Cyclic Representations of $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$ at $q^N=1$

By

Etsuro Date*, Michio Jimbo**, Kei Miki*** and Tetsuji Miwa****

§ 1. Introduction

In this article we deal with the $q$-analog of the universal enveloping algebra $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$ when $q=\omega$ is a primitive $N$-th root of 1 with odd $N$. We shall give an explicit construction of finite-dimensional irreducible representations having $n(n+2)$ continuous parameters.

Our motivation in this problem originates in the chiral Potts model [AMPT], [BPA]. This is a solvable lattice model built upon solutions to the Yang-Baxter equation whose spectral parameters live on certain algebraic curves of genus greater than 1. Bazhanov and Stroganov [BS] showed in effect that these solutions can be derived as intertwiners between tensor products of the representations of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ with $q$ a root of 1 (see [BS], [DJMM] for details). Attempts for extending their construction to the case of $U_q(\mathfrak{sl}(n+1, \mathbb{C}))$ have been initiated in [BK], [DJMM] for the case $n=2$.

Representations of $U_q(\mathfrak{g})$ at roots of 1 have been studied recently by De Concini and Kac [DK] for an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$. They showed that the irreducible representations of $U_q(\mathfrak{g})$ are generically parametrized by $\dim \mathfrak{g}$ number of continuous parameters which are the values of certain central elements. Our aim here is to write down such representations in the case $\mathfrak{g}=\mathfrak{sl}(n+1, \mathbb{C})$. 

Received June 19, 1990.

* Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka, 560, Japan.
** Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606, Japan.
*** Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, 606, Japan.
**** Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan.
**** Fellow of the Japan Society for the Promotion of Science for Japanese Junior Scientists.
Here is the outline of the paper. In section 2 we consider a Weyl algebra $\mathcal{W}$ with generators $x_{jk}, z_{jk}$ $(1 \leq j \leq k \leq n)$ such that $z_{jk}x_{jk} = qx_{jk}z_{jk}$ and all others pairwise commute. We shall construct (for generic $q$) an algebra map $\rho_{rs}: U_q(\mathfrak{sl}(n+1, C)) \to \mathcal{W}$ depending on arbitrary non-zero complex numbers $r_i, s_i$ $(1 \leq i \leq n)$. Explicitly it reads

$$\rho_{rs}(e_i) = \sum_{k=1}^{n} \{r_i z_{ik} z_{i-k-1} z_{i-k}^{-1} z_{i-k+1}^{-1} z_{i-k+1} \} x_{ik} x_{i-k+1} \cdots x_{in},$$

$$\rho_{rs}(f_i) = \sum_{k=1}^{n} \{s_i z_{i+1-k} z_{i-k} z_{i-k+1}^{-1} z_{i-k-1}^{-1} z_{i-k+1}^{-1} \} x_{i+1-k} x_{i-k+1} \cdots x_{in}^{-1},$$

$$\rho_{rs}(t_i) = \frac{r_i}{s_i} z_{in} z_{i-1-n} z_{i-1}^{-1}.$$

When $q=\omega$, $\mathcal{W}$ admits an $N^m$-dimensional irreducible representation $\sigma_{gh}: \mathcal{W} \to \text{End}((C^N)^{\otimes m})$ with $m = n(n+1)/2$. Let $X, Z$ be $N \times N$ matrices given by

$$Xu_i = u_{i+1} \quad (u_N = u_0), \quad Zu_i = \omega^i u_i,$$

where $\{u_i\}_{0 \leq i \leq N-1}$ denotes the standard basis of $C^N$. Let $X_{jk}, Z_{jk} \in \text{End}((C^N)^{\otimes m})$ denote the matrices acting as $X, Z$ on the $(j, k)$-component and as identity on the other components. Then we have

$$\sigma_{gh}(x_{jk}) = g_{jk} X_{jk}, \quad \sigma_{gh}(z_{jk}) = h_{jk} Z_{jk}.$$  

Here again $g_{jk}, h_{jk}$ $(1 \leq j \leq k \leq n)$ are arbitrary non-zero complex numbers. Composing $\sigma_{gh}$ with $\rho_{rs}$ above, we obtain in section 3 a representation of $U_q(\mathfrak{sl}(n+1, C))$ at $q=\omega$. The parameters $r_i, s_i, g_{jk}$ and $h_{jk}$ are not mutually independent, and there are altogether $n(n+2)$ continuous parameters. We show next that the central elements take values in an open set of $C^n_{n+2}$. From the results of De Concini-Kac [DK] we then conclude that these representations are generically irreducible.

§ 2. Algebra Homomorphism $U_q(\mathfrak{sl}(n+1, C)) \to \mathcal{W}$

Let $C(q)$ denote the field of rational functions in an indeterminate $q$. In this section we construct a $C(q)$ algebra $\mathcal{W}$ and an algebra homomorphism $\rho_{rs}: U_q(\mathfrak{sl}(n+1, C)) \to \mathcal{W}$.

We use the following notations:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k][k-1] \cdots [1], \quad \begin{bmatrix} N \\ k \end{bmatrix} = \frac{[N]!}{[k]![N-k]!}.$$
Suppose that $L$ is a logical expression. We define

$$
\theta(L) = \begin{cases} 
1 & \text{if } L \text{ is true} \\
0 & \text{if } L \text{ is false}.
\end{cases}
$$

The algebra $\mathcal{W}$ is generated by $x_{jk}, z_{jk}$ ($1 \leq j \leq k \leq n$) and the inverses $x_{jk}^{-1}, z_{jk}^{-1}$ satisfying

$$
[x_{jk}, x_{j'k'}] = [x_{jk}, z_{j'k'}] = [z_{jk}, z_{j'k'}] = 0 \quad \text{if } (j, k) \neq (j', k') , \quad (2.1a)
$$

$$
z_{jk} x_{jk} = q x_{jk} z_{jk} . \quad (2.1b)
$$

We define a $C(q)$ linear involution $*$ by

$$
x_{jk}^* = x_{k+1-j, k}^{-1}, \quad z_{jk}^* = z_{k+1-j, k}^{-1} .
$$

We also define a $C$ linear involution $\hat{\cdot}$ by

$$
\hat{q} = q^{-1}, \quad \hat{x}_{jk} = x_{jk}, \quad \hat{z}_{jk} = z_{jk}^{-1} .
$$

Let $(a_{ij})$ be the Cartan matrix of type $A_n$. By definition the algebra $U_q(\mathfrak{sl}(n+1, C))$ is a $C(q)$ algebra generated by $e_i, f_i, t_i$ ($1 \leq i \leq n$) and the inverses $t_i^{-1}$ satisfying

$$
[t_i, t_j] = 0 , \quad (2.2a)
$$

$$
t_i e_j t_i^{-1} = q^{\delta_{ij}} e_j , \quad (2.2b)
$$

$$
t_i f_j t_i^{-1} = q^{-\delta_{ij}} f_j , \quad (2.2c)
$$

$$
[e_i, f_j] = \delta_{ij} \{ t_i \} , \quad (2.2d)
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = 0 \quad \text{if } i \neq j , \quad (2.2e)
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \quad \text{if } i \neq j . \quad (2.2f)
$$

We denote the following $C(q)$ linear involution of $U_q(\mathfrak{sl}(n+1, C))$ by $*$.

$$
e_i^* = f_{n+1-i}, \quad f_i^* = e_{n+1-i}, \quad t_i^* = t_{n+1-i}^{-1} .
$$

We also denote the following $C$ linear involution by $\hat{\cdot}$.

$$
\hat{q} = q^{-1}, \quad \hat{e}_i = e_i, \quad \hat{f}_i = f_i, \quad \hat{t}_i = t_i^{-1} .
$$

We define the root vectors $e_{ij}$ ($1 \leq i \neq j \leq n+1$) inductively as follows.

$$
e_{i+1} = e_i , \quad (2.3a)$$
\[ e_{ij} = e_{ik} e_{kj} - q e_{kj} e_{ik} \quad \text{if} \quad i < k < j , \quad (2.3b) \]
\[ e_{ij} = (e_{n+2-i,n+2-j})^{n} \quad \text{if} \quad i > j . \quad (2.3c) \]

In particular
\[ e_{i+1i} = f_{i} . \quad (2.3d) \]

The consistency of this definition follows from \((2.2e)\) for \(i \neq j \pm 1\). Among the commutation relations of \(e_{ij}\)(cf. \([Y]\)) we shall need the following.

\[ e^{(2)}_{ik} e_{kj} - e_{ik} e_{kj} e_{ik} + e_{kj} e^{(2)}_{ik} = 0 \quad \text{if} \quad i < k < j , \quad (2.4a) \]
\[ e^{(2)}_{kj} e_{ik} - e_{kj} e_{ik} e_{kj} + e_{ik} e^{(2)}_{kj} = 0 \quad \text{if} \quad i < k < j . \quad (2.4b) \]

For \(r = (r_{1}, \ldots, r_{n}) \in (\mathbb{C}^{\times})^{n}\) we define
\[ r^{*} = (r_{n}, \ldots, r_{1}) , \]
\[ \tilde{r} = (r_{n}^{-1}, \ldots, r_{1}^{-1}) . \]

We construct a family of \(\mathcal{C}(q)\) algebra homomorphisms
\[ \rho_{r,s} : U_{q}(\mathfrak{gl}(n+1, \mathcal{C})) \rightarrow \mathcal{W} \]
depending on \(r, s \in (\mathbb{C}^{\times})^{n}\). Fix \(r, s\) and define
\[ \xi_{ik} = x_{ik} x_{i+1} \cdots x_{in} , \]
\[ \zeta_{ik} = z_{i} z_{i+1} \cdots z_{i+n} , \]
where \(z_{ik} = 1\) unless \(1 \leq i \leq k \leq n\). We use the following abbreviations.
\[ \zeta_{i+1-k} = \zeta_{i+1-k} \cdots \zeta_{i+k} \cdots \zeta_{i+1-k} , \]
\[ \xi_{i+1-k} = \xi_{i+1-k} \cdots \xi_{i+k} \cdots \xi_{i+1-k} . \]

It is easy to check the following commutation relations.
\[ \xi_{ik} \xi_{ik'} = q^{-2} \xi_{ik} \xi_{ik'} \quad \text{if} \quad k < k' \]
\[ = q^{-1} \xi_{ik'} \xi_{ik} \quad \text{if} \quad k = k' \]
\[ = \xi_{ik'} \xi_{ik} \quad \text{if} \quad k > k' , \quad (2.5a) \]
\[ \xi_{i+1-k} \xi_{ik'} = q \xi_{i+1-k} \xi_{ik'} \quad \text{if} \quad k \leq k' \]
\[ = \xi_{ik'} \xi_{i+1-k} \quad \text{if} \quad k > k' \quad , \quad (2.5b) \]
\[ \xi_{i-1-k} \xi_{ik'} = q \xi_{i-1-k} \xi_{ik'} \quad \text{if} \quad k < k' \]
\[ = \xi_{ik'} \xi_{i-1-k} \quad \text{if} \quad k > k' , \quad (2.5c) \]
\[ \xi_{ik} \xi_{i'k'} = \xi_{i'k'} \xi_{ik} \quad \text{if} \quad i \neq i' , i' \perp 1 \quad . \quad (2.5d) \]

We define
Note that (2.6b) means
\[ \rho_{r,s}(f_i) = \sum_{k=1}^{l} \{ s_k z_{i+1-k} z_{i+1-k} z_{i+1-k} \cdots x_{i+1-k} \} x_{i+1-k} x_{i+1-k} \cdots x_{i+1-k}. \]

**Proposition 2.1.** For \( a = e_i, f_i, t_i \) we have
\[ \rho_{r,s}(a)^* = \rho_{r,s}(a^*) \quad \rho_{r,s}(a)^- = \rho_{r,s}(a). \]

**Proof.** Straightforward. \( \square \)

**Theorem 2.2.** \( \rho_{r,s} \) defines a \( C(q) \) algebra homomorphism.

**Proof.** We shall check the relations (2.2). The relation (2.2a) is obvious. By using Proposition 2.1, (2.2c) follows from (2.26) and (2.2f) follows from (2.2e).

It is easy to see that
\[ \rho_{r,s}(t_i) x_j \rho_{r,s}(t_i)^{-1} = x_{j+1}, \]
\[ \rho_{r,s}(t_i) x_j \rho_{r,s}(t_i)^{-1} = d_{ij} x_j. \]
From this follows (2.2b).

Let us show (2.2d). Set
\[ (i, k) = \{ s_k z_{i+1-k} z_{i+1-k} z_{i+1-k} \cdots x_{i+1-k} \} x_{i+1-k} x_{i+1-k} \cdots x_{i+1-k}. \]
\[ (j, l)' = \{ s_j z_{j+1-l} z_{j+1-l} z_{j+1-l} \cdots x_{j+1-l} \} x_{j+1-l} x_{j+1-l} \cdots x_{j+1-l}. \]

Then we have
\[ \rho_{r,s}(e_i) = \sum_{k=1}^{l} (i, k), \quad \rho_{r,s}(f_j) = \sum_{i=1}^{l} (j, l)'. \]
Consider the product \( (i, k) (j, l)' \). Using the commutation relations (2.1) we can move all the \( x_{ab} \)'s in \( (i, k) \) to the right of \( z_{ab} \)'s in \( (j, l)' \). This procedure picks up nonzero power of \( q \) in the following two cases.

**Case 1:** \( l = j - i + 1, k + l = n + 1, \)
\[ x_{ik} \cdots x_{in} (z_{j+1-l} z_{j+1-l} z_{j+1-l} \cdots x_{j+1-l}) = q^i (z_{j+1-l} z_{j+1-l} z_{j+1-l} \cdots x_{j+1-l}) x_{ik} \cdots x_{in}. \]

**Case 2:** \( l = j - i, k + l = n + 1, \)
\[ x_{ik} \cdots x_{in} (z_{j-1} z_{j-1} z_{j-1} \cdots x_{j-1}) = q^{-i} (z_{j-1} z_{j-1} z_{j-1} \cdots x_{j-1}) x_{ik} \cdots x_{in}. \]
The situation is the same for the product \((j, l)' (i, k)\).

**Case 1:** 
\[ l=j-i+1, k+l=n+1, \]
\[ x_{j+1-l}^{-1}x_{j+1-l-n+1-l} \cdots x_{jn}^{-1}(z_{i-k} z_{i-k-1}) = q(z_{i-k} z_{i-k-1}) x_{j+1-l-n+1-l} \cdots x_{jn}^{-1}. \]

**Case 2:** 
\[ l=j-i, k+l=n+1, \]
\[ x_{j+1-l}^{-1}x_{j+1-l-n+1-l} \cdots x_{jn}^{-1}(z_{i-k} z_{i-k+1}) = q^{-1}(z_{i-k} z_{i-k+1}) x_{j+1-l-n+1-l} \cdots x_{jn}^{-1}. \]

If \(i>j\), neither Case 1 nor Case 2 occurs. If \(i=j\), Case 1 occurs but Case 2 does not. If \(i<j\), both Case 1 and Case 2 occur. Therefore, if \(i>j\) then
\[ [\rho_r(e_i), \rho_r(f_j)] = 0. \quad (2.7) \]

When \(i \leq j\), we use the following formulas for commutative \(x, y\).
\[
\{x\} \{gy\} - \{gx\} \{y\} = \{xy^{-1}\}, \\
\{x\} \{q^{-1}y\} - \{q^{-1}x\} \{y\} = \{x^{-1}y\}, \\
\{x\} + \{x^{-1}\} = 0.
\]

If \(i<j\) (2.7) follows from these identities. Finally, for \(i=j\) we have
\[ [\rho_r(e_i), \rho_r(f_j)] = \left\{ \frac{r_i}{s_i} z_{i-n} z_{i-n-k_{i-1}n} z_{i-1}^{-1} \right\} = \{e_{r,s}(t_i)\}. \]

From (2.5d) and (2.6a) we have \([e_i, e_j]=0\) if \(i \neq j \pm 1\). This is (2.2e) for \(i \neq j \pm 1\). In order to finish the proof we must show (2.2e) for \(i=j \pm 1\). Before proceeding to that proof we determine the image of \(e_{ij}\). Since we have shown (2.2e) for \(i \neq j \pm 1\), the map \(\rho_{r,s}\) is well-defined on \(e_{ij}\).

**Proposition 2.3.** For an integer \(l\) such that \(1 \leq l \leq n+1-i\) we have
\[ \rho_{r,s}(e_{i+l}) = \sum_{k_{l} \geq \cdots \geq k_{1} \geq 0} (-q)^{j-1} \theta(k_{1} \geq \cdots \geq k_{j} < \cdots < k_{l}) \times \xi_{i+k_{j}-k_{j-1}} \{\zeta_{i+j-k_{j-1}k_{j}}\} (\zeta_{i+j-k_{j+1}k_{j+1}})^{-1} \xi_{i+k_{j}-k_{l}}. \quad (2.8) \]

**Proof.** We use induction on \(l\). If \(l=1\) this is (2.6a). Suppose that (2.8) is shown for \(l\). By the definition we have
\[ e_{i+l+1} = e_{i+l} e_{i+l+1}^{-1} q e_{i+l+1} e_{i+l}. \]

Note that
\[ \rho_{r,s}(e_{i+l+1}) = \sum_{k_{l+1} \geq 0} \xi_{i+k_{l+1}k_{l+1}} \{\zeta_{i+l+k_{l+1}}\} \xi_{i+k_{l+1}k_{l+1}}. \]
Consider the product $\rho_{r,s}(e_{i+1})\rho_{r,s}(e_{i+1})$. When we move $\xi_{i,k_1-1}$ to the right of $\rho_{r,s}(e_{i+1})$, it may pick up some power of $q$ from $\zeta_{i+1,k_1+1}$. In fact, we have

$$\xi_{i,k_1-1} \zeta_{i+1,k_1+1} = q^{\theta(k_1 \leq k_1+1)} \zeta_{i+1,k_1+1} \xi_{i,k_1-1}.$$

Similarly, for the product $\rho_{r,s}(e_{i+1})\rho_{r,s}(e_{i+1})$, $\xi_{i+1,k_1+1}$ may pick up some power of $q$ from $\zeta_{i+1,k_1}$:

$$\xi_{i+1,k_1+1} \xi_{i+1,k_1} = q^{\theta(k_1 \geq k_1+1)} \zeta_{i+1,k_1} \xi_{i+1,k_1+1}.$$

We use the following formulas for commutative $x, y$.

$$x^{-1}\{y\} - qx^{-1}\{y\} = x^{-1}y^{-1},$$

$$\{x\} \{qy\} - q\{x\} \{y\} = \{x\} y^{-1},$$

$$x^{-1}\{y\} - q(x^{-1}\{y\} = 0,$$

$$\{x\} \{y\} - q\{qx\} \{y\} = -qx\{y\}.$$

From these formulas follows (2.8).

Proof of Theorem 1 (continued). If $i=j-1$, (2.2e) is equivalent to

$$e_{i+2} = qe_i e_{i+2} \quad (2.9a)$$

and if $i=j+1$ it is equivalent to

$$e_{j+1} e_{j+2} = qe_{j+2} e_{j+1} \quad (2.9b)$$

We shall prove (2.9b). The proof of (2.9a) is similar. Note the following formulas.

$$e_{j+1} = \sum_{k \geq j+1} \{\zeta_{j+1,k}\} \xi_{j+1,k},$$

$$e_{j+2} = \sum_{k \geq j+1} (\theta(k_1 < k_2) \{\zeta_{j_k,k_2}\} \xi_{j+1,k_2} \xi_{j+1,k_2} - q^2 \{k_1 \geq k_2\} \zeta_{j_k,k_2} \{\zeta_{j+1,k_2}\} \xi_{j_k,k_2} \xi_{j+1,k_2}).$$

Consider the term in (2.9b) corresponding to the summation indices $k, k_1, k_2$. There are 9 cases.

1. $k \leq k_1 < k_2$,
2. $k_1 < k < k_2$,
3. $k_1 < k = k_2$,
4. $k_1 < k_2 < k$,
5. $k < k_2 \leq k_1$,
6. $k = k_2 < k_1$,
7. $k_2 < k \leq k_1$,
(8) \( k_2 < k_1 < k \),
(9) \( k = k_2 = k_1 \).

Using (2.5) we have

\[
(1) + (8) = (2) + (4) = (3) = (5) + (7) = (6) = (9) = 0.
\]

For \( \lambda = (\lambda_{jk}) \in C(\ast)^{m} \), let \( S_\lambda, T_\lambda \) denote the automorphisms of \( \mathcal{H} \)

\[
S_\lambda(x_{jk}) = \lambda_{jk} x_{jk}, \quad S_\lambda(z_{jk}) = z_{jk},
T_\lambda(x_{jk}) = x_{jk}, \quad T_\lambda(z_{jk}) = \lambda_{jk} z_{jk}.
\]

**Proposition 2.4.** For \( r, s, \bar{r}, \bar{s} \in C(\ast)^{n} \), there exists a \( \lambda \in (C^\ast)^{m} \) such that

\[
T_\lambda \circ \rho_{r,s} = \rho_{\bar{r},\bar{s}}
\]

if and only if \( r_{s_{n+1-i}} = \bar{r}_{\bar{s}_{n+1-i}} \) \( (1 \leq i \leq n) \).

Before the proof we prepare

**Lemma 2.5.** Given \( d_j \in C^\ast \) \( (1 \leq j \leq l) \), consider the equations for the unknowns \( \lambda_{jk} \) \( (1 \leq j \leq k \leq l) \)

\[
d_j = \lambda_{jk} \lambda_{j(k-1)} \lambda_{j-1(k-1)} \lambda_{j+1(k+1)}^{-1} \quad (1 \leq j \leq k \leq l)
\]

where we set \( \lambda_{jk} = 1 \) unless \( 1 \leq j \leq k \leq l \). Define \( d_{jk} = d_j d_{j+1} \cdots d_k \) for \( 1 \leq j \leq k \leq l \). Then (2.11) has a unique solution given by

\[
\lambda_{jk} = d_{jk} d_{j+1} d_{j+1} \cdots d_{k-j+1}.
\]

Setting \( \lambda_{jk} = \lambda_{k+1-j, k} \) we have

\[
\lambda_{jk} \lambda_{jk}^\ast = 1.
\]

**Proof.** Set

\[
\mu_{jk} = \lambda_{jk}/\lambda_{j-1k-1}.
\]

The equation reads as

\[
d_j = \mu_{jk}/\mu_{j+1k}.
\]

This equation has a unique solution given by

\[
\mu_{jk} = d_{jk}.
\]

Therefore, from (2.14) we have

\[
\lambda_{jk} = d_{jk} \lambda_{j-1k-1} = d_{jk} d_{j-1k-1} \cdots d_{k-j+1}.
\]

This is (2.12). Substituting the solution we obtain
\[
\frac{\lambda_{jk}}{\lambda_{j,k-1}} = \frac{d_{jk}d_{j-1,k-1} \cdots d_{1,k-j+1}}{d_{j-1,k-2} \cdots d_{1,k-j}} = d_{k+1-j,k}
\]

From this follows (2.13).

**Proof of Proposition 2.4.** The condition (2.10) is equivalent to the following equations.

\[
\begin{align*}
\frac{\bar{r}_i}{s_i} &= r_i \lambda_{i,k-1} \lambda_{i-1,k-1}^{-1} \lambda_{i+1,k}^{-1}, & (1 \leq i \leq k \leq n) \quad (2.15a) \\
\frac{\bar{s}_i^*}{s_i} &= s_i^* \lambda_{i,k}^* \lambda_{i-1,k}^{-1} \lambda_{i+1,k}^{-1}, & (1 \leq i \leq k \leq n) \quad (2.15b) \\
\frac{\bar{r}_i}{s_i} &= r_i \lambda_{i,n}^* \lambda_{i-1,n}^{-1} \lambda_{i+1,n}^{-1}. & (1 \leq i \leq n) \quad (2.15c)
\end{align*}
\]

Set \(d_i = \bar{r}_i/r_i, d_i^* = s_i^*/s_i^*\) and \(l = n\) in Lemma 2.5. The equation (2.15a) is solved by

\[\lambda_{ik} = d_{ik}d_{i-1,k-1} \cdots d_{i,k-i+1},\]

and the equation (2.15b) is solved by

\[\lambda_{ik}^* = d_{ik}^*d_{i-1,k-1}^* \cdots d_{i,k-i+1}^*.\]

Since \(\lambda_{ik} = \lambda_{k+1-i,k}\), these two solutions are consistent with the definition \(\lambda_{ik}^* = \lambda_{k+1-i,k}^{-1}\) if and only if \(d_{ik}d_{i} = 1\). This is equivalent to \(r_is_{n+1-i} = \bar{r}_i s_{n+1-i}\).

Finally, (2.15c) follows from (2.15a) and (2.15b) with \(k = n\).

§3. Finite-dimensional Representations

Fix a positive odd integer \(N \geq 3\). Let \(\omega\) be a primitive \(N\)-th root of unity, and let \(\Phi_N(q)\) denote the \(N\)-th cyclotomic polynomial so that \(\Phi_N(\omega) = 0\). We set

\[\mathcal{A} = \{f \in C(q) \mid f \text{ is regular at } \Phi_N(q) = 0\}.\]

Let \(U_{\mathcal{A}}\) denote the \(\mathcal{A}\)-subalgebra of \(U_q\) generated by \(e_i, f_i, t_i (1 \leq i \leq n)\). Let further \(U_w = U_{\mathcal{A}} \otimes_{\mathcal{A}} C_w\), where \(C_w\) denotes the \(\mathcal{A}\)-algebra \(C\) on which \(q\) acts as \(\omega\). We define \(\mathcal{W}_{\mathcal{A}}, \mathcal{Y}_{\mathcal{A}}\) analogously.

Consider an \(N\)-dimensional vector space with fixed basis \(u_i (0 \leq i \leq N-1)\).
We define the following representation \( \sigma \) of the Weyl algebra \( \mathcal{W}_u \) with generators \( x, z \):

\[
\sigma : \mathcal{W}_u \rightarrow \text{End}(V_u),
\]

\[
\sigma(x)u_i = u_{i+1} \quad (u_N = u_0), \quad \sigma(z)u_i = \omega^i u_i.
\]

Set \( m = n(n+1)/2 \) and \( V = (V_u)^{\otimes m} \). Then we have a representation \( \sigma^\otimes m : \mathcal{W}_u \cong (\mathcal{W}_u)^{\otimes m} \rightarrow \text{End}(V) \) by letting the generators \( x_{jk}, z_{jk} \) act on the \( (j, k) \)-component of \( V \) as \( \sigma(x), \sigma(z) \) and as identity on the other components. Composition with \( \rho_{rs} : U_u \rightarrow \mathcal{W}_u \) and the automorphisms \( S_g, T_h \) gives rise to a representation

\[
\pi : U_u \xrightarrow{\rho_{rs}} \mathcal{W} \xrightarrow{S_g \circ T_h} \mathcal{W} \xrightarrow{\sigma^\otimes m} \text{End}(V).
\] (3.1)

Besides \( r, s \in (C^\times)^n \), \( \pi \) contains \( n(n+1) \) arbitrary parameters \( g = (g_{jk}), h = (h_{jk}) \in (C^\times)^m \). In view of Proposition 2.4, these parameters are not all independent, and we can set e.g. \( s_i = 1 \) \( (1 \leq i \leq n) \) without loss of generality.

The goal of this section is to show that (3.1) is irreducible for generic choice of the parameters \( r_i, g_{jk} \) and \( h_{jk} \). For this purpose we prepare some lemmas.

In what follows we set

\[
\left\{ x; \mu \right\} = \frac{xq^\mu \{xq^{\mu-1}\} \cdots \{xq^{\mu-l+1}\}}{[\nu]!}.
\]

**Lemma 3.1.** For any positive integer \( l \) we have in \( \mathcal{W} \)

\[
\rho_{rs}(e^{(l)}) = \sum_{k \geq 1} \sum_{\ell = \ell_1}^{l} \prod_{r} \{ \zeta_{ik} \nu_r - \nu_{r+1} \} \prod_{r} \xi_{ikr}^{-\nu_r+\nu_{r+1}}.
\]

where \( \nu_{l+1} = 0 \).

**Proof.** We use the induction on \( l \). The case \( l = 1 \) follows from the definition.

Assuming the Lemma for \( l \), we can calculate \( \rho_{rs}(e^{(l+1)})/[l!] \) as

\[
\sum_{k \geq 1} \sum_{\ell = \ell_1}^{l} \sum_{i, j, s, t, y \geq 1} \{ \zeta_{ij} \} \xi_{ik} \prod_{r} \{ \zeta_{ikr} \nu_r - \nu_{r+1} \} \prod_{r} \xi_{ikr}^{-\nu_r+\nu_{r+1}}.
\]

Since \( \xi_{ik} \zeta_{ij} = \zeta_{ij} \xi_{ik} q^{-\theta(i \geq k) - \theta(j > k)} \) the summand becomes

\[
\{ \zeta_{ik} \} \prod_{r} \{ \zeta_{ikr} \nu_r - \nu_{r+1} \} \xi_{ik} \prod_{r} \xi_{ikr}^{-\nu_r+\nu_{r+1}}.
\] (3.2)

We divide the sum into two parts according to whether (i) \( k = k_s \) with some \( s \),
or (ii) \( k_{s-1} > k > k_s \) with some \( s \). In the first case let \( k'_j = k_j \) (1 \( \leq j \leq p \)), \( \nu'_j = \nu_j + \theta(j \leq s) \). In the second case replace \( p \) by \( p-1 \), and set \( k'_j = k_j \), \( \nu'_j = \nu_j + 1 \) (\( j < s \)), \( k'_j = k_{j-1} \), \( \nu'_j = \nu_j \) (\( s < j \leq p \)). Then \( l+1 \geq \nu'_i > \cdots > \nu'_p \geq 1 \), and the cases (i), (ii) correspond to \( \nu'_i - \nu'_i+1 > 1 \) or \( =1 \), respectively.

Dropping primes and rewriting the summand, we get in both cases

\[
\prod_{r<s} \left\{ \zeta_{ik_r} : -\nu_{r+1} - 1 \right\} \cdot \left\{ \zeta_{ik_s} : -\nu_{s+1} - 1 \right\} \cdot \prod_{r \geq s} \left\{ \zeta_{ik_r} : -\nu_{r+1} \right\} \prod_{r=1}^p \zeta_{ik_r}^{\nu_r - \nu_{r+1}}.
\]

It remains to show that

\[
\sum_{s=1}^p \prod_{r<s} \left\{ \zeta_{ik_r} : -\nu_{r+1} - 1 \right\} \cdot \left\{ \zeta_{ik_s} : -\nu_{s+1} - 1 \right\} \cdot \prod_{r \geq s} \left\{ \zeta_{ik_r} : -\nu_{r+1} \right\} = \prod_{r=1}^{p-1} \left\{ \zeta_{ik_r} : -\nu_{r+1} \right\}.
\]

The sum for \( s=p-1, p \) reads

\[
\prod_{r<p} \left\{ \zeta_{ik_r} : -\nu_{r+1} - 1 \right\} \cdot \left\{ \zeta_{ik_{p-1}} : -\nu_{p-1} - 1 \right\} \cdot \prod_{r \geq p} \left\{ \zeta_{ik_r} : 0 \right\} \cdot \prod_{r+1} \left\{ \zeta_{ik_1} : 0 \right\}[
u_p] \]

\[
= \prod_{r<p} \left\{ \zeta_{ik_r} : -\nu_{r+1} - 1 \right\} \cdot \left\{ \zeta_{ik_{p-1}} : -\nu_{p-1} - 1 \right\} \cdot \prod_{r=p} \left\{ \zeta_{ik_r} : -\nu_{p+1} + 1 \right\}.
\]

The assertion follows by repeating this procedure. \( \square \)

**Lemma 3.2.** Let \( A, B \) be elements in an associative algebra over \( C(q) \), satisfying the relations

\[
A^{(3)}B - ABA + BA^{(3)} = 0,
B^{(3)}A - BAB + AB^{(3)} = 0.
\]

Set \( C = AB - qBA \). Then for any positive integers \( k, l \) we have

(i) \( C^{(l)} = \sum_{j=0}^l (-q)^{l-j} A^{(j)}B^{(l-j)}A^{(l-j)} \),

(ii) \( B^{(k)}A^{(l)} = \sum_{0 \leq j \leq s} (-1)^jq^{-j(k-1)(l-1)}A^{(l-j)}C^{(j)}B^{(k-j)} \),

(iii) \( AB^{(k)}A^{(l)} = B^{(k-1)}A^{(l+1)}B + [l-k+1]B^{(k)}A^{(l+1)} \).

**Proof.** Formulas (i) and (ii) are proved in [Lu]. Formula (ii) with \( k=1 \) together with \( AC = q^{-1}CA \) gives

\[
BA^{(l)} = q^{-l}(A^{(l)}B - CA^{(l-1)})
\]

Multiplying by \( A \) from the left and using the definition of \( C \), we get
ABA^{(l)} = A^{(l+1)}B + [l]BA^{(l+1)} \quad (3.3)

Similarly we have

\[ B^{(l)}A = AB^{(l+1)} + [l]B^{(l+1)}A \].

Then multiplying eq. (3.3) by \( B^{(k)} \) from the left and using the above identity we obtain (iii).

Now consider the following 'monomials' in \( W \):

\[ \Pi \xi_{i,j}^a \Pi \xi_{j,k}^b \quad \alpha_{jk}, \beta_{jk} \in \mathbb{Z} \]

For convenience we shall call an element \( f \in W \) of type I if it is an \( A \)-linear combination of monomials with \( \alpha_{jk}, \beta_{jk} \in \mathbb{N} \), and of type II if \( f \in \mathcal{W}_A \).

**Lemma 3.3.** For any \( i \neq j \), \( \rho_{r,s}(e_{i,j}^N) \in W \) is a sum of elements of type I and of type II.

**Proof.** Assuming \( i < j \) we prove the Lemma by induction on \( j-i \). The case \( i > j \) follows by applying the involutions \( \Phi^{-}\).

Let \( j-i = 1 \). From Lemma 3.1 we have

\[ \rho_{r,s}(e_{i,j}^N + 1) = \sum_{i \leq k \leq n} \xi_{i,j}^{-1} \xi_{i,k} + \text{(elements of type II)} \].

In view of the formula

\[ \Pi_{j=0}^{N-1} (xq^{-j} - x^{-N} q^j) = x^N q^{-N(N-1)/2} - x^{-N} q^{N(N-1)/2} \]

we find that the assertion is true in this case.

Next let us prove the case \( j-i > 1 \). Taking

\[ A = e_{ik}, \quad B = e_{kj}, \quad C = e_{ij} \quad (i < k < j) \]

we can apply Lemma 3.2 to obtain

\[ e_{i,j}^N = e_{i+1,j}^N e_{i+1,j}^N - \frac{q^N e_{i+1,j}^N e_{i+1,j}^N}{[N]!} + \sum_{0 < k < N} (-q)^{N-k} e_{i+1,j}^N e_{i+1,j}^N \]

\[ = \left[ e_{i,j}^N, e_{i+1,j}^N \right] \left( 1 - q^N \right) e_{i+1,j}^N e_{i+1,j}^N \]

\[ + \sum_{0 < k < N} (-q)^{N-k} e_{i+1,j}^N e_{i+1,j}^N + \frac{e_{i+1,j}^N e_{i+1,j}^N}{[k]!(N-k)!} \].
From the induction hypothesis it suffices to show the following.

(i) If \( f, g \) are of type I, then so are \( fg \) and \( [f, g]/[N]! \).
(ii) If \( f \) is of type I and \( g \) is of type II, then \( fg, gf \) and \( [f, g]/[N]! \) are of type II.
(iii) If \( f \) is either of type I or type II and \( g \) is a monomial, then \( [f, g] \) is of type II.

To show (i)-(iii), let \( f = f_0 \prod x_{jk}^{\alpha_{jk}} \prod x_{jk}^{\beta_{jk}}, g = g_0 \prod x_{jk}^{\alpha_{jk}} \prod x_{jk}^{\beta_{jk}}, f_0, g_0 \in \mathcal{A} \).

Then we have

\[
fg = f_0 g_0 \prod q^{-\alpha_{jk} \beta_{jk} + \alpha_{jk}'} x_{jk}^{\beta_{jk} + \beta_{jk}'},
\]

\[
[f, g]/[N]! = q^{-\sum \alpha_{jk} \beta_{jk} - q^{-\sum \alpha_{jk} \beta_{jk}'}} f_0 g_0 \prod x_{jk}^{\alpha_{jk} + \alpha_{jk}'} x_{jk}^{\beta_{jk} + \beta_{jk}'}.
\]

The assertions (i)-(iii) are clear from these. \( \square \)

We now specialize \( g \) to \( \omega \). It is clear that the image under \( \sigma^m \) of elements of type I are scalar, while those of type II vanish. From this we have the following.

**Proposition 3.4.** We have

\[
\pi(t_i^N) = Y_{ii} \text{id}, \quad (1 \leq i \leq n),
\]

\[
\pi(t_i^{N}) = (\omega - \omega^{-1})^{-N} Y_{ij} \text{id}, \quad (1 \leq i \neq j \leq n+1).
\]

where \( Y_{ij} \in \mathbb{C} \) are given as follows.

\[
Y_{ii} = (r_i s_i^{-1} h_{i}^{-1} h_{i}^{-1} h_{i}^{-1})^N \quad (3.4a)
\]

\[
Y_{i,i+1} = \sum_{l_1 \geq i, \ldots, l_{i+1-i} \geq i} (-1)^{i-1} \theta(k_1 \geq \cdots \geq k_{i+1-i} \geq k_{i+2-i} \geq \cdots \geq k_i)
\]

\[
(\eta_{i-k_i-1} \eta_{i-j-1} - \eta_{i-j-1} \eta_{i-k_i-1}) \theta_{i-k_i-1} \theta_{i-j-1} \quad (3.4b)
\]

\[
Y_{n-i+2,n-i+2} = \sum_{l_1 \geq i, \ldots, l_{n-i+2-i} \geq i} (-1)^{i-1} \theta(k_1 \geq \cdots \geq k_{n-i+2-i} \geq k_{n-i+2-i} \geq k_{n-i+2-i})
\]

\[
(\eta_{i-k_i-1} \eta_{i-j-1} - \eta_{i-j-1} \eta_{i-k_i-1}) \theta_{i-k_i-1} \theta_{i-j-1} \quad (3.4c)
\]

where we set \( \bar{g}_{jk} = g_{k+1-j}^{-1}, \bar{h}_{jk} = h_{k+1-j}^{-1}, \bar{s} = s_{n+1-i}^{-1} \),

\[
\eta_{ik} = (r_i h_{i} h_{i-1}^{-1} h_{i-1}^{-1})^N
\]

\[
\eta_{ik} = (s_i h_{i} h_{i-1}^{-1} h_{i-1}^{-1})^N
\]

\[
\theta_{ik} = \prod_{l=k}^{k} g_{i,l}^{N}, \quad \theta_{ik} = \prod_{l=k}^{k} (\bar{g}_{i,l})^{N}
\]

and \( \phi_{i,k_1 \cdots k_i} = \phi_{i,k_1 \cdots k_i} \cdots \phi_{i+l-1-k_i} \) for \( \phi = \eta, \bar{\eta}, \theta, \bar{\theta} \).

Let us now examine the irreducibility of the above representations of \( U_{\omega} \).
Hereafter we assume $s_i = 1$ without loss of generality. We denote by $Y$ the map $C^{2n+n} \to C^{2n+n}$, $(g_{ij}, h_{ij}, r_i)\to (Y_{ij})$ defined in Proposition 3.4.

Let $Z_0$ denote the subring of the center of $U_\omega$ generated by $t^N$ and $e^N$, and let $\text{Rep}$ be the set of equivalence classes of irreducible representations of $U_\omega$. De Concini and Kac show [DK] that

(i) $Z_0$ is a polynomial ring in $n(n+2)$ indeterminates,

(ii) the natural map $\chi': \text{Rep} \to \text{Spec } Z_0 = C^{n(n+2)}$ has finite fibre,

(iii) $\dim R \leq N^m$ for all $R \in \text{Rep}$, the equality being true for generic $\chi'(R)$.

To show the irreducibility of our representations $\pi$ (for generic values of parameters), it is therefore sufficient to show that the image of $Y$ contains an open set, since our representations are $N^m$ dimensional. We shall show that the Jacobian determinant of $Y$ at the point $P: g_{ij} = h_{ij} = 1$ ($1 \leq i \leq j \leq n$), $r_i = t$ ($1 \leq i \leq n$) does not vanish (for generic $t$).

It is easy to see that the Jacobian matrix at $P$ is of triangular form.

$$\left. \frac{\partial((Y_{ij})_{1 \leq i \leq j \leq n+1})}{\partial((h_{ij})_{1 \leq i \leq j \leq n}, (r_i)_{1 \leq i \leq n}, (g_{ij})_{1 \leq i \leq j \leq n})} \right|_P$$

$$= \begin{pmatrix}
(Y_{ij})_{1 \leq i \leq j \leq n+1} & (Y_{ij})_{1 \leq i \leq n} & (Y_{ij})_{1 \leq i \leq j \leq n+1} \\
(H) & * & * \\
0 & R & * \\
0 & 0 & G
\end{pmatrix}.$$ 

We shall show that $H, R, G$ are non-singular. From (3.4a) it follows that $R$ is $Nt^{n-1}\text{id}_N (\text{id}_N: N \times N$ identity matrix).

Proposition 3.5. The Jacobian matrix $H$ is non-singular.

Proof. The change of variables $(h_{ij})_{1 \leq i \leq j \leq n} \to (\tilde{h}_{ij})_{1 \leq i \leq j \leq n}$ is invertible at $h_{ij} = 1$ ($1 \leq i \leq j \leq n$). The change of variables $(\tilde{h}_{ij})_{1 \leq i \leq j \leq n} \to (\bar{h}_{ij})_{1 \leq i \leq j \leq n}$ where $\eta_{ij} = h_{ij} \eta_{i,j-1} \eta_{i-1,j-1} \eta_{i-1,j}$ is also invertible. In fact, the inverse map is given inductively by

$$\bar{h}_{ij} = \eta_{ij} \eta_{i+1,j} \cdots \eta_{jj} \eta_{i-1,j-1}.$$ 

Therefore, it is sufficient to show that the Jacobian matrix

$$\left. \frac{\partial((Y_{\eta+2-i,n+1-j})_{1 \leq i \leq n})}{\partial((\eta_{ij})_{1 \leq i \leq n})} \right|_P$$

is non-singular. We have
\[
\frac{\partial Y_{i+1,i+1}}{\partial \eta_{k,l}} \bigg|_P = -2 \sum_{k_1 \geq l}^{j-i+1} \sum_{j=1}^{i-1} (-1)^{j-1} \delta_{k_i+l-j} \theta(k_1 \geq \cdots \geq k_j < \cdots < k_{j-i+1}) \times \left( \begin{array}{c} j-k \\ j-k \end{array} \right)
\]

Here we have adopted the following convention for the usual binomial coefficients

\[
\left( \begin{array}{c} j \\ k \end{array} \right) = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad j < k.
\]

If \( j < k \) then \( \partial Y_{i+1,i+1} \big|_{\partial \eta_{k,l}} = 0 \) at \( P \). Therefore, it is sufficient to show that the diagonal blocks \( \left( \begin{array}{c} n-l \\ j-k \end{array} \right) \bigg|_{k \leq j \leq n} \) are non singular. Note that

\[
\left( \begin{array}{c} n-l \\ j-k \end{array} \right) = 0 \quad \text{if} \quad l+j > n+k \\
= 1 \quad \text{if} \quad l+j = n+k.
\]

The assertion follows immediately.

\[\Box\]

**Proposition 3.8.** The Jacobian matrix \( G \) is non-singular.

**Proof.** We have

\[
\frac{\partial Y_{i+1}}{\partial g_{r,s}} = \sum_{k_i \geq i, \ldots, k_{j-i+1} \geq j-i+1} \left( -1 \right)^{j-i+1} \delta_{k_i \geq \cdots \geq k_{j-i+1}} \theta(k_i \geq \cdots \geq k_{j-i+1}) \\
\times \left( \begin{array}{c} n-k \\ n-k \end{array} \right) \bigg|_{k \leq j \leq n} \frac{\partial \theta_{j,k-i}}{\partial g_{r,s}},
\]

\[
\frac{\partial \theta_{j,k-i}}{\partial g_{r,s}} = N \delta_{s \geq k} \theta(s \geq k). 
\]

In the limit \( t \to 0 \), the leading terms of the right hand side come from \( j=1 \). Setting \( l = j+1-i \) and \( r' = r+i+1 \) we get

\[
\frac{\partial Y_{i+1}}{\partial g_{r,s}} = -N \left( -i^{j-i+1} \right) \left( D_{ij,rs} + O(t) \right)
\]

where

\[
D_{ij,rs} = \sum_{i \leq k_1 < \cdots < k_{j-i+1} \leq k_{j-i+1}} \theta(s \geq k_r) \\
= \sum_{r \leq k \leq s} \sum_{i \leq k_1 < \cdots < k_{j-i+1} \leq n} \sum_{k \leq k_{j-i+1} \leq n} \frac{1}{n-k} \\
= \sum_{r \leq k \leq s} \left( \begin{array}{c} k-l \\ j-r \end{array} \right) \left( \begin{array}{c} n-k \\ j-r \end{array} \right). 
\]

(3.5)
It remains to show that $\det(D_{ij})_{1 \leq i, j \leq n} = 0$. Since the summand of (3.5) is independent of $s$, it is easy to see that $\det(D_{ij}) = \det(D'_{ij})$ where $D'_{ij} = (s-l)^{n-s-j-i-r}$. This matrix is block triangular since $D'_{ij} = 0$ if $r < i$. Further if $r = i$ we have $\det(D'_{ij})_{j \leq i, i \leq n} = 1$. This completes the proof. \qed

If we specialize the parameters, we get invariant subspaces. Let $l$ be an integer such that $0 \leq l \leq n-1$ and let $i$ be an integer such that $0 \leq i \leq N-1$. Set

$$u_{i,l} = \underbrace{u_i \otimes \cdots \otimes u_i}_{(j,k)-\text{component, } 1 \leq j \leq k \leq l}$$

and

$$V_{i, l} = u_{i,l} \otimes \underbrace{V^1 \otimes \cdots \otimes V^1}_{(j,k)-\text{component, } 1 \leq j \leq k \leq n}$$

Fix the parameters $r_j (1 \leq j \leq l)$ and $h_{jk} (1 \leq j \leq k \leq l)$ in such a way that

$$r_j = \omega^{-2}, \quad h_{jk} = (1 \leq j \leq k \leq l)$$

and

$$h_{jk-1} h_{j-1,k-1} = \omega^{1-ik}, \quad h_{jk-1} h_{j-1,k-1} = \omega^{-1+jk}$$

where $h_{jk} = h_{k-j-1,k}$. From Lemma 2.5, (3.6b) and (3.6c) are consistent and have a unique solution. The parameters $r_l (1 \leq l \leq n)$, $g_{ik} (1 \leq i \leq k \leq n)$ and $h_{ik} (1 \leq j \leq k, l < k \leq n)$ are free.

**Proposition 3.9.** The subspace $V_{i, l}$ is $U_\omega$ invariant.

**Proof.** It is easy to see that $V_{i, l}$ is invariant by $e_j, f_{n+1-j}$ with $l < j \leq n$ and $t_i$ for all $i$. If $1 \leq j \leq l$, we have

$$\rho_{r_i}(e_j) = \sum_{k \geq j} \xi_{jk} \{ \omega^{-1} z_{jk} z_{j-1,k-1} z_{j+1,k} \}$$

and

$$\rho_{r_j}(f_{n+1-j}) = \sum_{k \geq j} \xi_{jk} \{ \omega z_{jk} z_{j-1,k-1} z_{j+1,k} \}$$

If $1 \leq j \leq k \leq l$, then

$$z_{jk} z_{j-1,k-1} z_{j+1,k} = \omega^{jk+1} u_{i,l}, \quad z_{jk} z_{j-1,k-1} z_{j+1,k} = \omega^{-jk} u_{i,l}$$

and therefore, by using (3.6b), (3.6c) and $\{1\} = 0$,

$$\sigma^\omega S_m T_k(\{ \omega^{-1} z_{jk} z_{j-1,k-1} z_{j+1,k} \}) u_{i,l} = 0,$$

$$\sigma^\omega S_m T_k(\{ \omega z_{jk} z_{j-1,k-1} z_{j+1,k} \}) u_{i,l} = 0.$$
Thus we have shown that $V_{i,t}$ is invariant by the action of $e_j$ and $f_j$. 

In [DJMM], we have constructed an $N^n$ dimensional irreducible representation. We shall show that it is isomorphic to the representation on $V_{0,n-1}$. First we recall the construction in [DJMM]. Let $W=(V^i)^{\otimes(n+1)}$ and let $Z_j$ and $X_j$ ($1 \leq j \leq n+1$) be the operators acting as $\sigma(z)$ and $\sigma(x)$ on the $j$-th component of $W$. Set $W^{(0)} = \{ u \in W \mid \prod_{j=1}^{n+1} Z_j u = u \}$. Define $\pi' : U_n \rightarrow \text{End}_C(W^{(0)})$ by

$$
\pi'(e_j) = b_j \{ a_j Z_j \} X_j X_{j+1}^{-1}, \\
\pi'(f_j) = b_j^{-1} \{ a_{j+1} Z_{j+1} \} X_{j+1}^{-1} X_j, \\
\pi'(t_j) = \frac{a_j}{a_{j+1}} Z_j Z_{j+1}^{-1}.
$$

Now consider $V_{0,n-1} \cong (V^i)^{\otimes n}$. Let $C_j$ and $Q_j$ ($1 \leq j \leq n$) be the operators acting as $\sigma(z)$ and $\sigma(x)$ on the $j$-th component of $(V^i)^{\otimes n}$. Then the subrepresentation of $\pi$ on $V_{0,n-1}$, which we denote also by $\pi$, reads as

$$
\pi(e_j) = g_j \{ r_j h_{j,n-1} h_{j-1,n-1} h_{j-1,n} C_j C_{j+1} \} Q_j, \\
\pi(f_j) = g_j^{-1} \{ h_{j,n-1}^{-1} h_{j-1,n-1} h_{j-1,n} C_{j-1}^{-1} C_{j-1} \} Q_j^{-1}, \\
\pi(t_j) = r_j h_{j,n}^{-1} h_{j-1,n} C_{j-1} C_{j+1}^{-1} C_{j+1}.
$$

Choose $d_j$ ($1 \leq j \leq n$) in such a way that

$$
\frac{d_j^N}{d_{j+1}^N} \frac{a_j^N - a_{j+1}^{-N}}{a_{j+1}^N - a_j^{-N}} = 1. \tag{3.7}
$$

For $1 \leq j \leq n$, define mutually commuting operators

$$
\Phi_j = (f_j(Z_j)X_j)(f_{j+1}(Z_{j+1})X_{j+1})^{-1} = \frac{f_j(Z_j)}{f_{j+1}(\omega Z_{j+1})} X_j X_{j+1}^{-1} = \left( \frac{f_{j+1}(Z_{j+1})}{f_j(\omega Z_j)} X_{j+1} X_j^{-1} \right)^{-1},
$$

where

$$
f_j(Z_j) = d_j \{ a_j Z_j \}.
$$

The equation (3.7) implies $\Phi_j^N = 1$.

Define a linear isomorphism

$$
\psi : (V^i)^{\otimes n} \rightarrow W^{(0)}
$$

by

$$
\psi(u_1 \otimes \cdots \otimes u_n) = \prod_{j=1}^{n} \Phi_j^{u_j} u_j^{\otimes(n+1)}.
$$
We denote also by $\psi$ the induced algebra homomorphism

$$\psi: \text{End}_c((V^1)^{\otimes n}) \to \text{End}_c(W^{(0)}) .$$

Then we have

$$\psi(C_j C^{-1}_{j-1}) = Z_j ,$$  
(3.8a)

$$\psi(\delta_j) = \Phi_j .$$  
(3.8b)

We shall show that by a suitable choice of the parameters $h_{jn}$, $g_{jn}$, $1 \leq j \leq n$ and $r_n$ we have

$$\psi \circ \pi(e_j) = \pi'(e_j) ,$$  
(3.9a)

$$\psi \circ \pi(f_j) = \pi'(f_j) ,$$  
(3.9b)

$$\psi \circ \pi(t_j) = \pi'(t_j) .$$  
(3.9c)

The equations (3.9a) and (3.9b) give rise to

$$b_j \{a_j Z_j\} = g_{jn} \{r_j h_{jn} h_{j-1}^{-1} h_{j-1,n-1} h_{j-1,n}^{-1} Z_{j-1}^{-1}\} \frac{d_j \{a_j Z_j\}}{d_{j+1}\{a_{j+1} Z_{j+1}\}} ,$$

$$b^{-1}_{j+1} \{a_{j+1} Z_{j+1}\} = g_{jn}^{-1} \{h_j h_{j-1}^{-1} n^{-1} h_{j-1,n-1} h_{j-1,n}^{-1} Z_{j-1}^{-1}\} \frac{d_{j+1} \{a_{j+1} Z_{j+1}\}}{d_j \{a_j Z_j\}} .$$

These are satisfied if we choose

$$g_{jn} = -b_j d_{j+1}/d_j ,$$

$$h_{jn} h_{j-1,n}^{-1} = \omega a_j h_{j-1,n}^{-1} h_{j-1,n-1} ,$$

$$r_n h_{nn} = \omega^{-1} a_{n+1} h_{n-1,n-1} .$$

Finally, (3.9c) is checked easily.

In closing we shall explain how we get the above representations.

For $\lambda=(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, let $M_\lambda=U_q/I$ denote the Verma module where $I=\sum_i U_q e_i + \sum_i U_q(t_i-q^{\lambda_i})$. Let $v_\lambda=1 \text{ mod } I$ be the highest weight vector. Let further $F^k(j_{1k} \cdots j_{kk})=f^k(j_{1k}) \cdots f^k(j_{kk}) (j_{rh} \in \mathbb{Z}_{\geq 0}, 1 \leq r \leq k \leq n)$. We define

$$v_\lambda[j] = F^k(j_{1m}, \cdots, j_{nn}) \cdots F^k(j_{10}) v_\lambda, \quad j = (j_{rs})_{1 \leq r \leq s \leq n} \in \mathbb{Z}_{\geq 0} .$$

From now on, $j_{rs}=0$ unless $1 \leq r \leq s \leq n$, and by convention $F^k(j_{1k}, \cdots, j_{kk})=0$, $v_\lambda[j]=0$ if a negative value of some $j_{rs}$ appears in the expressions. Then we have

**Proposition 3.10.** Set $e_{jk}=(\delta_{jr}, \delta_{kr})_{1 \leq r \leq s \leq n}$. The action of $e_i, f_i$ and $t_i$ on vectors $v_\lambda[j]$ is given by
CYCLIC REPRESENTATIONS

\[ e_i \nu_k[j] = \sum_{k=1}^{n} \left[ 1 + \lambda_i - j_{ik} - 2 \sum_{l=1}^{k-1} j_{il} + \sum_{l=1}^{k-1} j_{i-1,l} \right] \nu_k[j - \epsilon_{ik}], \]

\[ f_i \nu_k[j] = \sum_{k=1}^{n} \left[ 1 + j_{ik} + k + \lambda_i - j_{ik} + - j_{ik} - j_{ik - 1,k} \right] \nu_k[j + \sum_{l=1}^{k} \epsilon_{i+1,l} - \sum_{l=1}^{k} \epsilon_{i+1,l-1,k} + \epsilon_{i+1,l-1,k} \nu_k[j]. \]

\[ t_i \nu_k[j] = q^{\lambda_i - 2} \sum_{k=1}^{n} j_{i-1} \nu_k[j + \sum_{l=1}^{k} j_{i-1,l} + \sum_{l=1}^{k} j_{i+1,l} \nu_k[j]. \]

**Proof.** From \([e_i, f_j] = \delta_{ij} f_i^{(j-1)} \{ q^{i-j} t_i \} \)** and Lemma 3.2 with \( A = f_i \) and \( B = f_{i-1} \), we have

\[ [e_i, f_i^{(j)}] = \theta(k \geq i) \times F^k(j, i, j_{i-1}, k, j_{i+1}, \ldots, j_{jk}) \{ q^{i-j} t_i \}, \]

and for \( 1 \leq i \leq k \)

\[ f_i f_i^{(j)}(j, i, j_{i-1}, j_{i-1,k} + j_{i, k} - 1, j_{i,k} + 1, j_{i+1, k}, \ldots, j_{jk}) = [j_{ik} - j_{i+1,k} + 1] f_i^{(j-1)}(j, i, j_{i-1,k} + j_{i,k} + 1, j_{i+1, k}, \ldots, j_{jk}). \]

Here \( f_0 \) is understood to be zero. From these the proposition follows. \( \square \)

Returning to the Weyl algebra \( \mathcal{W} \), let us consider the following representation of \( \mathcal{W} \) on the linear span of the symbols \( V[J], J \in \mathbb{Z}^m \):

\[ x_{jk} V[J] = V[J - \epsilon_{jk}], \quad z_{jk} V[J] = q^{f_j} V[J]. \]

This induces a representation of \( U_q \) via \( \rho_{r,s} \) with \( r_i = q^{h_i}, s_i = 1 \). Explicitly it reads

\[ e_i V[J] = \sum_{k=1}^{n} \left[ 1 + \lambda_i - j_{ik} + j_{i+1,k} + j_{i-1,k} - j_{ik} - 1 \right] V[J - \sum_{l=1}^{k} \epsilon_{il}], \]

\[ f_i V[J] = \sum_{k=1}^{n} \left[ 1 + j_{ik} + k + \lambda_i - j_{ik} + j_{i+1,k} - j_{ik} + j_{i+1,k} - j_{ik} + j_{i+1,k} \right] \times V[J + \sum_{l=1}^{k} \epsilon_{il} - 1], \]

\[ t_i V[J] = q^{\lambda_i - 2} f_{j_{i+1}} + j_{i+1}^+ j_{i+1}^+ V[J]. \]

If we formally identify \( V[J] = \nu_k[j] \) with \( J_{rs} = \sum_{l=1}^{n} f_{l} \), we see that this representation has the same form as the one for the Verma module described above. (Note that in the latter case the vectors \( \nu_k[j] \) are not linearly independent in general.)

**Acknowledgement**

The authors would like to thank V.V. Bazhanov, M. Kashiwara, G. Lusztig, B. M. McCoy and N. Yu. Reshetikhin for discussions. They also thank
C. De Concini and V. G. Kac for sending their preprint. K.M. is supported in part by the Grant-in-aid for Scientific Research from the Ministry of Education, Science and Culture of Japan No. 01790204.

References


[DJMM] Date, E., Jimbo, M., Miki, K. and Miwa, T., $R$-matrix for cyclic representations of $\widehat{U}_q(\mathfrak{sl}(3, C))$ at $q^3=1$, Phys. Lett. A 148 (1990), 45–49.

