A Characterization for Fourier Hyperfunctions

By

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Abstract

The space of test functions for Fourier hyperfunctions is characterized by two conditions \( \sup_{\varphi(x)} |\exp k |x| < \infty \) and \( \sup_{\hat{\varphi}(\xi)} |\exp h |\xi| < \infty \) for some \( k, h > 0 \). Combining this result and the new characterization of Schwartz space in [1] we can easily compare two important spaces \( \mathcal{F} \) and \( \mathcal{S} \) which are both invariant under Fourier transformations.

§ 0. Introduction

The purpose of this paper is to give new characterization of the space \( \mathcal{F} \) of test functions for the Fourier hyperfunctions.

In [6], K. W. Kim, S. Y. Chung and D. Kim introduce the real version of the space \( \mathcal{F} \) of test functions for the Fourier hyperfunctions as follows,

\[
\mathcal{F} = \left\{ \varphi \in C^\omega : \sup_{a, \xi} \frac{|\partial^a \varphi(x)| \exp_{a} |x|}{h^{a_{1}}!} < \infty \text{ for some } k, h \right\}.
\]

They also show the equivalence of the above definition and Sato-Kawai’s original definition in complex form.

Also, in [1] J. Chung, S. Y. Chung and D. Kim give new characterization of the Schwartz space \( \mathcal{S} \), i.e., show that for \( \varphi \in C^\omega \) the following are equivalent:

1. \( \varphi \in \mathcal{S} \);
2. \( \sup_{x} |x^{a} \varphi(x)| < \infty \), \( \sup_{x} |\partial^{\alpha} \varphi(x)| < \infty \) for all multi-indices \( \alpha \) and \( \beta \);
3. \( \sup_{x} |x^{a} \varphi(x)| < \infty \), \( \sup_{\xi} |\xi^{\beta} \hat{\varphi}(\xi)| < \infty \) for all multi-indices \( \alpha \) and \( \beta \).

In a similar fashion as above we will give new characterization of the space \( \mathcal{F} \) of test functions for the Fourier hyperfunctions as the main theorem in this paper which says that for \( \varphi \in C^\omega \) the following are equivalent:

1. \( \varphi \in \mathcal{F} \);
2. \( \sup_{x} |\varphi(x)| \exp_{a} |x| < \infty \), \( \sup_{\xi} |\hat{\varphi}(\xi)| \exp_{h} |\xi| < \infty \) for some \( h, k > 0 \).

Observing the above growth conditions we can easily see that the space \( \mathcal{F} \) which is invariant under the Fourier transformation is much smaller than...
Since an element in the strong dual $\mathcal{S}'$ of the space $\mathcal{S}$ is called a Fourier hyperfunction, the space $\mathcal{S}'$ of Fourier hyperfunctions which is also invariant under the Fourier transformation is much bigger than the space $\mathcal{S}'$ of tempered distributions.

Section 1 is devoted to providing the necessary definitions and preliminaries. We prove the main theorem in Section 2.

§ 1. Preliminaries

We use the multi-index notations; for $x = (x_1, \ldots, x_n), \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n, \partial^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n}, |\alpha| = \alpha_1 + \cdots + \alpha_n$ with $\partial_j = \partial / \partial x_j,$ and $\mathbb{N}_0$ the set of non-negative integers.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\hat{f}$ is the bounded continuous function in $\mathbb{R}^n$ defined by

\begin{equation}
\hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n
\end{equation}

Definition 1.1. We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that

\begin{equation}
\sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty
\end{equation}

for all multi-indices $\alpha$ and $\beta$.

We need the following characterization to compare the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions with the above space.

Theorem 1.2 [1]. (i) The Schwartz space $\mathcal{S}$ consists of all $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying the conditions

\begin{equation}
\sup_x |x^\alpha \varphi(x)| < \infty, \\
\sup_x |\partial^\beta \varphi(x)| < \infty
\end{equation}

for all multi-indices $\alpha$ and $\beta$.

(ii) Also, the Schwartz space can be characterized by the following two conditions

\begin{equation}
\sup_x |x^\alpha \varphi(x)| < \infty, \\
\sup_x |\xi^\beta \hat{\varphi}(\xi)| < \infty
\end{equation}

for all multi-indices $\alpha$ and $\beta$.

Now, we are going to introduce the original complex version and new real definition of test functions for the Fourier hyperfunctions as in [6], and state their equivalence.
Definition 1.3 [6]. A real valued function $\varphi$ is in $\mathcal{S}$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and if there are positive constants $h$ and $k$ such that

$$|\varphi|_{k,h} = \sup_{x,z} \frac{\partial^a \varphi(x)}{h^{\alpha} \alpha!} \exp k|x| < \infty.$$ 

Definition 1.4 [5]. A complex valued function $\varphi(z)$ is in $\mathcal{S}^\ast$ if $\varphi(z)$ is holomorphic in a tubular neighborhood $\mathbb{R}^n + i\{|y| \leq r\}$, for some $r$, of $\mathbb{R}^n$ and if for some $k > 0$

$$\sup_{z \in \mathbb{R}^n + i\{|y| \leq r\}} |\varphi(z)| \exp k|z| < \infty.$$

Theorem 1.5 [6]. The space $\mathcal{S}$ is isomorphic to the space $\mathcal{S}^\ast$.

Definition 1.6. We denote by $\mathcal{S}'$ the strong dual space of $\mathcal{S}$ and call its elements Fourier hyperfunctions.

Thus the global theory of the Fourier hyperfunctions is nothing but the duality theory for the space $\mathcal{S}$.

§2. Main Theorem

Now we shall give new characterization of the space $\mathcal{S}$ of test functions for the Fourier hyperfunctions which is the main result in this paper. First, we prove

Theorem 2.1. The following conditions for $\varphi \in C^\infty$ are equivalent:

(i) There are positive constants $k$ and $h$ such that

$$\sup_{x} \frac{\partial^a \varphi(x) \exp k|x|}{h^{\alpha} \alpha!} < \infty. \tag{2.1}$$

(ii) There are positive constants $C$, $k$ and $h$ such that

$$\sup_{x} |\varphi(x)| \exp k|x| < \infty, \tag{2.2}$$

$$\sup_{x} |\partial^a \varphi(x)| \leq C h^{\alpha} \alpha!. \tag{2.3}$$

(iii) There are positive constants $k$ and $h$ such that

$$\sup_{x} |\varphi(x)| \exp k|x| < \infty, \tag{2.4}$$

$$\sup_{\xi} |\hat{\varphi}(\xi)| \exp h|\xi| < \infty. \tag{2.5}$$

Proof. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) are trivial. So it suffices to prove the implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) in order.

(iii) $\Rightarrow$ (ii): By the inequality (2.5) we have
for some positive constants $M$, $A$ and $C$. Thus, we obtain the condition (2.3) which completes the proof of the implication (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i): First, we can assume that $\varphi$ is real valued. By integration by parts we obtain that

$$\|x^\beta \varphi(x)\|_2 = \left|\int_{\mathbb{R}^n} \partial^\alpha \varphi(x) \varphi(x) dx\right|.$$

Note that the boundary terms tend to zero by Theorem 1.2. Therefore, applying the Leibniz formula we have, for some constant $A$,

$$\|x^\beta \varphi(x)\|_2 \leq \sum_{\gamma \leq \beta \alpha} \left(\begin{array}{c} \beta \\ \gamma \end{array}\right) \gamma! \|x^{\beta - \gamma} \varphi(x)\| \varphi(x) \int dx.$$

Thus we obtain that for some positive constants $C_0$, $C_1$ and $C_2$ such that

$$\frac{\|x^\beta \varphi(x)\|_2}{C_2^{\beta !} \beta !} \leq C_0 C_1^{\alpha !} \alpha !.$$

Therefore, summing up with respect to $\beta$ we can choose a positive constant
\[ k \text{ such that } \| \partial^a \phi(x) \exp k \cdot x \|_{L^2} \leq C_s C_4^{a1} \alpha !. \]

By the Cauchy-Schwarz inequality there exists a positive constant \( C_s \) such that
\[
\left\| \partial^a \phi(x) \exp k \cdot x \right\|_{L^1} \leq \left\| \partial^a \phi(x) \exp \left\| \exp \left( -k \cdot x \right) \right\|_{L^2} \right\| \left[ \int \exp(-k \cdot x) \, dx \right]^{1/2} \leq C_s C_4^{a1} \alpha !.
\]

Also, there exist positive constants \( k \) and \( C_1 \) such that
\[
\| \partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2} \|_{L^1} \leq C_s C_4^{a1} \alpha !.
\]

Hence
\[
| \partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2} | = \left| \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \partial_1 \cdots \partial_n (\partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2}) \, dx \right|
\]
\[
= \left| \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \partial_n \cdots \partial_1 [ \partial_n \partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2} ] \, dx \right|
\]
\[
+ \partial^a \phi(x) \cdot \partial_1 \left( \exp k \sqrt{1 + \left| x \right|^2} \right) \, dx \right|
\]
\[
\leq \left| \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \sum | \partial_{j_1} \cdots \partial_{j_r} \partial^a \phi(x) \partial_{j_{r+1}} \cdots \partial_{j_n} (\partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2}) \, dx \right|
\]
where the summation is taken over all \( r = 0, 1, \ldots, n \) and \( \{j_1, \ldots, j_n\} \) is a permutation of \( \{1, \ldots, n\} \).

We can prove by induction
\[
| (\partial_{j_1} \cdots \partial_{j_r}) \partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2} | \leq P_r(k) \exp k \sqrt{1 + \left| x \right|^2}
\]
where \( P_r(k) \) is a polynomial of \( k \) of \( r \)-th degree. Hence we derive that
\[
| \partial^a \phi(x) \exp k \sqrt{1 + \left| x \right|^2} | \leq \int C \Sigma | P_{n-r}(k) | | (\partial_{j_1} \cdots \partial_{j_r}) \partial^a \phi | \exp k \sqrt{1 + \left| x \right|^2} \, dx
\]
\[
\leq C \Sigma | P_{n-r}(k) | C_s C_4^{a1+r} (\alpha + \beta) !
\]
\[
\leq C(k, n) C_4^{a1} \alpha !
\]
where \( \beta \) is a multi-index with \( | \beta | = r \). Therefore, using the relation
\[
\exp k | x | \leq \exp k \sqrt{1 + \left| x \right|^2} \leq e^k \exp k | x | \]
we obtain
\[
\sup_x | \partial^a \phi(x) | \exp k | x | \leq C(k, n) C_4^{a1} \alpha !
\]
which completes the proof.

Now we can rephrase Theorem 2.1 as follows.
Theorem 2.2. The space $\mathcal{S}$ of test functions for the Fourier hyperfunctions consists of all locally integrable functions such that for some $h, k > 0$

$$\sup_x |\phi(x)| \exp k|x| < \infty,$$

$$\sup_\xi |\hat{\phi}(\xi)| \exp h|\xi| < \infty.$$ 

Remark. Combining Theorem 1.2 on the Schwartz space $S$ and Theorem 2.2 on the space $\mathcal{S}$ we can easily compare the spaces $S$ and $\mathcal{S}$ which are both invariant under the Fourier transformations as follows:

(i) The space $S$ consists of all $C^\infty$ functions $\phi$ such that $\phi$ itself and its Fourier transform $\hat{\phi}$ are both rapidly decreasing.

(ii) The space $\mathcal{S}$ consists of all $C^\infty$ functions $\phi$ such that $\phi$ itself and its Fourier transform $\hat{\phi}$ are both exponentially decreasing.

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References


