is proved similarly.

Let us prove (A.1.12). Assume that \(\langle h, \alpha_i \rangle = -2\). This is the most complicated case. The other cases can be proved similarly.

We have following formulas:

\[E^+_i(z_1) E^+_i(z_2) E^+_i(z_3) T^+_i(w) \]

\[= \langle z_1 - z_3 \rangle \langle z_1 - q^{2} z_2 \rangle \langle z_1 - q^{2} z_3 \rangle \langle z_2 - z_3 \rangle \langle z_2 - q^{2} z_3 \rangle \]

\[\left(\kappa^{1/2 \kappa^{m_u} z_1 - k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_1 - q^{-2} k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_2 - k^{1/2 \kappa^{m_u} w}} \right) \]

\[\times \frac{1}{\left(\kappa^{1/2 \kappa^{m_u} z_2 - q^{-2} k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_3 - k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_3 - q^{-2} k^{1/2 \kappa^{m_u} w}} \right)} \]

\[\times E^+_i(z_1) E^+_i(z_2) E^+_i(z_3) E^+_i(w) T^+_i(z_1) T^+_i(z_2) T^+_i(z_3) T^+_i(w) \]

\[\times e^{3 \alpha_i z^{H_{1,0} + 1} z^{H_{1,0} + 1}} z^{H_{1,0} + 1} w^{H_{1,0} + 1}. \]

\[E^+_i(z_1) E^+_i(z_2) T^+_i(w) E^+_i(z_3) \]

\[= \langle z_1 - z_3 \rangle \langle z_1 - q^{2} z_2 \rangle \langle z_1 - q^{2} z_3 \rangle \langle z_2 - z_3 \rangle \langle z_2 - q^{2} z_3 \rangle \]

\[\left(\kappa^{1/2 \kappa^{m_u} z_1 - k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_1 - q^{-2} k^{1/2 \kappa^{m_u} w}} \right) \left(\kappa^{1/2 \kappa^{m_u} z_2 - k^{1/2 \kappa^{m_u} w}} \right) \]
Therefore it is enough to show that

\[(A.1.14)\]

\[
\sum_{\sigma \in S_j} \prod_{1 \leq i < j} \left( \frac{z_{\sigma(i)} - z_{\sigma(j)} - q^{-2}z_{\sigma(i)}}{\kappa^{2m_{\sigma(i)}} - \kappa^{2m_{\sigma(j)}} - \kappa^{2m_{\sigma(i)}} - \kappa^{2m_{\sigma(j)}}} \right)
\times \left\{ \begin{array}{l}
\frac{1}{(\kappa^{2m_{\sigma(1)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}})} \times \frac{q^2+1+q^2}{(\kappa^{2m_{\sigma(1)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}} - q^{-2}\kappa^{2m_{\sigma(2)}})} + q^2+1+q^2
\end{array} \right.
\]

Therefore it is enough to show that

\[(A.1.14)\]
This identity is proved by a direct calculation. Thus the proposition is proved.

Acknowledgment

The author would like to thank Masaki Kashiwara, Tetsuji Miwa and Michio Jimbo for their encouragement and valuable discussions. He also thanks Kenji Iohara, Norio Suzuki, Eric Vasserot and Hiroshi Yamada for stimulating discussions.

References


[V] Vasserot, E., Private communication.

