A New Method Using the Circles of Curvature for Solving Equations in $\mathbb{R}^1$

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Abstract

In this paper, we propose a numerical method using the circles of curvature for solving the equations in $\mathbb{R}^1$, whose order of convergence is cubic. Some numerical examples are given, for which the method works well, while there is shown an example failed by means of the Newton-Raphson's method.

§ 1. A New Method for Solving Equations in $\mathbb{R}^1$

Consider the equation

$$F(x) = 0$$

in $\mathbb{R}^1$ and let $x_0$ be an approximate solution for (1). As have been well known, the circle of curvature at the point $(x_0, y_0) = (x_0, F(x_0))$ on the curve $y = F(x)$ is given by

$$(x - x_0 + \frac{y_0'(1+y_0'^2)}{y_0''})^2 + (y - y_0 - \frac{1+y_0'^2}{y_0''})^2 = \left(1+\frac{y_0'^2}{y_0''}\right)^3,$$  \hspace{1cm} (2)

provided that $F \in C^2$. Therefore, if we define the next approximation $x_1$ by the $x$-coordinate of the point at which the circle (2) intersects the $x$-axis, then we obtain the following iterative procedure for solving the equation (1):

$$(x_{n+1} - x_n + \frac{y_n'(1+y_n'^2)}{y_n''})^2 = \left(1+\frac{y_n'^2}{y_n''}\right)^3 - \left(y_n + \frac{1+y_n'^2}{y_n''}\right)^2,$$

i.e.,

$$(x_{n+1} - x_n)^2 + 2B_n(x_{n+1} - x_n) + C_n = 0$$ \hspace{1cm} (3)


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where \( y_n = F(x_n), y^{(i)}_n = F^{(i)}(x_n), i = 1, 2, \)

\[
B_n = y_n'(1+y_n'^2)/y''_n
\]

and

\[
C_n = (y_n^2y_n''+2y_n(1+y_n'^2))/y''_n.
\]

Substituting \((y_n/y_n')^2\) for \((x_{n+1} - x_n)^2\) in (2), we have

\[
2B_n(x_{n+1} - x_n) = -C_n - y_n^2/y_n'^2,
\]

i.e.,

\[
x_{n+1} = x_n - (y_n^2y_n'' + 2y_ny_n'^2)/(2y_n'^2) \quad (n \geq 0).
\]

(4)

We shall call the procedure (4) as the first method of the curvature iteration.

On the one hand, if we solve the quadratic equation (3) with respect to

\[x_{n+1} - x_n,\]

then we have

\[
x_{n+1} - x_n = -B_n + \text{sign}(B_n)(B_n^2 - C_n)^{1/2}.
\]

(5)

To avoid the loss of significant digits, we modify (5) as follows;

\[
x_{n+1} = x_n - C_n/(B_n + \text{sign}(B_n)(B_n^2 - C_n)^{1/2}),
\]

(6)

which we shall call the second method of the curvature iteration.

§2. Order of Convergence

To examine the order of convergence for (4), let \( \alpha \) be a root of (1) and put

\[
\varphi(x) = x - (y^2 y'' + 2yy'^2)/(2y'^2).
\]

Then, the procedure (4) may be written in the form of \( x_{n+1} = \varphi(x_n) \), so that

\[
x_{n+1} - \alpha = \varphi(x_n) - \varphi(\alpha)
\]

\[
= \varphi'(\alpha)(x_n - \alpha) + \frac{1}{2}\varphi''(\alpha)(x_n - \alpha)^2 + \frac{1}{6}\varphi'''(\alpha)(x_n - \alpha)^3 + \cdots
\]

\[
= \frac{1}{6}\varphi'''(\alpha)(x_n - \alpha)^3 + \cdots,
\]

since, as is easily seen, we have

\[
\varphi'(\alpha) = \varphi''(\alpha) = 0,
\]

and

\[
\varphi'''(\alpha) = (2y'(\alpha)y'''(\alpha) + 3y''(\alpha)^2)/(2y'(\alpha)^2),
\]

provided that \( F \in C^5 \). This implies that the order of convergence for (4) is
cubic, if \( F \in C^5 \).

§3. Numerical Examples

Example 1. Table 3–1 shows the results of computation for both the Newton-Raphson method and the second method applied to the equation \( \sin(2.1x-0.6)=0 \) with the approximation, \( x_0=1 \). As we have observed, the second method converges, but the Newton-Raphson’s method fails. The same situation occurs for the value \( x_0 \) in the interval \([1, 2] \).

![Table 3-1](image)

* Here “1” is given as an initial approximation.

Example 2. In Table 3–2, we compare the methods (4) and (5) with the Newton-Raphson’s method for the following equations.

1. \( F(X) = X^3 - X^2 - 1 = 0 \)
2. \( F(X) = X^4 - 3X^3 - X^2 + 2X + 3 = 0 \)
3. \( F(X) = X^5 - 2X^4 - 4X^3 + X^2 + 5X + 3 = 0 \)
4. \( F(X) = X^6 - 8X^4 - 4X^3 + 7X^2 + 13X + 6 = 0 \)
5. \( F(X) = X^7 + X^6 - 8X^5 - 12X^4 + 3X^3 + 20X^2 + 19X + 6 = 0 \)

![Table 3-2](image)

(* Initial approximation)
As confirmed in Table 3–2, the number of iterations with which our method converges is fewer than the Newton-Raphson’s method.