Scattering Theory for Wave Equations with Dissipative Terms

By

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§ 1. Introduction

We shall consider wave equations of the form

\[ w_{tt}(x, t) + b(x, t)w_t(x, t) - \Delta w(x, t) = 0, \]

where \( x \in \mathbb{R}^n \) \((n \neq 2)\), \( t \geq 0, \)

\( w_t = \partial w/\partial t, \ w_{tt} = \partial^2 w/\partial t^2 \) and \( \Delta \) is the \( n \)-dimensional Laplacian. \( b(x, t) \) is a non-negative function and is assumed to satisfy the following conditions:

(A1) There exist constants \( C_1 > 0 \) and \( \delta > 0 \) such that

\[ 0 \leq b(x, t) \leq C_1 (1 + |x|)^{-1 - \delta} \]

for any \( x \in \mathbb{R}^n, \ t \geq 0. \)

(A2) \( b_t(x, t) \) is bounded continuous in \( x \in \mathbb{R}^n \) and \( t \geq 0. \)

In the following we assume that \( \delta \leq 1 \) without any loss of generality.

Since \( b(x, t) > 0, \) \( b(x, t)w_t(x, t) \) represents the resistance of viscous type. Our aim of this note is to show that the solutions of (1.1) are asymptotically equal for \( t \rightarrow \infty \) to those of the free wave equation

\[ w_{tt}(x, t) - \Delta w(x, t) = 0. \]

More precisely, we shall show the existence of the Møller wave operators.

We restrict ourselves to solutions with finite energy. For pairs \( f = \{f_1, f_2\} \) of functions in \( \mathbb{R}^n \) the energy is defined by

\[ \|f\|_E^2 = \int_{\mathbb{R}^n} (|Df_1|^2 + |f_2|^2) \, dx. \]

where \( Df_i = (D_{x_1} f_i, \ldots, D_{x_n} f_i) \) \((D_j = \partial/\partial x_j)\) and \( |Df_i|^2 = \sum_{j=1}^n |D_j f_i|^2. \) The Hilbert space \( \mathcal{H} \) is defined as the completion in the energy norm of

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smooth data with bounded support in $\mathbb{R}^n$. Put $u = \{w, w_t\}$. Then (1.1) can be expressed in the matrix notation as

\begin{equation}
(1.4) \quad u_t = A(t)u = A_0u - V(t)u,
\end{equation}

where

\[ A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & b(x, t) \end{pmatrix}. \]

Put $u^0 = \{w^0, w^0_t\}$. Then (1.2) is expressed as

\begin{equation}
(1.5) \quad u^0_t = A_0 u^0.
\end{equation}

$A_0$ determines a skew-selfadjoint operator in $\mathcal{H}$ with domain

\begin{equation}
(1.6) \quad \mathcal{D}(A_0) = \{f \in \mathcal{H}; \mathcal{A}f, D\mathcal{A}_f \in L^2(\mathbb{R}^n) \ (j=1, \ldots, n)\},
\end{equation}

where all the derivatives are considered in the distribution sense. Thus, $A_0$ generates a one-parameter group $\{U_0(t) = e^{tA}; t \in \mathbb{R}\}$ of unitary operators. Under the above conditions on $b(x, t)$, $A(t)$ determines for each $t \geq 0$ a closed operator in $\mathcal{H}$ with domain $\mathcal{D}(A(t)) = \mathcal{D}(A_0)$. Moreover, positive numbers belong to the resolvent set of each $A(t)$ and $A(t)(A(0) - I)^{-1}$, where $I$ is the identity in $\mathcal{H}$, is continuously differentiable in $t$ in operator norm. Thus applying results of Kato [2], we see that there exists a unique family $\{U(t, s); t \geq s \geq 0\}$ of contraction evolution operators which is defined as mapping solution data of (1.4) at time $s$ into those at time $t$.

Now the main results can be stated as follows:

**Theorem 1.** (a) *The wave operator*

\begin{equation}
(1.7) \quad Z = \lim_{t \to \infty} U_0(-t)U(t, 0)
\end{equation}

exists. (b) *$Z$ is a not identically vanishing contraction operator in $\mathcal{H}$. (c) If we denote by $Z^*$ the adjoint of $Z$, then*

\begin{equation}
(1.8) \quad Z^* = \lim_{t \to \infty} U(t, 0)^*U_0(t).
\end{equation}

We also consider the special case where $b(x, t)$ is independent of $t$. Then the operator $A = A_0 - V$, where $V = \begin{pmatrix} 0 & 0 \\ 0 & b(x) \end{pmatrix}$, generates a semi-group $\{U(t); t \geq 0\}$ of contraction operators.
In this case we have the following

**Theorem 2.** (a) The wave operators

\begin{equation}
W = \lim_{t \to \infty} U(t) U_0(-t),
\end{equation}

\begin{equation}
Z = \lim_{t \to \infty} U_0(-t) U(t)
\end{equation}

exist. (b) They both are not identically vanishing contraction operators in \( \mathcal{H} \). (c) \( U_0(t) \) and \( U(t) \) are intertwined by both \( W \) and \( Z \), i.e.,

\begin{equation}
W U_0(t) = U(t) W, \quad Z U(t) = U_0(t) Z \quad \text{for any } t \geq 0.
\end{equation}

(d) The scattering operator, defined by \( S = ZW \), commutes with \( U_0(t) \):

\begin{equation}
S U_0(t) = U_0(t) S \quad \text{for any } t \in \mathbb{R}.
\end{equation}

The proof of these theorems will be based on the "smooth perturbation theory" developed by Kato [3].

The above theorems generalize some results already announced in Mochizuki [7], where the main concern was in the local energy decay for wave equations with non-linear dissipative terms. The scattering theory has been developed by Lax-Phillips [4] for wave equation: \( \frac{\partial^2 w}{\partial t^2} = \Delta w \) in an exterior domain of \( \mathbb{R}^n \) (\( n \geq 2 \)) with lossy boundary conditions: \( \alpha(x) \frac{\partial w}{\partial t} = Q \), \( \alpha(x) > Q \). Some related problems has been studied in [1] and [5].

§ 2. Preliminaries

First we shall show an inequality for \( L^2 \)-solutions of the Helmholtz equation

\begin{equation}
-\Delta u - \kappa^2 u = f(x) \quad \text{in } \mathbb{R}^n,
\end{equation}

where \( \kappa \) is a complex number such that \( \text{Im } \kappa \neq 0 \) and \( f(x) \) is a function such that \( (1 + |x|)^{n+\alpha/2} f(x) \in L^2(\mathbb{R}^n) \).

**Lemma 2.1.** Let \( \text{Im } \kappa \geq 0 \). Then we have for any \( \rho > 0 \)
\[
(2.2) \quad \frac{1}{2} \int_{S_s} \left( \frac{\partial u}{\partial r} + \frac{n-1}{2r} u \right)^2 + |\kappa|^2 |u|^2 \, dS
+ |\text{Im} \ \kappa| \int_{S_s} \left( |Du|^2 + \frac{n-1}{2r} |u|^2 + |\kappa|^2 |u|^2 \right) \, dx
= \frac{1}{2} \int_{S_s} |\theta_\pm|^2 \, dS + \int_{S_s} \text{Re} \left[ f i \kappa u \right] \, dx,
\]
where \( r = |x|, S_o = \{ x; |x| = \rho \}, K_o = \{ x; |x| < \rho \} \) and
\[
(2.3) \quad \theta_\pm = \frac{\partial u}{\partial r} + \frac{n-1}{2r} u + i \kappa u.
\]

Proof. Note the identity
\[
-\text{Re} \left[ \frac{\partial u}{\partial r} i \kappa u \right] = -\text{Im} \ \kappa \ \frac{n-1}{2r} |u|^2 + \frac{1}{2} |\theta_\pm|^2 + \frac{1}{2} \left( \left| \frac{\partial u}{\partial r} + \frac{n-1}{2r} u \right|^2 + |\kappa|^2 |u|^2 \right).
\]
Then (2.2) follows from the integration by parts of (2.1) multiplied by \( i \kappa u \).

Lemma 2.2. Let \( \text{Im} \ \kappa \geq 0 \). Then we have
\[
(2.4) \quad |\text{Im} \ \kappa| \int_{R^s} r^3 \left\{ |\zeta_\pm|^2 + \frac{(n-1)(n-3)}{4r^3} |u|^4 \right\} \, dx
+ \int_{R^s} r^{-1-\delta} \left\{ \left( 1 - \frac{\delta}{2} \right) (|\zeta_\pm|^2 - |\theta_\pm|^2) + \frac{\delta}{2} |\theta_\pm|^2 \right\} \, dx
+ \frac{(n-1)(n-3)(2-\delta)}{8} \int_{R^s} r^{-3+2\delta} |u|^4 \, dx
= \int_{R^s} r^4 \text{Re} \left[ f \bar{\theta}_\pm \right] \, dx,
\]
where
\[
(2.5) \quad \zeta_\pm = Du + \frac{n-1}{2r} \frac{x}{r} u + i \kappa \frac{x}{r} u.
\]

Proof (cf., Mochizuki [6]). Put \( v = e^{-i \kappa r} r^{(n-1)/2} u \). Then
\[
(2.6) \quad -4v + \left( \frac{n-1}{2} + 2i \kappa \right) \frac{\partial v}{\partial r} + \frac{(n-1)(n-3)}{4r^2} v = e^{-i \kappa r} r^{(n-1)/2} f.
\]
Multiply by \( e^{+i \text{Im} \ \kappa r^{-n+1+\delta}} \left( \frac{\partial \bar{v}}{\partial r} \right) \) on both sides and take the real parts. Then the repeated use of integration by parts gives (2.4) if we note
\[ (2.7) \quad \zeta_\pm = e^{\pm i\omega r} r^{-(n-1)/2} Dv \quad \text{and} \quad \theta_\pm = \sum_{j=1}^n \frac{\zeta_j}{r} [\zeta_\pm]_j, \]

where \([\zeta_\pm]_j\) is the \(j\)-th component of \(\zeta_\pm\).

**Proposition 2.1.** Let \(u\) be a \(L^2\)-solution of (2.1). Then there exists a constant \(C_2 > 0\) such that for any \(\kappa \in C_R\)

\[ (2.8) \quad |\kappa|^2 \int_{R^*} (1 + r)^{-1-\delta} |u|^2 dx \leq C_2 \int_{R^*} (1 + r)^{1+\delta} |f|^2 dx. \]

**Proof.** Multiply by \((1 + \rho)^{-2\delta} r^{-1+\delta}\) on both sides of (2.2) and integrate over \([0, \infty)\). Then we have

\[ (2.9) \quad \frac{1}{2} |\kappa|^2 \int_{R^*} (1 + r)^{-1-\delta} |u|^2 dx \leq \frac{1}{2} \int_{R^*} r^{-1+\delta} |\theta_\pm|^2 dx + C(\kappa) \int_{R^*} |f i\kappa u|^2 dx. \]

On the other hand, noting that \(n \neq 2, 0 < \delta \leq 1\) and \(|\zeta_\pm| \geq |\theta_\pm|\), we have from (2.4)

\[ (2.10) \quad \int_{R^n} r^{-1+\delta} |\theta_\pm|^2 dx \leq \left( \frac{2}{\delta} \right)^2 \int_{R^n} r^{-1+\delta} |f|^2 dx. \]

Inequality (2.8) then follows if we note \((1 + r)^{-1-\delta} \leq (1 + r)^{-2\delta} r^{-1+\delta}\).

**§ 3. Proof of Theorem 1**

(a) Let \(f = \{f_1, f_2\} \in \mathcal{H}\). Then \(u(t) = U(t, 0) f\) satisfies (1.4) and the initial condition \(u(0) = f\). Since \(A_0\) is skew-selfadjoint, we have from (1.4)

\[ (3.1) \quad U_0(-t) U(t, 0) f = f - \int_0^t U_0(-\tau) V(\tau) U(\tau, 0) f d\tau \]

and

\[ (3.2) \quad \|U(t, 0) f\|_{\mathcal{B}}^2 + 2 \int_0^t \|V(\tau) U(\tau, 0) f\|_{\mathcal{B}}^2 d\tau = \|f\|_{\mathcal{B}}^2. \]

We put

\[ (3.3) \quad A = \begin{pmatrix} 0 & 0 \\ 0 & a(x) \end{pmatrix}, \quad a(x) = \sqrt{C_1 (1 + |x|)^{-(1+\delta)/\delta}}. \]
Note that $A \geq \sqrt{V(t)}$. Then for any $g \in \mathcal{H}$

\begin{equation}
\int_0^t |(U_0(-\tau)V(\tau) U(\tau,0)f, g)_{\mathcal{H}}| d\tau \\
\leq \left( \int_0^t |\sqrt{V(\tau)} U(\tau,0) f|_{\mathcal{H}}^2 d\tau \right)^{1/2} \left( \int_0^t |AU_0(\tau) g|_{\mathcal{H}}^2 d\tau \right)^{1/2},
\end{equation}

where $(\cdot, \cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. Thus, to see the existence of the strong limit of (3.1) as $t \to \infty$, it is sufficient to prove that there exists a constant $C > 0$ such that

\begin{equation}
\int_0^\infty \|AU_0(t)g\|_{\mathcal{H}}^2 dt \leq C \|g\|_{\mathcal{H}}^2
\end{equation}

for any $g \in \mathcal{H}$.

The following result is due to Kato [3].

**Proposition 3.1.** There exists a $C > 0$ satisfying (3.5) if the operator $A$ satisfies the condition

\begin{equation}
\sup_{\epsilon \in \mathbb{C} - \mathbb{R}} \|A(A_0 - i\epsilon I)^{-1}A\|_{\mathcal{H}} < \infty.
\end{equation}

For $g = \{g_1, g_2\} \in \mathcal{H}$ put

\begin{equation}
u = \{u_1, u_2\} = (A_0 - i\epsilon I)^{-1}Ag.
\end{equation}

Then, as is easily seen, the second component $u_2$ satisfies equation (2.1) with $f = -i\alpha(x)g_2$. Thus, by Proposition 2.1 we have

\begin{equation}
|\nu|^2 \int_{\mathbb{R}^n} (1 + r)^{-1-\xi}|u_2|^2 dx \leq C \int_{\mathbb{R}^n} (1 + r)^{1+\xi}|i\alpha(x)g_2|^2 dx
\end{equation}

\begin{equation}
\leq C_1 C_2 |\nu|^2 \int_{\mathbb{R}^n} |g_2|^2 dx.
\end{equation}

Since $A(A_0 - i\epsilon I)^{-1}Ag = \{0, \alpha(x)u_2\}$, it follows from (3.8) that

\begin{equation}
\|A(A_0 - i\epsilon I)^{-1}Ag\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^n} |\alpha(x)u_2|^2 dx
\end{equation}

\begin{equation}
\leq C_1^2 C_2 \int_{\mathbb{R}^n} |g_2|^2 dx \leq C_1 C_2 \|g\|_{\mathcal{H}}^2.
\end{equation}

This proves that $A$ satisfies condition (3.6). Hence, (3.5) holds and the wave operator $Z$ exists.

(b) To show the existence of $f \in \mathcal{H}$ such that $Zf \neq 0$, we assume
contrary, i.e., for any $f \in \mathcal{H}$ \(\|U(t, 0)f\|_E \to 0\) as $t \to \infty$. Then we have from (3.2)

\[(3.10) \quad \|f\|_E^2 = 2 \int_0^\infty \|\sqrt{V(t)} U(t, 0) f\|_E^2 \, dt.\]

Further, by (3.1) and (3.4)

\[(3.11) \quad \|f\|_E^2 \leq \left( \int_0^\infty \|\sqrt{V(t)} U(t, 0) f\|_E^2 \, dt \right)^{1/2} \left( \int_0^\infty \|A U_0(t) f\|_E^2 \, dt \right)^{1/2}.\]

Hence, it follows that

\[(3.12) \quad \|f\|_E^2 \leq \frac{1}{2} \int_0^\infty \|A U_0(t) f\|_E^2 \, dt.\]

Put $f = U_0(s) g$, where $\|g\|_E = 1$. Then by (3.12)

\[(3.13) \quad \|U_0(s) g\|_E^2 = \frac{1}{2} \int_0^\infty \|A U_0(t) g\|_E^2 \, dt \to 0, \quad \text{as} \quad s \to \infty\]

(cf., (3.5)). This is a contradiction and (b) is proved.

(c) It follows from (3.5) that in (3.4)

\[(3.14) \quad \int_s^t \|A U_0(\tau) g\|_E^2 d\tau \to 0 \quad \text{as} \quad s, t \to \infty.\]

On the other hand, we have from (3.2)

\[(3.15) \quad \int_0^\infty \|\sqrt{V(t)} U(t, 0) f\|_E^2 \, dt \leq \frac{1}{2} \|f\|_E^2 \quad \text{for any} \quad f \in \mathcal{H}.\]

Thus, $U(t, 0) * U_0(t) g$ converges in $\mathcal{H}$ as $t \to \infty$ and (c) is proved.

\section*{§ 4. Proof of Theorem 2}

The assertions (a) and (b) for the operator $W$ can be proved by the same argument as in the proof of Theorem 1 if we note that the adjoint semigroup $U(t)^*$ has generator

\[(3.16) \quad A^* = -A_0 - V \quad \text{with domain} \quad \mathcal{D}(A^*) = \mathcal{D}(A_0).\]

(e) and (d) are obvious from the definition of $W$ and $Z$.

\section*{References}


**Added in Proof.** Recently, Mr. A. Matsumura (Dept. Appl. Math. Phys., Fac. Engi., Kyoto U.) obtained the following result: If $b(x, t)$ in (1-1) satisfies $t \geq 0$

$$b_r(x, t) \leq 0 \quad \text{and} \quad \min_{|x| \leq R + t} b(x, t) \geq \frac{1}{K + \varepsilon t},$$

where $R$, $K$, $\varepsilon$ are positive constants, and if the initial data $f = \{f_1, f_2\}$ has support contained in $\{x; |x| \leq R\}$, then the total energy of solution of (1-1) decays like

$$\|U(t, 0)f\|_\infty = O(t^{-1/2(1+\varepsilon)}) \quad \text{as} \quad t \to \infty.$$ 

By this result we can say that our assumption (A1) is settled in a sense.