$L^2$-well-posedness for Hyperbolic Mixed Problems

By

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Introduction

Strongly hyperbolic differential equations become $L^2$-well-posed mixed problems under suitable boundary conditions. A concept of uniform Lopatinski's condition, given by S. Agmon [1], gives a sufficient condition for $L^2$-well-posedness. Moreover, it is known that some types of mixed problems become $L^2$-well-posed, which do not satisfy uniform Lopatinski's condition (cf. [2], [3], [4]). On the other hand, in the case of constant coefficients and half-space, a necessary and sufficient condition for $L^2$-well-posedness is given by R. Agemi & T. Shirota [5] by the words of uniform $L^2$-well-posedness for boundary value problems of ordinary differential equations with parameters. But it is not so concrete to clear the role of uniform Lopatinski’s condition. This paper is a trial of more concrete characterizations of $L^2$-well-posedness for strongly hyperbolic mixed problems.

We consider the problem

$$
\begin{align*}
A(D_t, D_y, D_x)u &= \sum_{i+|\nu|+h \leq m} a_{i\nu h} D_\nu D_y^h D_x^u u = f \\
&\quad \text{in } t > 0, \ y \in R^{n-1}, \ x > 0, \\
B_j(D_t, D_y, D_x)u &= \sum_{i+|\nu|+h \leq r_j} b_{ji\nu h} D_\nu D_y^h D_x^u u = 0 \\
&\quad \text{on } t > 0, \ y \in R^{n-1}, \ x = 0, \\
&\quad (j = 1, 2, \ldots, \mu, \ 0 \leq r_j \leq m-1), \\
D_1^u &= 0 \quad \text{on } t = 0, \ y \in R^{n-1}, \ x > 0, \\
&\quad (j = 0, 1, \ldots, m-1).
\end{align*}
$$

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The problem (P) is said to be $L^2$-well-posed if there exists $C_T > 0$ for any $T > 0$ such that $C_T \to +0$ as $T \to +0$ and

$$\sum_{i+|\gamma|+k \leq m-1} \int_0^T dt \int_{R^n} |D^n t^k \exp D_t \phi(t, \gamma, x)|^2 dy dx \leq C_T \int_0^T dt \int_{R^n} |f(t, \gamma, x)|^2 dy dx.$$ 

It is obvious that $L^2$-well-posedness is characterized only by the principal parts of $\{A, B_j\}$. Therefore, hereafter, we consider the case when $\{A, B_j\}$ are homogeneous. Assumptions are as follows:

i) $A$ is strongly hyperbolic with respect to $t$-axis,

ii) $t = 0$ is non-characteristic of $A$,

iii) $\{A, B_j\}$ satisfy Lopatinski's condition, that is,

$$R(\tau, \gamma) = \det \left( \frac{B_i(\tau, \gamma; \xi)}{A_+(\tau, \gamma; \xi)} \frac{d\xi}{2\pi i} \right)_{j=1}^{\mu} = 0$$

for $\text{Im} \tau < 0$, $\gamma \in R^{n-1}$,

where

$$A(\tau, \gamma; \xi) = e \prod_{j=1}^{\mu} (\xi - \xi_j^+(\tau, \gamma)) \prod_{j=\mu + 1}^{\mu} (\xi - \xi_j^-(\tau, \gamma))$$

$$\text{Im} \xi_j^+(\tau, \gamma) > 0 \quad \text{for} \quad \text{Im} \tau < 0, \quad \gamma \in R^{n-1},$$

$$A_+(\tau, \gamma; \xi) = \prod_{j=1}^{\mu} (\xi - \xi_j^+(\tau, \gamma)).$$

Here we remark that iii) is a necessary condition for $L^2$-well-posedness for (P) under the assumptions i) and ii).

By Laplace-Fourier transform with respect to $(t, \gamma)$, the problem (P) becomes to

$$\hat{(P)} \quad \left\{ \begin{array}{l}
A(\tau, \gamma; D_x) \hat{u}(x) = \hat{f}(x) \quad \text{for} \ x > 0, \\
B_j(\tau, \gamma; D_x) \hat{u}(x)|_{x=0} = 0 \quad (j = 1, 2, \ldots, \mu).
\end{array} \right.$$ 

Let $G(\tau, \gamma; x, \gamma)$ be Green's function of (P), that is, the solution $\tilde{u} \in L^2$
of (P) for $f \in L^2$ is represented by

$$u(x) = \int_0^\infty G(\tau, \eta; x, y) f(y) \, dy.$$  

Then we have from the results of R. Agemi and T. Shirota.

**Lemma.** *In order that (P) is $L^2$-well-posed, it is necessary and sufficient that*

$$\frac{\sum_{k=0}^{m-1} \left\| \left( \frac{\partial}{\partial x} \right)^k G(\tau, \eta; x, y) \right\|_{L^2(L^2, L^2)}}{|\text{Im} \tau|} \leq C$$

*for* $\tau \in \mathbb{C}^1$, $\text{Im} \tau < 0$, $\eta \in \mathbb{R}^{n-1}$, $|\tau|^2 + |\eta|^2 = 1$, where $C$ is independent of $(\tau, \eta)$.

**Remark.** Let

$$G_0(\tau, \eta; x-y) = \frac{1}{2\pi} \int_\infty^\infty \frac{e^{i(x-y)\xi}}{A(\tau, \eta; \xi)} \, d\xi,$$

$$G(\tau, \eta; x, y) = G_0(\tau, \eta; x-y) - G_0(\tau, \eta; x, y),$$

then $G$ in Lemma may be replaced by $G_c$.

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**§1. Necessary Conditions**

**1.1. Preliminary.** Let $(\sigma_0, \eta_0)$ be a real fixed point and let $\{\xi_i = \xi_i(\sigma_0, \eta_0)\}_{i=1,...,N} (N = N(\sigma_0, \eta_0))$ be real distinct roots of $A(\sigma_0, \eta_0, \xi) = 0$ with multiplicities $\{m_i = m_i(\sigma_0, \eta_0)\}_{i=1,...,N}$. Then there exists a complex neighbourhood $U$ of $(\sigma_0, \eta_0)$ such that

$$A(\tau, \eta; \xi) = \prod_{i=1}^N H_i(\tau, \eta; \xi) E(\tau, \eta; \xi) = H(\tau, \eta; \xi) E(\tau, \eta; \xi),$$

$$H_i(\tau, \eta; \xi) = (\xi - \xi_i)^{m_i} + a_{i1}(\tau, \eta)(\xi - \xi_i)^{m_i-1} + \cdots + a_{im_i}(\tau, \eta)$$

in $U$, where $a_{ij}(\sigma_0, \eta_0) = 0$ and $a_{ij}(\tau, \eta)$ are holomorphic. Moreover from the assumption i), $a_{ij}(\sigma, \eta)((\sigma, \eta) \in \mathbb{R}^n \cap U)$ are real valued and $\frac{\partial}{\partial \tau} a_{im_i}(\tau, \eta) \equiv 0$. 

Now we denote
\[ \alpha_i = \frac{\partial a_{\text{im}}}{\partial \tau} (\sigma_0, \eta_0), \quad \beta_i = \frac{\partial a_{\text{im}}}{\partial \eta} (\sigma_0, \eta_0), \]
and we denote
\[
\begin{align*}
\Delta_{i\delta} &= \{ (\tau, \eta) \in V_{\delta} ; |\alpha_i(\tau - \sigma_0) + \beta_i(\eta - \eta_0)| \\
&\geq \cos \theta_0 (|\alpha_i|^2 + |\beta_i|^2)^{\frac{1}{2}} d(\tau, \eta) \quad \text{if } m_i \geq 2, \\
\Delta_{i\delta} &= V_{\delta} \quad \text{if } m_i = 1,
\end{align*}
\]
and
\[ \Delta_{i\delta} = \bigcap_i \Delta_{i\delta}, \]
where
\[ d = d(\tau, \eta) = \text{dis} \{ (\tau, \eta), (\sigma_0, \eta_0) \}, \]
\[ V_{\delta} = \{ \tau, \eta \}; \text{ Im } \tau < 0, \eta \in R^{n-1}, d < \delta, \]
and \( \theta_0 (0 \leq \theta_0 < \pi) \) is an arbitrarily fixed number. Let \( \{ \xi_{ij}^\phi(\tau, \eta) \}_{j=1, \ldots, m_i} \) (Im \( \xi_{ij}^\phi \geq 0 \)) be roots of \( H_i(\tau, \eta; \xi) = 0 \), and \( \{ \xi_{ij}^\phi(\tau, \eta) \}_{j=1, \ldots, m_i} \) (Im \( \xi_{ij}^\phi \geq 0 \)) be roots of
\[ (\xi - \xi_i)^m + \alpha_i(\tau - \sigma_0) + \beta_i(\eta - \eta_0) = 0. \]
Then we have

**Lemma 1.1.** There exists \( \delta > 0 \) such that
\[ \xi_{ij}^\phi(\tau, \eta) = \xi_{ij}^\phi(\tau, \eta) + O(\delta^{\frac{2}{m_i}}) \quad \text{in } \Delta_{i\delta}, \]
\[ \frac{\partial \xi_{ij}^\phi}{\partial \tau}(\tau, \eta) = \frac{\partial \xi_{ij}^\phi}{\partial \tau}(\tau, \eta) + O(\delta^{\frac{2}{m_i}}) \quad \text{in } \Delta_{i\delta}. \]

**Corollary 1.**
\[ c_1 d^{\frac{1}{m_i}} \leq |\xi_{ij}^\phi(\tau, \eta) - \xi_i| \leq c_2 d^{\frac{1}{m_i}} \quad \text{in } \Delta_{i\delta}, \]
\[ c_1 d^{\frac{1}{m_i}} \leq |\xi_{ij}^\phi(\tau, \eta) - \xi_{jk}^\phi(\tau, \eta)| \leq c_2 d^{\frac{1}{m_i}} (j \neq k) \quad \text{in } \Delta_{i\delta}. \]
Next we consider \( \{ \text{Im} \xi_i^+(\tau, \eta) \} \). Let us denote \( i \in I \) if \( m_i \) is even, \( i \in J \) if \( m_i \) is odd, and moreover \( i \in J_\pm \) if \( i \in I \) and \( \alpha_i \geq 0 \). Let
\[
\Delta_{I_3}^+ = A_{I_3} \cap \{ \alpha_i (\text{Re} \tau - \sigma_0) + \beta_i (\eta - \gamma_0) \geq 0 \},
\]
and let
\[
* = (*_1, *_2, \ldots, *_N) \quad (*_i = \pm), \quad \Delta_{I_3}^* = \bigcap_{i=1}^N \Delta_{I_3}^{*_i},
\]
then
\[
\Delta_{I_3} = \bigcup_* \Delta_{I_3}^*.
\]
Let \((\tau, \eta) \in \Delta_{I_3}^*\) and \(\text{Im} \tau \to 0\), we have the followings:

i) if \( i \in I, *_i = + \), then none of \( \{ \xi_i^+(\tau, \eta) \} \) has a real limit,

ii) if \( i \in I, *_i = - \), then only one of \( \{ \xi_i^+(\tau, \eta) \} \) and only one of \( \{ \xi_i^-(\tau, \eta) \} \) have real limits, which we denote especially by \( \xi_i^{+*}(\tau, \eta) \),

iii) if \( i \in J_+, \) then only one of \( \{ \xi_i^+(\tau, \eta) \} \) has a real limit, which we denote by \( \xi_i^{+*}(\tau, \eta) \),

iv) if \( i \in J_- \), then only one of \( \{ \xi_i^-(\tau, \eta) \} \) has a real limit, which we denote by \( \xi_i^{-*}(\tau, \eta) \).

Here we denote
\[
\delta_{ij, \pm}(\tau, \eta) = \begin{cases} 
\left( \frac{I}{d} \right)^{m_i} d & \text{if } \xi_i^+ = \xi_i^{+*}, \\
\text{otherwise},
\end{cases}
\]
for \( i = 1, \ldots, N, j = 1, \ldots, m_i^\pm \), where \( \gamma = -\text{Im} \tau \), then we have

**Corollary 2.**
\[
c_1 (\delta_{ij, \pm})^{m_i} \leq |\text{Im} \xi_i^+(\tau, \eta)| \leq c_2 (\delta_{ij, \pm})^{m_i} \quad \text{in } \Delta_{I_3}^*.
\]

**Proof.**
\[
\xi_i^{+*}(\tau, \eta) = \xi_i^{+*}(\sigma, \eta) - i\gamma \frac{\partial \xi_i^{+*}}{\partial \tau} (\sigma - i\theta \gamma, \eta) \quad (0 < \theta < 1)
\]
\[
= \xi_i^{+*}(\sigma, \eta) - i\gamma \left\{ \frac{\partial \xi_i^{+*}}{\partial \tau} (\sigma - i\theta \gamma, \eta) + O \left( d^{-1+\frac{2}{m_i}} \right) \right\},
\]
\[
\frac{\partial \xi_{i}^{\pm}(\sigma - i \theta \gamma, \eta)}{\partial \tau} = -\frac{\alpha_i}{m_i(\xi_{i}^{\pm}(\sigma - i \theta \gamma, \eta) - \xi_i)^{m_i-1}}
\]
\[
= -\frac{\alpha_i}{m_i(\xi_{i}^{\pm}(\sigma, \eta) - \xi_i + 0(\frac{1}{m_i}))^{m_i-1}}
\]
\[
= -\frac{\alpha_i}{m_i(\xi_{i}^{\pm}(\sigma, \eta) - \xi_i)^{m_i-1}} \cdot \left\{ 1 + 0\left( \frac{1}{m_i} \right) \right\}.
\]

Therefore we have
\[
c_1 \frac{\gamma}{d^{1-\frac{1}{m_i}}} \leq |\text{Im} \xi_{i1}^{\pm}(\tau, \eta)| \leq c_2 \frac{\gamma}{d^{1-\frac{1}{m_i}}} \quad \text{in } A^T
\]
for \( \gamma \ll d \). In other cases, the required results follow from
\[
\xi_{i1}^{\pm}(\tau, \eta) = \xi_{i1}^{\pm}(\tau, \eta) + 0(d^{-\frac{2}{m_i}}).
\]
Q.E.D.

Let
\[
E(\tau, \eta; \xi) = E_+(\tau, \eta; \xi)E_-.(\tau, \eta; \xi),
\]
where roots of \( E_+(\tau, \eta; \xi) = 0 \) are on the upper (resp. lower) half plane and \( M \) is the degree of \( E_+ \).

**Corollary 3.**

\[
k \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} C_{i,j}^{k}(\tau, \eta) \cdot \left( \begin{array}{c} B_{1}(\tau, \eta; \xi_{i1}^{+}(\tau, \eta)) \\ \vdots \\ B_{n}(\tau, \eta; \xi_{i1}^{+}(\tau, \eta)) \end{array} \right)
\]
\[
+ \sum_{j=1}^{M} C_{0,j}^{k}(\tau, \eta) \left( \begin{array}{c} \int \frac{1}{2\pi i} \frac{B_{1}(\tau, \eta; \xi)\xi^{j-1}}{E_+(\tau, \eta; \xi)} d\xi \\ \vdots \\ \int \frac{1}{2\pi i} \frac{B_{n}(\tau, \eta; \xi)\xi^{j-1}}{E_+(\tau, \eta; \xi)} d\xi \end{array} \right),
\]

where \( \{C_{i,j}^{k}(\tau, \eta), C_{0,j}^{k}(\tau, \eta)\} \) are bounded in \( A_8 \).
Proof. Let us denote
\[ f(0)(x) = f(x), \]
\[ f(k)(x_1, x_2, \ldots, x_{k+1}) = \frac{f(k-1)(x_1, x_2, \ldots, x_k) - f(k-1)(x_2, x_3, \ldots, x_{k+1})}{x_1 - x_{k+1}} \]
\( (k = 1, 2, 3, \ldots). \)

Then

\[
\begin{align*}
B_{1\overline{0}}(\tau, \eta; \xi^+_{11}(\tau, \eta)) & B_{1(1)}(\tau, \eta; \xi^+_{11}(\tau, \eta), \xi^+_{12}(\tau, \eta)) \\
& \vdots \\
B_{\mu(0)}(\tau, \eta; \xi^+_{11}(\tau, \eta)) & B_{\mu(1)}(\tau, \eta; \xi^+_{11}(\tau, \eta), \xi^+_{12}(\tau, \eta)) \\
& \vdots \\
B_{1(m-1)}(\tau, \eta; \xi^+_{11}(\tau, \eta), \ldots, \xi^+_{1m-1}(\tau, \eta)) & B_{1(1)}(\tau, \eta; \xi^+_{11}(\tau, \eta), \ldots, \xi^+_{1m-1}(\tau, \eta)) \\
& \vdots \\
B_{\mu(0)}(\tau, \eta; \xi^+_{11}(\tau, \eta)) & B_{\mu(1)}(\tau, \eta; \xi^+_{11}(\tau, \eta), \ldots, \xi^+_{1m-1}(\tau, \eta)) \\
& \vdots \\
B_{1(0)}(\tau, \eta; \xi^+_{21}(\tau, \eta)) & \frac{1}{2\pi i} \int \frac{B_1(\tau, \eta; \xi)}{E_+(\tau, \eta; \xi)} d\xi \\
& \frac{1}{2\pi i} \int \frac{B_1(\tau, \eta; \xi) \xi^{M-1}}{E_-(\tau, \eta; \xi)} d\xi \\
& \vdots \\
B_{\mu(0)}(\tau, \eta; \xi^+_{21}(\tau, \eta)) & \frac{1}{2\pi i} \int \frac{B_\mu(\tau, \eta; \xi)}{E_+(\tau, \eta; \xi)} d\xi \\
& \frac{1}{2\pi i} \int \frac{B_\mu(\tau, \eta; \xi) \xi^{M-1}}{E_-(\tau, \eta; \xi)} d\xi \\
& \vdots \\
& \lim_{\xi \to \infty} - B_1(\sigma_0, \gamma_0; \xi_1) B_1(\sigma_0, \gamma_0; \xi_1) \cdots (m-1)! B_1^{(m-1)}(\sigma_0, \gamma_0; \xi_1) B_1(\sigma_0, \gamma_0; \xi_2) \\
& \vdots \\
& B_\mu(\sigma_0, \gamma_0; \xi_1) B_\mu(\sigma_0, \gamma_0; \xi_1) \cdots (m-1)! B_\mu^{(m-1)}(\sigma_0, \gamma_0; \xi_1) B_\mu(\sigma_0, \gamma_0; \xi_2) \\
& \vdots \\
& \frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi)}{E_+(\sigma_0, \gamma_0; \xi)} d\xi \\
& \frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi) \xi^{M-1}}{E_-(\sigma_0, \gamma_0; \xi)} d\xi \\
& \vdots \\
& \cdots \\
& \lim_{\xi \to \infty}.
\end{align*}
\]

Since
rank \[ \begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_1)B'_{1\xi}(\sigma_0, \eta_0; \xi_1) \cdots (m_1 - 1)! B_{0\xi}(m_1 - 1)(\sigma_0, \eta_0; \xi_1) \\ \vdots \\ B_{\mu}(\sigma_0, \eta_0; \xi_1)B'_{\mu\xi}(\sigma_0, \eta_0; \xi_1) \cdots (m_1 - 1)! B_{\mu\xi}(m_1 - 1)(\sigma_0, \eta_0; \xi_1) \end{pmatrix} = \mu, \]

we have

\[
\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ B_{\mu(0)}(\tau, \eta; \xi_{11}(\tau, \eta)) \cdots B_{\mu(m_1 - 1)}(\tau, \eta; \xi_{11}(\tau, \eta), \cdots, \xi_{1m_1}(\tau, \eta)) \end{pmatrix} = \begin{pmatrix} \alpha_{11}(\tau, \eta) \\ \vdots \\ \alpha_{1m_1-1}(\tau, \eta) \end{pmatrix},
\]

where \( \{\alpha_{ij}(\tau, \eta)\} \) are bounded.

Q.E.D.

1.2. Representation of Green's function in \( \mathcal{A}_\delta \). Let us denote

\[
\tilde{E}_+(\tau, \eta; \xi) = \left( \frac{1}{\xi - \xi_{11}(\tau, \eta)}, \ldots, \frac{1}{\xi - \xi_{1m_1}(\tau, \eta)}, \ldots, \frac{1}{E_{\pm}(\tau, \eta; \xi)}, \ldots, \frac{\xi_{M-1}}{E_{\pm}(\tau, \eta; \xi)} \right),
\]

\[
E_+(\tau, \eta; x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ix\xi} \tilde{E}_+(\tau, \eta; \xi) d\xi,
\]

\[
E_-(\tau, \eta; x) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-ix\xi} \tilde{E}_-(\tau, \eta; \xi) d\xi,
\]

\[
B_\pm(\tau, \eta) = \frac{1}{2\pi i} \int \begin{pmatrix} B_1(\tau, \eta; \xi) \\ \vdots \\ B_{\mu}(\tau, \eta; \xi) \end{pmatrix} \tilde{E}_\pm(\tau, \eta; \xi) d\xi
\]
Let Poisson's kernels \( \{P_h(\tau, \eta; x)\}_{h=1, \ldots, \mu} \) be the \( L^2 \)-solutions of

\[
\begin{aligned}
\left\{ \begin{array}{l}
A(\tau, \eta; D_x) P_h(\tau, \eta; x) = 0 \quad & \text{for } x > 0, \\
B_j(\tau, \eta; D_x) P_h(\tau, \eta; x) |_{x=0} = \delta_{j,k} \quad & (j=1, \ldots, \mu),
\end{array} \right.
\end{aligned}
\]

that is,

\[
(P_1(\tau, \eta; x), \ldots, P_\mu(\tau, \eta; x)) = \mathcal{E}_+ (\tau, \eta; x) \{ B_1(\tau, \eta) \}^{-1}.
\]

Then we have

\[
G(\tau, \eta; x, \gamma) = (P_1(\tau, \eta; x), \ldots, P_\mu(\tau, \eta; x)) \left\{ \begin{array}{l}
B_1(\tau, \eta; D_x) \\
\vdots \\
B_\mu(\tau, \eta; D_x)
\end{array} \right. \\
G_0(\tau, \eta; x - \gamma) \bigg|_{x=0},
\]

where

\[
\left( \begin{array}{c}
B_1(\tau, \eta; D_x) \\
\vdots \\
B_\mu(\tau, \eta; D_x)
\end{array} \right) G_0(\tau, \eta; x - \gamma) \bigg|_{x=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \begin{array}{c}
B_1(\tau, \eta; \xi) \\
\vdots \\
B_\mu(\tau, \eta; \xi)
\end{array} \right) e^{-iy\xi} A(\tau, \eta; \xi) d\xi.
\]

By the way, we have

\[
\frac{1}{i A(\tau, \eta; \xi)} \left( \begin{array}{c}
B_1(\tau, \eta; \xi) \\
\vdots \\
B_\mu(\tau, \eta; \xi)
\end{array} \right) = P_+(\tau, \eta) \tilde{E}_+(\tau, \eta; \xi) + P_-(\tau, \eta) \tilde{E}_-(\tau, \eta; \xi),
\]

where
\[ B_{\pm}(r, \eta) = P_{\pm}(r, \eta) \cdot \frac{1}{2\pi} \int E_{\pm}(r, \eta; \xi) A(r, \eta; \xi) d\xi \]
\[ = P_{\pm}(r, \eta) \cdot Q_{\pm}(r, \eta), \]
therefore we have
\[ \left( B_{1}(r, \eta; D_{x}) \right) G_{0}(r, \eta; x-y)|_{x=0} = P_{-}(r, \eta) E_{-}(r, \eta; \gamma) \]
\[ = B_{-}(r, \eta) Q_{-}(r, \eta) E_{-}(r, \eta; \gamma). \]

Here we remark
\[ Q_{-}(r, \eta) = \begin{pmatrix}
\frac{1}{iA_{1}^{*}(r, \eta; \xi_{12}(r, \eta))} \\
\frac{1}{iA_{1}^{*}(r, \eta; \xi_{12}(r, \eta))} \\
\vdots \\
\frac{1}{2\pi} \int \frac{E_{+}(r, \eta; \xi) H(r, \eta; \xi)}{E_{-}(r, \eta; \xi)} d\xi \\
\vdots \\
\frac{1}{2\pi} \int \frac{E_{+}(r, \eta; \xi) H(r, \eta; \xi) \xi^{M-1}}{E_{-}(r, \eta; \xi)} d\xi \\
\frac{1}{2\pi} \int \frac{E_{-}(r, \eta; \xi) H(r, \eta; \xi) \xi^{2M-2}}{E_{-}(r, \eta; \xi)} d\xi \\
\end{pmatrix}^{-1}. \]

Let us denote
\[ B(r, \eta) = (B_{+}(r, \eta))^{-1} B_{-}(r, \eta) Q_{-}(r, \eta) = \beta(r, \eta) Q_{-}(r, \eta), \]
then we have

**Lemma 1.2.**

\[ G_{\xi}(r, \eta; x, \gamma) = E_{+}(r, \eta; x) B(r, \eta) E_{-}(r, \eta; \gamma) \]
1.3. Estimates of Green's function in $\mathcal{A}_0$. It is obvious that it holds

$$\|G_c\|_{\mathcal{A}(L^2 \times L^2)} \leq \|G_c\|_{\mathcal{A}(L^2 \times L^2)} \leq \|G_c\|_{L^2 \times L^2}$$

in general. On the other hand, we show that it holds

$$\|G_c\|_{\mathcal{A}(L^2 \times L^2)} \geq c_1 \|G_c\|_{\mathcal{A}(L^2 \times L^2)} \geq c_2 \|G_c\|_{L^2 \times L^2}$$

in $\mathcal{A}_0$, where $c_1$, $c_2$ are positive constants independent of $(\tau, \eta)$.

Now let

$$N_{\pm}(\tau, \eta) = \left| \begin{array}{ccc} |\text{Im} \xi_{+1}(\tau, \eta)|^{-\frac{1}{2}} & \cdots & |\text{Im} \xi_{m_1}(\tau, \eta)|^{-\frac{1}{2}} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{array} \right|$$

and

$$F_{\pm}(\tau, \eta; x) = N_{\pm}(\tau, \eta)^{-1} E_{\pm}(\tau, \eta; x),$$

then $L^2$-norms of $F_{\pm}(\tau, \eta; x)$ are bounded in $\mathcal{A}_0$, therefore we have

$$\left| \left\{ F_{\pm}(\tau, \eta; x) G_c(\tau, \eta; x, \gamma) \bar{F}_{\pm}(\tau, \eta; \gamma) \right\} d\tau d\eta \right| \leq C \|G_c\|_{\mathcal{A}(L^2 \times L^2)}$$

in $\mathcal{A}_0$, where $C$ is independent of $(\tau, \eta)$. Let

$$S_{\pm}(\tau, \eta) = \left( \int_0^\infty F_{\pm}(\tau, \eta; x) F_{\pm}(\tau, \eta; x) \, dx \right)$$

then we have
Lemma 1.3. \( S_{\pm}(\tau, \eta) \) are positive hermitian matrices in \( \Delta_3 \) and

\[
 c_1 I < S_{\pm}(\tau, \eta) < c_2 I \quad (I: \text{identity matrix}),
\]

where \( c_1, c_2 \) are positive constants independent of \( (\tau, \eta) \).

Proof. Let

\[
 S_{\pm}(\tau, \eta) = \left( \begin{array}{c}
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\Im \xi_{\pm}^1(\tau, \eta)|}{(\xi - \xi_1^1(\tau, \eta))(\xi - \xi_2^1(\tau, \eta))} \, d\xi \\
 \vdots \\
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\Im \xi_{\pm}^m(\tau, \eta)|}{(\xi - \xi_1^1(\tau, \eta))(\xi - \xi_2^1(\tau, \eta))} \, d\xi \\
 \vdots \\
 \end{array} \right)
\]

\[
 = \pm i \left( \begin{array}{ccc}
 \frac{\Im \xi_{\pm}^1(\tau, \eta)}{\xi_1^1(\tau, \eta) - \xi_2^1(\tau, \eta)} & \frac{\Im \xi_{\pm}^2(\tau, \eta)}{\xi_1^1(\tau, \eta) - \xi_2^1(\tau, \eta)} & \cdots \\
 \vdots & \vdots & \ddots \\
 \frac{\Im \xi_{\pm}^m(\tau, \eta)}{\xi_1^1(\tau, \eta) - \xi_2^1(\tau, \eta)} & \frac{\Im \xi_{\pm}^m(\tau, \eta)}{\xi_1^1(\tau, \eta) - \xi_2^1(\tau, \eta)} & \cdots \\
 \end{array} \right)
\]
then we have
\[
\det S_\pm(\tau, \eta) = \prod_{i=1}^{N} \det S_{i\pm}(\tau, \eta) \cdot \det S_{0\pm}(\tau, \eta) + 0(1)
\]
as \(d \to 0\). From Lemma 1.1, we have
\[
|\det S_{i\pm}(\tau, \eta)| = \frac{1}{2^{m+1}} \left| \prod_{j<k} \frac{\xi_{ij}(\tau, \eta) - \xi_{jk}(\tau, \eta)}{\xi_{ij}(\tau, \eta) - \xi_{jk}(\tau, \eta)} \right| > c > 0
\]
in \(D_3\), where \(c\) is independent of \((\tau, \eta)\). Obviously since
\[
|\det S_{0\pm}(\tau, \eta)| > c > 0,
\]
we have
\[
|\det S_\pm(\tau, \eta)| > c > 0 \quad \text{in} \quad D_3.
\]
On the other hand, we have easily that \(S_\pm(\tau, \eta)\) are positive, hermitian and bounded in \(D_3\). Therefore we have \(S_\pm(\tau, \eta) > c \cdot I\). Q.E.D.

It follows from Lemma 1.3
\[
c_1 |N_+(\tau, \eta)B(\tau, \eta)N_-(\tau, \eta)|
\]
\[
\leq \left| \int \bar{F}_+(\tau, \eta; x)G_c(\tau, \eta; x, \gamma)^T F_-(\tau, \eta; \gamma) dx dy \right|
\]
\[
\leq c_2 |N_+(\tau, \eta)B(\tau, \eta)N_-(\tau, \eta)| \quad \text{in} \quad D_3.
\]
On the other hand, we have
\[ \|G_c\|_{L^1 \times L^1} = \|E_+(\tau, \eta; x)B(\tau, \eta)E_-(\tau, \eta)\|_{L^1 \times L^1} \leq C|N_+(\tau, \eta)B(\tau, \eta)N_-(\tau, \eta)| \]

in \(\mathcal{A}_8\), and moreover
\[ \left\| \left( \frac{\partial}{\partial x} \right)^k G_c \right\|_{L^1 \times L^1} \leq C_k |N_+(\tau, \eta)B(\tau, \eta)N_-(\tau, \eta)| \]

in \(\mathcal{A}_8\). Hence we have

**Lemma 1.4.**
\[ \sum_{k=0}^{m-1} \left\| \left( \frac{\partial}{\partial x} \right)^k G_c \right\|_{L^1 \times L^1} \leq C|N_+(\tau, \eta)B(\tau, \eta)N_-(\tau, \eta)| \]
\[ \leq C'\|G_c\|_{\mathcal{A}(L^1 \times L^1, \mathbb{C}^1)} \quad \text{in} \ \mathcal{A}_8. \]

Let us denote
\[ D_{\pm}^\#(\tau, \eta) = \begin{pmatrix}
\{\delta_{1\pm}(\tau, \eta)\} & \{\delta_{1\pm}(\tau, \eta)\}^{-\frac{1}{2m_1}} \\
& \ddots & \ddots \\
& & \{\delta_{1\pm}(\tau, \eta)\}^{-\frac{1}{2m_1}} & 1 \\
& & & 1
\end{pmatrix} \]

in \(\mathcal{A}_8\), then we have

**Proposition 1.**
\[ c_1 |D_{\pm}^\#(\tau, \eta)B(\tau, \eta)D_{\pm}^\#(\tau, \eta)| \leq \|G_c\|_{\mathcal{A}(L^1, H^{m-1})} \]
\[ \leq c_2 |D_{\pm}^\#(\tau, \eta)B(\tau, \eta)D_{\pm}^\#(\tau, \eta)| \quad \text{in} \ \mathcal{A}_8. \]

Let us denote
\[ \mathcal{D}_-(\tau, \eta) = \begin{pmatrix}
\{d(\tau, \eta)\}^{-\frac{1}{m_1}} \\
& \ddots & \ddots \\
& & \{d(\tau, \eta)\}^{-\frac{1}{m_1}} & 1 \\
& & & 1
\end{pmatrix} \]
then we have

**Proposition 1'.**

\[
c_1 |D^*_2(\tau, \eta) \mathcal{F}(\tau, \eta) \mathcal{D}^*_2(\tau, \eta)| \leq c_2 |D^*_2(\tau, \eta) \mathcal{F}(\tau, \eta) \mathcal{D}^*_2(\tau, \eta)| \quad \text{in } \Delta^*_2.
\]

Here we have

**Theorem I.** Let (P) be \(L^2\)-well-posed. Then, at any real point \((\sigma_0, \eta_0)\), there exists \(\delta > 0\) such that

\[
|D^*_2(\tau, \eta) \mathcal{F}(\tau, \eta) \mathcal{D}^*_2(\tau, \eta)| < \frac{C}{\Gamma}
\]

in \(\Delta^*_\delta\), where \(C\) is a positive constant independent of \((\tau, \eta)\).

Let us denote

\[
\mathcal{F}(\tau, \eta) = \begin{pmatrix}
\beta_{11,11}(\tau, \eta) & \cdots & \beta_{11,1m_1}(\tau, \eta) \\
\vdots & & \vdots \\
\beta_{1m_1,11}(\tau, \eta) & \cdots & \beta_{1m_1,1m_1}(\tau, \eta) \\
\cdots & & \cdots \\
\beta_{01,01}(\tau, \eta) & \cdots & \beta_{01,0m}(\tau, \eta) \\
\beta_{0M,01}(\tau, \eta) & \cdots & \beta_{0M,0m}(\tau, \eta)
\end{pmatrix},
\]

then Theorem I says that a necessary condition for \(L^2\)-well-posedness is

\[
|\beta_{ij,1h}(\tau, \eta)| \delta^*_{ij} - \frac{1}{2m} \delta_{ij}^* - \frac{1}{2m} C \left( \begin{array}{c}
i = 1, \ldots, N \\
l = 1, \ldots, N \end{array} \right) < \frac{C}{\Gamma} \left( \begin{array}{c}
j = 1, \ldots, M, h = 1, \ldots, m_i^+ \end{array} \right),
\]

\[
|\beta_{ij,1h}(\tau, \eta)| \delta_{ij}^* - \frac{1}{2m} \delta_{ij}^* - \frac{1}{2m} C \left( \begin{array}{c}
j = 1, \ldots, M, l = 1, \ldots, N, h = 1, \ldots, m_i^+ \end{array} \right),
\]

\[
|\beta_{ij,0h}(\tau, \eta)| \delta^*_{ij} - \frac{1}{2m} C \left( \begin{array}{c}
i = 1, \ldots, N, h = 1, \ldots, M \end{array} \right),
\]

\[
|\beta_{ij,0h}(\tau, \eta)| < \frac{C}{\Gamma} \left( \begin{array}{c}
j = 1, \ldots, M, h = 1, \ldots, M \end{array} \right)
\]

in \(\Delta^*_\delta\).
Since \( \delta_{ij} \leq d \), we have

**Corollary 1.** If \((P)\) is \(L^2\)-well-posed, then there exists \( \delta > 0 \) at every real point \((\sigma_0, \eta_0)\) such that

\[
\begin{align*}
|\beta_{ij,th}(\tau, \eta)| d^{-\frac{1}{2m}} \left( 1 - \frac{1}{2m} \right) &< \frac{C}{T} \quad (i \neq 0, l \neq 0), \\
|\beta_{0l,th}(\tau, \eta)| d^{-\frac{1}{2m}} &< \frac{C}{T} \quad (l \neq 0), \\
|\beta_{ij,0h}(\tau, \eta)| d^{-\frac{1}{2m}} &< \frac{C}{T} \quad (i \neq 0), \\
|\beta_{0l,0h}(\tau, \eta)| &< \frac{C}{T}
\end{align*}
\]

in \( \Delta_3 \).

Let us denote

\[
\{B_+(\tau, \eta)\}^{-1} = \begin{pmatrix} r_{11,1}(\tau, \eta) & \cdots & r_{11,n}(\tau, \eta) \\
\vdots & \ddots & \vdots \\
r_{1m,1}(\tau, \eta) & \cdots & r_{1m,n}(\tau, \eta) \\
r_{01,1}(\tau, \eta) & \cdots \\
r_{0m,1}(\tau, \eta) & \cdots \end{pmatrix},
\]

then we have from the Corollary 3 of Lemma 1.1 and the Corollary 1 of Theorem I

**Corollary 2.** A necessary condition for \(L^2\)-well-posedness is

\[
\begin{align*}
|r_{ij,k}(\tau, \eta)| d^{-\frac{1}{2m}} &< \frac{C}{T} \quad (i \neq 0), \\
|r_{0j,k}(\tau, \eta)| &< \frac{C}{T}
\end{align*}
\]

in \( \Delta_3 \).

**Corollary 3.** Let \((P)\) be \(L^2\)-well-posed, and let \(A(\sigma_0, \eta_0; \xi)\) have not real multiple roots. Let us assume that \(m_i^+ = 1 (i = 1, \ldots, N_0), m_i^- = 1 (i = \ldots)\).
\[ N_0 + 1, \ldots, N \), and

\[
\begin{pmatrix}
\frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi)}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi \\
\vdots \\
\frac{1}{2\pi i} \int \frac{B_\mu(\sigma_0, \gamma_0; \xi)}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi)\xi^{M-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi \\
\vdots \\
\frac{1}{2\pi i} \int \frac{B_\mu(\sigma_0, \gamma_0; \xi)\xi^{M-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi
\end{pmatrix} = M_0,
\]

then we have

i) \[ \left\{ \begin{array}{c}
B_1(\sigma_0, \gamma_0; \xi) \\
B_\mu(\sigma_0, \gamma_0; \xi)
\end{array} \right\}_{j=1, \ldots, N_0}
\]

are linearly independent modulo the space spanned by

\[
\begin{pmatrix}
\frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi)\xi^{j-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi \\
\vdots \\
\frac{1}{2\pi i} \int \frac{B_\mu(\sigma_0, \gamma_0; \xi)\xi^{j-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi
\end{pmatrix}_{j=1, \ldots, M}
\]

ii) \[ \left\{ \begin{array}{c}
B_1(\sigma_0, \gamma_0; \xi) \\
B_\mu(\sigma_0, \gamma_0; \xi)
\end{array} \right\}_{j=N_0+1, \ldots, N}
\]

belong the space spanned by

\[
\begin{pmatrix}
B_1(\sigma_0, \gamma_0; \xi) \\
\vdots \\
B_\mu(\sigma_0, \gamma_0; \xi)
\end{pmatrix}_{j=1, \ldots, N_0}
\begin{pmatrix}
\frac{1}{2\pi i} \int \frac{B_1(\sigma_0, \gamma_0; \xi)\xi^{j-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi \\
\vdots \\
\frac{1}{2\pi i} \int \frac{B_\mu(\sigma_0, \gamma_0; \xi)\xi^{j-1}}{E_+^{(\sigma_0, \gamma_0; \xi)}} d\xi
\end{pmatrix}_{j=1, \ldots, M}
\]

iii) \[ R(\sigma_0, \gamma_0) = R_1'(\sigma_0, \gamma_0) = \cdots = R_1^{(M_1-1)}(\sigma_0, \gamma_0) = 0, \quad R_1^{(M_1)}(\sigma_0, \gamma_0) \neq 0,
\]

where \( M_1 = M - M_0 \).
§ 2. Sufficient Conditions

2.1. Preliminary. Let us say that a real point \((\sigma_0, \eta_0)\) is a regular point, when \(m_i\)-ple real root \(\xi = \xi_i\) of \(A(\sigma_0, \eta_0; \xi) = 0\) may be \(m_i\)-ple or simple in a neighbourhood of \((\sigma_0, \eta_0)\). Let \((\sigma_0, \eta_0)\) be a regular point, then \(m_i \geq 2\)-ple real roots are just all over a real analytic surface \(S_i: \sigma = \varphi_i(\eta)\).

Now let \((\sigma_0, \eta_0)\) be a regular point. Already we have had a decomposition of \(A\) in \(U\) with center \((\sigma_0, \eta_0)\):

\[
A(\tau, \eta; \xi) = \prod_{i=1} H_i(\tau, \eta; \xi)E(\tau, \eta; \xi),
\]

where

\[
H_i(\tau, \eta; \xi) = (\xi - \xi_i)^{m_i} + a_{i1}(\tau, \eta)(\xi - \xi_i)^{m_i-1} + \cdots + a_{imi}(\tau, \eta),
\]

\[
a_{ij}(\sigma_0, \eta_0) = 0.
\]

Let \((\bar{\sigma}_0, \bar{\eta}_0) \in S_i \cap U\) and let \(\bar{\xi}_i\) be the \(m_i\)-ple root of \(H_i(\bar{\sigma}_0, \bar{\eta}_0; \xi) = 0\), then we have

\[
H_i(\tau, \eta; \xi) = (\xi - \bar{\xi}_i)^{m_i} + \bar{a}_{i1}(\tau, \eta)(\xi - \bar{\xi}_i)^{m_i-1} + \cdots + \bar{a}_{imi}(\tau, \eta),
\]

\[
\bar{a}_{ij}(\bar{\sigma}_0, \bar{\eta}_0) = 0.
\]

Since

\[
\bar{a}_{imi-k}(\tau, \eta) = \frac{1}{k!} \frac{\partial^k H_i(\tau, \eta; \bar{\xi}_i)}{\partial \xi_k},
\]

and \(\bar{\xi}_i\) is a continuous function of \((\bar{\sigma}_0, \bar{\eta}_0)\), we have a neighbourhood \(U' \subset U\) such that

\[
|\bar{a}_{imi-k}(\tau, \eta)|_{\mathcal{A}^k(U')} < C,
\]

\[
|\frac{\partial \bar{a}_{imi}}{\partial \tau}(\tau, \eta)|_{\mathcal{A}^k(U')} > c > 0,
\]

where \(C\) and \(c\) are independent of \((\bar{\sigma}_0, \bar{\eta}_0)\).

Let us denote
\[ \tilde{\alpha}_i = \frac{\partial a_i}{\partial \tau}(\tilde{\sigma}_0, \tilde{\eta}_0), \quad \tilde{\beta}_i = \frac{\partial a_i}{\partial \eta}(\tilde{\sigma}_0, \tilde{\eta}_0), \]

\[ \mathcal{A}_{i1} = \{(\tau, \eta) \in V_8; \, |\tilde{\alpha}_i(\tau - \tilde{\sigma}_0) + \tilde{\beta}_i(\eta - \tilde{\eta})| \geq \cos \theta_0(|\tilde{\alpha}_i|^2 + |\tilde{\beta}_i|^2)^{\frac{1}{2}} \cdot d \} \quad \text{if } m_i \geq 2, \]

\[ \mathcal{A}_{i2} = V_8 \quad \text{if } m_i = 1, \]

where \( d = \text{dis}\{(\tau, \eta), (\tilde{\sigma}_0, \tilde{\eta}_0)\} \).

Moreover let \( \{\xi_j^\pm \} \) be roots of

\[ (\xi - \xi_j^{\pm})^m + \tilde{\alpha}_i(\tau - \tilde{\sigma}_0) + \tilde{\beta}_i(\eta - \tilde{\eta}_0) = 0, \]

then we have

**Lemma 2.1.** Let \((\sigma_0, \eta_0)\) be a regular point, then there exist a neighbourhood \( U \) of \((\sigma_0, \eta_0)\) and \( \delta > 0 (V_8 \subset U) \) such that for any point \((\tilde{\sigma}_0, \tilde{\eta}_0) \in S_i \cap U \), we have

\[ |\tilde{\xi}_j^\pm(\tau, \eta) - \xi_j^\pm(\tau, \eta)| < C \tilde{d}^{\frac{2}{m_i}} \quad \text{in } \mathcal{A}_{i1}, \]

\[ \left| \frac{\partial \tilde{\xi}_j^\pm}{\partial \tau}(\tau, \eta) - \frac{\partial \xi_j^\pm}{\partial \tau}(\tau, \eta) \right| < C \tilde{d}^{-1 + \frac{2}{m_i}} \quad \text{in } \mathcal{A}_{i1}, \]

where \( C \) is independent not only of \((\tau, \eta)\) but also of \((\tilde{\sigma}_0, \tilde{\eta}_0)\).

**Corollary 1.** Let \((\sigma_0, \eta_0)\) be a regular point, then for any \((\tilde{\sigma}_0, \tilde{\eta}_0) \in S_i \cap U \)

\[ c_1 \tilde{d}^{\frac{1}{m_i}} \leq |\tilde{\xi}_j^\pm(\tau, \eta) - \xi_j^\pm(\tau, \eta)| \leq c_2 \tilde{d}^{\frac{1}{m_i}} \quad \text{in } \mathcal{A}_{i1}, \]

\[ c_1 \tilde{d}^{\frac{1}{m_i}} \leq |\tilde{\xi}_j^\pm(\tau, \eta) - \tilde{\xi}_k^\pm(\tau, \eta)| \leq c_2 \tilde{d}^{\frac{1}{m_i}} \quad (j \neq k) \quad \text{in } \mathcal{A}_{i1}, \]

where \( c_1 \) and \( c_2 \) are independent of \((\tau, \eta)\) and \((\tilde{\sigma}_0, \tilde{\eta}_0)\).

We define \( \tilde{\partial}_{i1}^\pm \) in the same way as \( \partial_{i1}^\pm \), only replacing \( d \) by \( \tilde{d} \), and \( \mathcal{A}_{i}^\pm = \mathcal{A}_{i1}^\pm \), then
Corollary 2. Let \((\hat{\sigma}_0, \eta_0)\) be a regular point, then for any \((\hat{\sigma}_0, \eta_0)\) \(\in S_i \cap U\)

\[ c_1 \delta^*_i \pm m_i \leq |\text{Im} \xi^*_i(\tau, \eta)| \leq c_2 \delta^*_i \pm m_i \quad \text{in } A_i \]

where \(c_1\) and \(c_2\) are independent of \((\tau, \eta)\) and \((\hat{\sigma}_0, \eta_0)\).

Now let us denote

\[ V^+_i = \bigcup (\hat{\sigma}_i, \eta_i), \quad V^*_i = \bigcap_{i=1}^{N} V^*_i, \]

then we have

\[ V_{\delta} = \bigcup_{\delta} V^*_{\delta}. \]

Moreover we denote for \(m_i \geq 2\)

\[ d_i(\tau, \eta) = \text{dis} ((\tau, \eta), S_i), \]

\[ \delta^*_i(\tau, \eta) = \begin{cases} \left( \frac{\tau}{d_i} \right)^{m_i} d_i & \text{if } \xi^*_i = \xi^*_i, \\ d_i & \text{otherwise}, \end{cases} \]

and \(\delta^*_i(\tau, \eta) = \tau\) for \(m_i = 1\), then we have

Corollary 1'. Let \((\hat{\sigma}_0, \eta_0)\) be a regular point, then we have

\[ c_1 d_i^{\frac{1}{m_i}} \leq |\xi^*_i(\tau, \eta) - \xi^*_i(\tau, \eta)| \leq c_2 d_i^{\frac{1}{m_i}} \quad (j \neq k) \]

in \(V_{\delta}\).

Corollary 2'. Let \((\hat{\sigma}_0, \eta_0)\) be a regular point, then we have

\[ c_1 \delta^*_i \pm m_i \leq |\text{Im} \xi^*_i(\tau, \eta)| \leq c_2 \delta^*_i \pm m_i \]

in \(V^*_i\).

Corollary 3. Let \((\sigma_0, \eta_0)\) be regular, then we have
\[
\begin{pmatrix}
0 \\
o \\
m \\
1 \\
0 \\
0
\end{pmatrix} = \sum_{i=1}^{N} \sum_{i=1}^{m_i} C_{ij,\pm}^{s}(\tau, \eta) d_i^{-1} \begin{pmatrix}
B_{1}(\tau, \eta; \xi_{1j}^{+}(\tau, \eta)) \\
\vdots \\
B_{\mu}(\tau, \eta; \xi_{1j}^{+}(\tau, \eta))
\end{pmatrix}
\]

\[
+ \sum_{j=1}^{M} C_{0j,\pm}^{s}(\tau, \eta) \begin{pmatrix}
\frac{1}{2i\pi} \int \frac{B_{1}(\tau, \eta; \xi) \xi^{j-1}}{E_{\pm}(\tau, \eta; \xi)} d\xi \\
\vdots \\
\frac{1}{2i\pi} \int \frac{B_{\mu}(\tau, \eta; \xi) \xi^{j-1}}{E_{\pm}(\tau, \eta; \xi)} d\xi
\end{pmatrix}
\]
in $V_{\delta}$, where \(\{C_{ij,\pm}^{s}(\tau, \eta), C_{0j,\pm}^{s}(\tau, \eta)\}\) are bounded in $V_{\delta}$.

### 2.2. Estimates of Green's function in $V_{\delta}^{*}$.
Let $(\sigma_0, \eta_0)$ be a regular point, then it is shown that
\[
c_1 I < S_{\pm}(\tau, \eta) < c_2 I \quad \text{in } V_{\delta},
\]
in the same way as the proof of Lemma 1.3, making use of Lemma 2.1. Hence we have

**Lemma 2.2.** Let $(\sigma_0, \eta_0)$ be regular, then
\[
\sum_{k=0}^{m-1} \left\| \left( \frac{\partial}{\partial k} \right)^k G_{\epsilon}(\tau, \eta; x, \gamma) \right\|_{L^1 L^2} \leq C |N_{\pm}(\tau, \eta)B(\tau, \eta)N_{-}(\tau, \eta)|
\]
\[
\leq C' \| G_{\epsilon}(\tau, \eta; x, \gamma) \|_{L^2 L^1 C^1} \quad \text{in } V_{\delta}.
\]

Let us denote
\[
\begin{pmatrix}
\delta_{11,\pm}^{s} & \frac{1}{2m_1} \\
\delta_{1m_1,\pm}^{s} & \frac{1}{2m_1} \\
\vdots & \vdots \\
0 & \frac{1}{2m_1} \\
0 & \frac{1}{2m_1} \\
\vdots & \vdots \\
1 & 1
\end{pmatrix}
\]
in $V_{\delta}^{*}$ and
\[ \tilde{\mathcal{D}}_-(\tau, \eta) = \left( \begin{array}{cccc} d_1^{-1+\frac{1}{m_1}} & & & m_1^- \\ \vdots & \ddots & \ddots & \vdots \\ d_1^{-1+\frac{1}{m_1}} & & & 1 \\ 1 & \cdots & 1 & \end{array} \right) \text{ in } V_\delta, \]

then we have

**Proposition 2.** Let \((\sigma_0, \eta_0)\) be regular, then there exist positive constants \(\delta, c_1\) and \(c_2\) such that

\[
c_1 |D^*_w(\tau, \eta)\mathcal{S}(\tau, \eta)\tilde{\mathcal{D}}_-(\tau, \eta)D^*_w(\tau, \eta)| \leq \| G_c(\tau, \eta; x, \gamma) \|_{L^2, H^{m-1}} \]

\[
\leq c_2 |D^*_w(\tau, \eta)\mathcal{S}(\tau, \eta)\tilde{\mathcal{D}}_-(\tau, \eta)D^*_w(\tau, \eta)| \quad \text{in } V^*_\delta. \]

**Theorem II.** Let every real point be regular. In order that \((P)\) is \(L^2\)-well-posed, it is necessary and sufficient that there exist positive constants \(\delta\) and \(C\) for each real point such that it holds

\[
|D^*_w(\tau, \eta)\mathcal{S}(\tau, \eta)\tilde{\mathcal{D}}_-(\tau, \eta)D^*_w(\tau, \eta)| < \frac{C}{\tau} \quad \text{in } V^*_\delta, \]

that is,

\[
\begin{align*}
|\beta_{ij,ih}(\tau, \eta)| & \delta^{ij}_{i+} \delta^{*} - 1 + \frac{1}{m_1} \tilde{\delta}^{*}_{i+} - \frac{1}{2m_1} < \frac{C}{\tau} \quad (i \neq 0, l \neq 0), \\
|\beta_{ij,ih}(\tau, \eta)| & d_i^{-1+\frac{1}{m_1}} \delta^{*}_{i+} - \frac{1}{2m_1} < \frac{C}{\tau} \quad (i \neq 0), \\
|\beta_{ij,0h}(\tau, \eta)| & \delta^{ij}_{i+} \delta^{*} - \frac{1}{2m_1} < \frac{C}{\tau} \quad (i \neq 0), \\
|\beta_{ij,0h}(\tau, \eta)| & < \frac{C}{\tau}
\end{align*}
\]

in \(V^*_\delta\).

Since \(\tilde{\delta}_{i+}^{*} \leq d_i\), we have

**Corollary 1.** Let every real point be regular. In order that \((P)\) is
$L^2$-well-posed, it is necessary that

\[
\begin{align*}
|\beta_{ij,k}(\tau, \eta)| d_i^\frac{1}{2m_i} d_i^{-\frac{1}{2m_i}} &< \frac{C}{r} \quad (i \neq 0, l \neq 0), \\
|\beta_{0j,k}(\tau, \eta)| d_i^{-1 + \frac{1}{2m_i}} &< \frac{C}{r} \quad (l \neq 0), \\
|\beta_{ij,0k}(\tau, \eta)| d_i^{-\frac{1}{2m_i}} &< \frac{C}{r} \quad (i \neq 0), \\
|\beta_{0j,0k}(\tau, \eta)| &< \frac{C}{r}
\end{align*}
\]

in $V_\delta$.

Making use of Corollary 3 of Lemma 2.1, it follows from Corollary 1 of Theorem II:

**Corollary 2.** Let every real point be regular. In order that $\mathcal{P}$ is $L^2$-well-posed, it is necessary that

\[
\begin{align*}
|\rho_{ij,k}(\tau, \eta)| d_i^{-\frac{1}{2m_i}} &< \frac{C}{r} \quad (i \neq 0), \\
|\rho_{0j,k}(\tau, \eta)| &< \frac{C}{r}
\end{align*}
\]

in $V_\delta$.

On the other hand, since $\bar{\delta}^*_{ij,kl}(\tau, \eta) = \left(\frac{r}{d_i}\right)^{m_i} d_i$, we have

**Corollary 3.** Let every real point be regular. In order that $\mathcal{P}$ is $L^2$-well-posed, it is sufficient that

\[
\begin{align*}
|\beta_{ij,k}(\tau, \eta)| d_i^{\frac{1}{2}} d_i^{-\frac{1}{2m_i}} (1 - \frac{1}{2m_i}) &< C \quad (i \neq 0, l \neq 0), \\
|\beta_{0j,k}(\tau, \eta)| d_i^{-\frac{1}{2} - \frac{1}{2m_i}} &< \frac{C}{r^2} \quad (l \neq 0), \\
|B_{ij,0k}(\tau, \eta)| d_i^{\frac{1}{2}} d_i^{-\frac{1}{2m_i}} &< \frac{C}{r^2} \quad (i \neq 0), \\
|\beta_{0j,0k}(\tau, \eta)| &< \frac{C}{r}
\end{align*}
\]
in $V_3$.

2.3. Semi-uniform Lopatinski's condition. Let us denote in $V_3$

$$R_{l_0-j_0-l_{0h}}(\tau, \eta) = \prod_{i=0}^{l_0} \prod_{j<k}^{l_{0h}} \left( \frac{1}{\prod_{j<k}^{l_{0h}} (\hat{\xi}_{i, j}(\tau, \eta) - \hat{\xi}_{i, k}(\tau, \eta))} \prod_{j<k}^{l_{0h}} (\hat{\xi}_{i, j}(\tau, \eta) - \hat{\xi}_{i, k}(\tau, \eta)) \right) \times \det \begin{pmatrix} B_1(\tau, \eta; \xi_{11}(\tau, \eta)) & \cdots & B_1(\tau, \eta; \xi_{1l_{0h}-1}(\tau, \eta)) \\ \vdots & \ddots & \vdots \\ B_\mu(\tau, \eta; \xi_{11}(\tau, \eta)) & \cdots & B_\mu(\tau, \eta; \xi_{1l_{0h}-1}(\tau, \eta)) \end{pmatrix} \times \det \begin{pmatrix} B_1(\tau, \eta; \xi_{1l_{0h}}(\tau, \eta)) & \cdots & B_1(\tau, \eta; \xi_{1l_{0h}+1}(\tau, \eta)) \\ \vdots & \ddots & \vdots \\ B_\mu(\tau, \eta; \xi_{1l_{0h}}(\tau, \eta)) & \cdots & B_\mu(\tau, \eta; \xi_{1l_{0h}+1}(\tau, \eta)) \end{pmatrix}$$

$$(i_0 \equiv 0, \ l_0 \equiv 0),$$

$$R_{0j_0-l_{0h}}(\tau, \eta) = \prod_{i}^{l_{0h}} \prod_{j<k}^{l_{0h}} \left( \frac{1}{\prod_{j<k}^{l_{0h}} (\hat{\xi}_{i, j}(\tau, \eta) - \hat{\xi}_{i, k}(\tau, \eta))} \prod_{j<k}^{l_{0h}} (\hat{\xi}_{i, j}(\tau, \eta) - \hat{\xi}_{i, k}(\tau, \eta)) \right) \times \det \begin{pmatrix} \cdots & \frac{1}{2\pi i} \int \frac{B_1(\tau, \eta; \xi, \xi_{j-2})d\xi}{E_s(\tau, \eta; \xi)} & \cdots \\ \vdots & \ddots & \vdots \\ \frac{1}{2\pi i} \int \frac{B_1(\tau, \eta; \xi_{l_{0h}})d\xi}{E_s(\tau, \eta; \xi)} & \cdots & \cdots \end{pmatrix}$$

$$(l_0 \equiv 0),$$
\[ R_{i_i, t_0, \theta_0}(\tau, \eta) = \prod_{i \neq i_0} \prod_{j < k} (\xi_i^+(\tau, \eta) - \xi_i^-(\tau, \eta)) \prod_{j < k} (\xi_i^{+\alpha}(\tau, \eta) - \xi_i^{+\beta}(\tau, \eta)) \]

\[ \times \det \begin{pmatrix}
\cdots \frac{1}{2\pi i} \int B_1(\tau, \eta; \xi) \xi^{h-1} E_+(\tau, \eta; \xi) d\xi \\
\cdots \\
B_1(\tau, \eta; \xi_i^{+\alpha-1}(\tau, \eta)) \\
\cdots \\
\end{pmatrix}
\]

\[ (i_0 \neq 0), \]

\[ R_{0,i_0,\theta_0}(\tau, \eta) = \prod_{i} \prod_{i < k} (\xi_i^+(\tau, \eta) - \xi_i^-(\tau, \eta)) \]

\[ \times \det \begin{pmatrix}
\cdots \frac{1}{2\pi i} \int B_1(\tau, \eta; \xi) \xi^{i-2} E_+(\tau, \eta; \xi) d\xi \\
\cdots \\
\end{pmatrix}
\]

then these are all bounded in $V_\theta$.

**Lemma 2.3.** Let $(\sigma_0, \eta_0)$ be regular, then there exist positive constants $\delta, c_1$ and $c_2$ such that

\[
\begin{align*}
&c_1 |\beta_{ji, lb}(\tau, \eta)| d_i^{m_i^{-1}} d_l^{m_l^{-1}} \leq \frac{R_{ij, lb}(\tau, \eta)}{R(\tau, \eta)} \\
\leq c_2 |\beta_{ji, lb}(\tau, \eta)| d_i^{m_i^{-1}} d_l^{m_l^{-1}} & (i \neq 0, l \neq 0), \\
&c_1 |\beta_{0j, lb}(\tau, \eta)| d_i^{m_i^{-1}} \leq \frac{R_{0j, lb}(\tau, \eta)}{R(\tau, \eta)} \leq c_2 |\beta_{0j, lb}(\tau, \eta)| d_i^{m_i^{-1}} & (l \neq 0), \\
&c_1 |\beta_{ij, 0b}(\tau, \eta)| d_i^{m_i^{-1}} \leq \frac{R_{ij, 0b}(\tau, \eta)}{R(\tau, \eta)} \leq c_2 |\beta_{ij, 0b}(\tau, \eta)| d_i^{m_i^{-1}} & (i \neq 0), \\
&c_1 |\beta_{0j, 0b}(\tau, \eta)| \leq \frac{R_{0j, 0b}(\tau, \eta)}{R(\tau, \eta)} \leq c_2 |\beta_{0j, 0b}(\tau, \eta)|
\end{align*}
\]
From Lemma 2.3, we have

**Theorem II'.** Let every real point be regular. In order that $(P)$ is $L^2$-well-posed, it is necessary and sufficient that

$$
\left| \frac{R_{ij,1h}(\tau, \eta)}{R(\tau, \eta)} \right| d_i \frac{m_i^{-1}}{m_i} \delta_{ij}^* \frac{1}{2m_i} d_i \frac{m_i^{-1}}{m_i} \delta_{ih}^* \frac{1}{2m_i} < \frac{C}{r} \quad (i \neq 0, l \neq 0),
$$

$$
\left| \frac{R_{ij,1h}(\tau, \eta)}{R(\tau, \eta)} \right| d_i \frac{m_i^{-1}}{m_i} \delta_{ij}^* \frac{1}{2m_i} < \frac{C}{r} \quad (l \neq 0),
$$

$$
\left| \frac{R_{ij,0h}(\tau, \eta)}{R(\tau, \eta)} \right| d_i \frac{m_i^{-1}}{m_i} \delta_{ij}^* \frac{1}{2m_i} < \frac{C}{r} \quad (i \neq 0),
$$

$$
\left| \frac{R_{ij,0h}(\tau, \eta)}{R(\tau, \eta)} \right| < \frac{C}{r}
$$
in $V_{\delta}$.

**Remark.** The conditions stated in Theorem II' are satisfied if uniform Lopatinski’s condition is satisfied, remarking that $\frac{m_i^+ - \frac{1}{2}}{m_i} \leq \frac{1}{2}$.

Now we consider a sufficient condition for $L^2$-well-posedness, which is stated only by the word of Lopatinski’s determinant $R(\tau, \eta)$. Let us denote

$$
\Omega = \{(\sigma, \eta) \in R^1 \times R^{n-1}, \sigma^2 + |\eta|^2 = 1, A(\sigma, \eta; \xi) \equiv 0 \text{ for any } \xi \in R^1\}.
$$

We say that semi-uniform Lopatinski’s condition is satisfied for $(P)$, when the following conditions are satisfied:

i) let $(\sigma_0, \eta_0) \in (\bar{\Omega})^c$, then $R(\sigma_0, \eta_0) \equiv 0$,

ii) let $(\sigma_0, \eta_0) \in \Omega$, then $R(\sigma_0, \eta_0) \equiv 0$ or $R_i(\sigma_0, \eta_0) \equiv 0$,

iii) let $(\sigma_0, \eta_0) \in \partial \Omega$, then there exists $V_{\delta}$ such that

$$
|R(\tau, \eta)| \geq c \frac{r}{d_0^{1-\frac{1}{m}}-1} \quad \text{if } (\text{Re } \tau, \eta) \in \Omega \cap V_{\delta},
$$

$$
|R(\tau, \eta)| \geq c d_0^{\frac{1}{m}} \quad \text{if } (\text{Re } \tau, \eta) \in (\bar{\Omega})^c \cap V_{\delta},
$$
where

$$\bar{m} = \max \{ m_i(\sigma_0, \eta_0) \}, \quad d_0 = \text{dis}((\tau, \eta), \partial \Omega).$$

**Theorem III.** Let every point of $\partial \Omega$ be regular. Then semiuniform Lopatinski’s condition is a sufficient condition for $L^2$-well-posedness for $(P)$.

**Proof.** Let $(\sigma_0, \eta_0) \in \partial \Omega(\equiv \phi)$, then all the indexes $\{ m_i(\sigma_0, \eta_0) \}$ are even, that is, $m^+(\sigma_0, \eta_0) = m^- (\sigma_0, \eta_0)$. Since

$$\begin{cases} 
  d_0 \leq d_i & \text{if } (\Re \tau, \eta) \in \Omega, \\
  d_0 \geq d_i & \text{if } (\Re \tau, \eta) \in (\bar{\Omega})^c,
\end{cases}$$

and

$$\begin{cases} 
  \delta_{ij}^* = d_i(\geq d_0) & \text{if } (\Re \tau, \eta) \in \Omega, \\
  \bar{\delta}_{ij}^* = \left( \frac{I}{d_i} \right)^{m_i} d_i & \text{if } (\Re \tau, \eta) \in (\bar{\Omega})^c,
\end{cases}$$

we have

$$\begin{cases} 
  d_i^{-\frac{1}{2} + \frac{1}{m_i}} \delta_{ij}^* \geq d_i^{-\frac{1}{2} + \frac{1}{2m_i}} \leq d_i^{-\frac{1}{2} + \frac{1}{2m_i}} & \text{if } (\Re \tau, \eta) \in \Omega, \\
  d_i^{-\frac{1}{2} + \frac{1}{m_i}} \bar{\delta}_{ij}^* \geq d_i^{-\frac{1}{2} + \frac{1}{2m_i}} \leq r^{-\frac{1}{2}} d_0^{-\frac{1}{2m}} & \text{if } (\Re \tau, \eta) \in (\bar{\Omega})^c,
\end{cases}$$

therefore we have

$$\frac{|\bar{R}_{ij, th}(\tau, \eta)|}{\bar{R}(\tau, \eta)} \leq \frac{C d_0^{-\frac{1}{2} + \frac{1}{m_i}} d_i^{-\frac{1}{2} + \frac{1}{m_i}} d_i^{-\frac{1}{2} + \frac{1}{2m_i}}}{|\bar{R}(\tau, \eta)|} \leq \frac{C'}{\gamma}$$

if $(\Re \tau, \eta) \in \Omega$,

$$\frac{|\bar{R}_{ij, th}(\tau, \eta)|}{\bar{R}(\tau, \eta)} \leq \frac{C' d_0^{-\frac{1}{2} + \frac{1}{m_i}} d_i^{-\frac{1}{2} + \frac{1}{m_i}} d_i^{-\frac{1}{2} + \frac{1}{2m_i}}}{|\bar{R}(\tau, \eta)|} \leq \frac{C'}{\gamma}$$

if $(\Re \tau, \eta) \in (\bar{\Omega})^c$,

and so on. Q.E.D.
Example. Let $(P)$ be defined by

\[
\begin{align*}
A &= (\xi^2 + |\eta|^2 - \alpha^2 \tau^2)(\xi^2 + |\eta|^2 - \beta^2 \tau^2) \quad (\alpha > \beta > 0), \\
B_1 &= \xi^2 + |\eta|^2 - \alpha^2 \tau^2, \\
B_2 &= \xi - i(\alpha \tau + b \eta) \quad (a, b : \text{real}).
\end{align*}
\]

Then uniform Lopatinski’s condition is never satisfied, but Lopatinski’s condition is satisfied if and only if

\[|b|^2 - \frac{a^2}{\alpha^2} \leq 1.\]

If \(b^2 - \frac{a^2}{\alpha^2} = 1\), then $(P)$ is not $L^2$-well-posed. In fact, let \(\{\xi_1^\pm(\tau, \eta)\}\) be roots of \(\xi^2 + |\eta|^2 - \alpha^2 \tau^2 = 0\), let \(\{\xi_2^\pm(\tau, \eta)\}\) be roots of \(\xi^2 + |\eta|^2 - \beta^2 \tau^2 = 0\), then

\[R(\tau, \eta) = \frac{1}{\xi_2^+(\tau, \eta) - \xi_1^+(\tau, \eta)} \det \begin{pmatrix} B_1(\tau, \eta; \xi_1^+(\tau, \eta)) & B_1(\tau, \eta; \xi_2^+(\tau, \eta)) \\ B_2(\tau, \eta; \xi_1^+(\tau, \eta)) & B_2(\tau, \eta; \xi_2^+(\tau, \eta)) \end{pmatrix}.\]

Let \(\sigma_0 = -\frac{a}{\alpha^2}, \eta_0 = b\), then \(R(\sigma_0, \eta_0) = R'(\sigma_0, \eta_0) = 0\), and

\[\det \begin{pmatrix} B_1(\sigma_0, \eta_0; \xi_1(\sigma_0, \eta_0)) & B_1(\sigma_0, \eta_0; \xi_2(\sigma_0, \eta_0)) \\ B_2(\sigma_0, \eta_0; \xi_1(\sigma_0, \eta_0)) & B_2(\sigma_0, \eta_0; \xi_2(\sigma_0, \eta_0)) \end{pmatrix} \neq 0\]

therefore

\[
\begin{align*}
\det \begin{pmatrix} B_1(\sigma_0 - i \gamma, \eta_0; \xi_1(\sigma_0 - i \gamma, \eta_0)) & B_1(\sigma_0 - i \gamma, \eta_0; \xi_2(\sigma_0 - i \gamma, \eta_0)) \\ B_2(\sigma_0 - i \gamma, \eta_0; \xi_1(\sigma_0 - i \gamma, \eta_0)) & B_2(\sigma_0 - i \gamma, \eta_0; \xi_2(\sigma_0 - i \gamma, \eta_0)) \end{pmatrix} &
\geq \frac{c}{\gamma^2} \\
\det \begin{pmatrix} B_1(\sigma_0 - i \gamma, \eta_0; \xi_1(\sigma_0 - i \gamma, \eta_0)) & B_1(\sigma_0 - i \gamma, \eta_0; \xi_2(\sigma_0 - i \gamma, \eta_0)) \\ B_2(\sigma_0 - i \gamma, \eta_0; \xi_1(\sigma_0 - i \gamma, \eta_0)) & B_2(\sigma_0 - i \gamma, \eta_0; \xi_2(\sigma_0 - i \gamma, \eta_0)) \end{pmatrix} &
= 0 < \gamma < \gamma_0,
\end{align*}
\]

which contradict to the necessary condition for $L^2$-well-posedness. If \(b^2 - \frac{a^2}{\alpha^2} < 1\), then semi-uniform Lopatinski’s condition is satisfied, therefore $(P)$ is $L^2$-well-posed.
References


