A Remark on the $C^\infty$-Goursat Problem II

By

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§1. Introduction and Results

Let us consider the following operator with constant coefficients:

$$(1.1) \quad \mathcal{L}(\partial_t, \partial_x, \partial_y) = \sum_{i+j+|\alpha| \leq m} a_{ij\alpha} \partial_t^i \partial_x^j \partial_y^\alpha, \quad a_{ij\alpha} : \text{constant},$$

$$\partial_t = \partial/\partial t, \quad \partial_x = \partial/\partial x, \quad \partial_y = (\partial/\partial y_1, \partial/\partial y_2, \ldots, \partial/\partial y_n).$$

In this paper we assume that the hypersurface $t = 0$ is $m_2$-tuple characteristic, namely

1) $a_{ij\alpha} = 0$ for $i + j + |\alpha| = m$, $i > m - m_2 \equiv m_1$,
2) $\sum_{j+|\alpha|=m_2} a_{m_1,i,j} \xi^i \eta^\alpha \neq 0$.

We consider the following Goursat problem:

$$\begin{cases}
\mathcal{L}u = 0, \\
\partial_t^i u(0, x, y) = \phi_i(x, y) \in \mathcal{E}(x,y), & 0 \leq i \leq m_1 - 1, \\
\partial_x^j u(t, 0, y) = \psi_j(t, y) \in \mathcal{E}(t,y), & 0 \leq j \leq m_2 - 1, \\
(x \in \mathbb{R}, y \in \mathbb{R}^n, t \in \mathbb{R}^+ \text{ (or } t \in \mathbb{R}^1))
\end{cases}$$

where $\partial_y^j \phi_i(0, y) = \partial_y^j \psi_j(0, y), \ 0 \leq i \leq m_1 - 1, \ 0 \leq j \leq m_2 - 1$.

We say that the Goursat problem (G) is $\mathcal{E}$-wellposed for $t \geq 0$ (or for $t \leq 0$) if for any data $\{\phi_i\} \{\psi_j\}$ there exists a unique solution $u(t, x, y) \in \mathcal{E}(t,x,y)$ for $t \geq 0$ (or $t \leq 0$). If the Goursat problem is $\mathcal{E}$-wellposed for $t \geq 0$ (or for $t \leq 0$) then the linear mapping $\{\{\phi_i\}, \{\psi_j\}\} \rightarrow u(t, x, y)$ is continuous from $\prod \mathcal{E}(x,y) \times \prod \mathcal{E}(t,y)$ into $\mathcal{E}(t,x,y)$ for $t \geq 0$ (or for $t \leq 0$). T. Nishitani [3] had considered the following operator:

$$(N) \quad \sum_{i+j+|\alpha| \leq m} a_{ij\alpha} \partial_t^i \partial_x^j \partial_y^\alpha, \quad a_{m-m_2, m_2, 0} \neq 0,$$

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and he had obtained a necessary and sufficient condition for $\mathcal{E}$-wellposedness. For this operator (N), we obtained Levi condition [2]. Let us call the operator (N) which was treated by Nishitani "N-type". In this paper we will show that if (G) is $\mathcal{E}$-wellposed then $\mathcal{L}$ is N-type.

Remark 1.1. "Operator $\mathcal{L}$ is N-type" means that
\[
a_{m_1, m_2, 0} \neq 0 \quad \text{and} \quad a_{i, j, x} = 0 \quad \text{for} \quad i > m_1.
\]

Theorem. If the Goursat problem (G) is $\mathcal{E}$-wellposed for $t \geq 0$ and for $t \leq 0$ then $\mathcal{L}$ is N-type.

Remark 1.2. In [1] we proved this theorem under some assumptions. The paper "A remark on the $C^\infty$-Goursat problem I" does not exist, but we regard [1] as "A remark on the $C^\infty$-Goursat problem I".

Proposition 1.3. If (G) is $\mathcal{E}$-wellposed for $t \geq 0$ then $a_{m_1, m_2, 0} \neq 0$.

The proof is given in [1].

§ 2. Simple Case (where $\mathcal{L}$ does not include $\partial_x$)

At first we treat the operator which does not include $\partial_y$. Taking account of Proposition 1.3, we consider the following operator:

\[
L(\partial_t, \partial_x) = \partial_t^{m_1} \partial_x^{m_2} - \sum_{i+j=m} a_{ij} \partial_t^i \partial_x^j,
\]
\[
m = m_1 + m_2, \quad a_{ij}: \text{constant}, \quad a_{ij} = 0 \quad \text{for} \quad i + j = m, \quad i \geq m_1,
\]

(2.1)

\[
L(\partial_t, \partial_x)u = 0.
\]

(2.2)

Let the following Goursat problem be (G'):

\[
(G') \begin{cases}
L(\partial_t, \partial_x)u = 0, \\
\partial_t^i u(0, x) = \phi_i(x) \in \mathcal{E}_x, & 0 \leq i \leq m_1 - 1, \\
\partial_t^j u(t, 0) = \psi_j(t) \in \mathcal{E}_t, & 0 \leq j \leq m_2 - 1, \\
\partial_t^i \phi_i(0) = \partial_t^j \psi_j(0), & 0 \leq i \leq m_1 - 1, \quad 0 \leq j \leq m_2 - 1,
\end{cases}
\]

\[(t \geq 0 \text{ or } t \leq 0, x \in R^1).\]

We set

\[
\Omega = \{(i, j); a_{ij} \neq 0\}, \quad ((m_1, m_2) \notin \Omega).
\]

(2.3)
Theorem 2.1. If there exists \((i, j)\) in \(\Omega\) such that \(i > m_1 (i + j < m)\) then the Goursat problem \((G')\) is not \(\varepsilon\)-wellposed for \(t \geq 0\) or for \(t \leq 0\).

For to prove Theorem 2.1, we are going to show that the continuity from data to solution does not hold under the assumption of the theorem 2.1. More precisely we construct Goursat data \(\{\phi_i(x; \xi), \psi_j(t; \xi)\}\) which have the following properties:

1) The growth order of Goursat data is at most polynomial of \(|\xi|\),
2) We denote the solution of \((G')\) with previous Goursat data by \(u(t, x; \xi)\). The growth order of \(\partial_t^{m_1}u(0, x; \xi)\) is exponential of \(|\xi|\), where \(x\) is bounded for large \(|\xi|\).

Under the assumption of theorem 2.1, we consider \(\Omega\) in \(\mathbb{R}^2_+\). Let \(\ell\) be the straight line through \((m_1, m_2)\) which has the following properties:

1) All elements \((i, j)\) in \(\Omega\) exist under \(\ell\) or on \(\ell\),
2) There exists at least one element \((i, j)\) in \(\Omega\) on \(\ell\).

Let the slope of \(\ell\) be \(-p/q\). \((0 < q < p, p\) and \(q\) are relatively prime). Here we put \(m_1p + m_2q = J\) and

\[
\Gamma(k) = \{(i, j); pi + qj = k, (i, j) \in \Omega\}, \quad k = 0, 1, \ldots, J.
\]

Obviously, it holds that

\[
\bigcup_{k=0}^{J} \Gamma(k) = \Omega.
\]

Let

\[
L_j(\tau, \zeta) = \tau^{m_1} \zeta^{m_2} - \sum_{(i, j) \in \Gamma(j)} a_{ij} \tau^i \zeta^j
\]

and consider the roots of \(L_j(1, \zeta) = 0\). We put

\[
L_j(1, \zeta) = \zeta^\rho (\zeta - \alpha_1)^{n(1)} (\zeta - \alpha_2)^{n(2)} \ldots (\zeta - \alpha_N)^{n(N)},
\]

where \(\rho \geq 0, \alpha_i \neq \alpha_j\) if \(i \neq j, |x_1| \geq |x_i| > 0, \alpha_i\) is \(n(i)\)-tuple root and \(\rho + n(1) + n(2) + \cdots + n(N) = m_2\). Let (2.8) be a formal solution of (2.2):

\[
(2.8) \quad u(t, x) = \sum_{r,s} u_{r,s} t^r x^s / r! s!.
\]

We consider the following Goursat data:

\[
(2.9) \quad \begin{cases} 
   u_{r,s} = 0, & 0 \leq s \leq \rho - 1, \\
   u_{r,s} = \alpha_1^s \zeta^{pr + qs}, & \rho \leq s \leq m_2 - 1, \\
   u_{r,s} = 0, & s \geq m_2, \quad 0 \leq r \leq m_1 - 1,
\end{cases}
\]

namely
Putting (2.8) in (2.2) and comparing the coefficients of \( t^r x^s \), we have

\[
\partial_t^\rho u(t,0) = 0, \quad 0 \leq j \leq \rho - 1,
\]

\[
\partial_t^\rho u(t,0) = \sum_r (u_{r,j}/r!) t^r = (\alpha_1 \xi^q)^j \sum_r (\xi^{pr}/r!) t^r = (\alpha_1 \xi^q)^j \exp(\xi^q t),
\]

(2.9')

\[
\rho \leq j \leq m_2 - 1,
\]

\[
\partial_t^\rho u(0,x) = \xi^{pl} \sum_{s=\rho}^{m_2-1} (\alpha_1^s \xi^{qs}/s!) x^s, \quad 0 \leq i \leq m_1 - 1.
\]

Putting (2.8) in (2.2) and comparing the coefficients of \( t^r x^s \), we have

\[
u_{r+m_1,s+m_2} = \sum_{(i,j) \in \Omega} a_{ij} u_{r+i,s+j}.
\]

Here we introduce the following notion.

**Definition 2.2.** The order of \( u_{rs} \) is higher than the order of \( u_{r's'} \) (or the order of \( u_{rs} \) is lower than that of \( u_{rs} \)) if and only if \( pr + qs > pr' + qs' \) or else \( pr + qs = pr' + qs' \) and \( s' < s \).

By (2.9) and (2.10), we can determine \( u_{rs} \) successively and we have the following estimate.

**Lemma 2.3.** \( u_{rs} \) is a polynomial of \( \xi \) with degree \( pr + qs \) and has the following estimate:

\[
|u_{r,s}| \leq C |\xi|^{pr+qs} \quad \text{for large } |\xi|,
\]

where \( C \) is a constant independent of \( r, s \) and \( \xi \).

**Proof.** We prove this by induction. Taking \( C \) as \( |\alpha_1| < C \), Goursat data (2.9) satisfy (2.11). Suppose that \( u_{r's'} \) satisfies (2.11) if its order is lower than that of \( u_{r+m_1,s+m_2} \), then the following holds:

\[
|u_{r+m_1,s+m_2}| \leq A \sum_{(i,j) \in \Omega} |u_{r+i,s+j}|
\]

\[
\leq A \sum_{(i,j) \in \Omega} C^{r+j} |\xi|^{p(r+i)+q(s+j)}, \quad \text{where } A = \max_{(i,j) \in \Omega} |a_{ij}|.
\]

If \( (i, j) \in \Gamma(J) \), \( (i, j) \) is on \( \ell \). So, we have

\[
\Gamma(J) \subset \{(m_1 + q, m_2 - p), (m_1 + 2q, m_2 - 2p), \ldots, (m_1 + h_0 q, m_2 - h_0 p)\},
\]

\[
h_0 = [m_2/p].
\]

Thus, (2.12) becomes

\[
|u_{r+m_1,s+m_2}| \leq C^{r+m_2} |\xi|^{p(r+m_1)+q(s+m_2)}
\]

\[
\times \left\{ A \sum_{(i,j) \in \Gamma(J)} C^{j-m_2} |\xi|^{-j+(p+i+q)} + \sum_{(i,j) \notin \Gamma(J)} A C^{j-m_2} |\xi|^{-j+(p+i+q)} \right\}
\]
\[ C^m GOURSAT PROBLEM  \\
\leq C^{s+m_2} |\xi|^{pr+qs+j} \{(Ah_0/C^p) + (AN_1 C^{m_1}/|\xi|)\} , \]

where \( N_1 \) is the number elements of \( \Omega \).

Here we can suppose \( C > 1 \). First we take \( C \) large enough to have \((Ah_0/C^p) < 1/2\). Next we take \(|\xi|\) large enough to have \((AN_1 C^{m_1}/|\xi|) < 1/2\). Then we have

\[ |u_{r+m_1,s+m_2}| < C^{s+m_2} |\xi|^{pr(r+m_1)+qs(s+m_2)} . \]

q.e.d.

By Lemma 2.3 we have

**Proposition 2.4.** The formal solution (2.8) of the Goursat problem (2.2)–(2.9) converges uniformly.

Then (2.8) is a true solution of (2.2)–(2.9). By Lemma 2.3, we can put

\[ u_{r,s} = \sum_{k=0}^{pr+qs} u_{r,s}^{(k)} \xi^{pr+qs-k} . \]

The following holds:

**Lemma 2.5.** For \( r \geq m_1 \) and \( s \geq \rho \), the leading coefficient of \( u_{r,s} \) is \( \alpha_1^s \), namely

\[ u_{r,s}^{(0)} = \alpha_1^s \quad \text{for} \quad r \geq m_1, s \geq \rho . \]

**Proof.** We prove this by induction. Putting (2.15) in (2.10) and comparing the coefficient of \( \xi^{pr+qs+j} \), we have

\[ u_{r+m_1,s+m_2}^{(0)} = \sum_{(i,j) \in \Gamma(J)} a_{ij} u_{r+i,s+j}^{(0)} . \]

Goursat data (2.9) satisfy (2.16). Suppose that the every term of right-hand side of (2.17) satisfies (2.16). We have

\[ u_{r+m_1,s+m_2}^{(0)} = \sum_{(i,j) \in \Gamma(J)} a_{ij} \alpha_1^{r+j} \]

\[ = \alpha_1^s \sum_{(i,j) \in \Gamma(J)} a_{0} a_{i} \alpha_1^{r_1} \alpha_1^{m_2} = \alpha_1^{r+m_2} . \]

Here we used the fact that \( \alpha_1 \) is a root of \( L_j(1, \xi) = 0 \). q.e.d.

By (2.8) we have

\[ \alpha_1^{m_1} u(0, x) = \sum_{s \geq 0} (u_{m_1,s}/s!) x^s . \]

So we must estimate \( u_{m_1,s} \). The lower order terms of \( u_{m_1,s} \) on \( \xi \) have the following estimate.
Proposition 2.6. For $0 \leq s < |\xi|^\mu$, $0 < \mu < 1/p'$, there exists constant $C$ and $\sigma > 0$ such that

\begin{equation}
|u_{m_1,s} - \alpha_1^s \xi^{pm_1 + qs}| \leq C |\alpha_1|^s |\xi|^{pm_1 + qs - \sigma}.
\end{equation}

Concerning $p$, refer to (2.7) and we set $p' = \max_{1 \leq h \leq N} n(h)$.

This proposition is the most important estimate to prove Theorem 2.1. The proof of proposition 2.6 is complicated. So we prove this later in § 3.

Now let us prove theorem 2.1. We set

\begin{equation}
x_\xi = 1/|\xi|^\mu, \quad \text{where} \quad 0 < q - \mu < \varepsilon < q.
\end{equation}

The following holds:

\begin{equation}
\left| \sum_{s \geq 0} u_{m_1,s} x_\xi^{s}/s! - \xi^{pm_1} \sum_{s \geq 0} (\alpha_1 \xi q)^s x_\xi^{s}/s! \right|
\leq \sum_{s \geq |\xi|^\mu} |u_{m_1,s} - \xi^{pm_1}(\alpha_1 \xi q)^s| x_\xi^{s}/s! + \sum_{s > |\xi|^\mu} |u_{m_1,s} x_\xi^{s}/s!|
+ |\xi|^{pm_1} \sum_{s > |\xi|^\mu} |\alpha_1 \xi q|^s x_\xi^{s}/s!.
\end{equation}

Let the argument of $\alpha_1$ be $\theta$ and let the argument of $\xi$ be $-\theta/q$, that is

\begin{equation}
\alpha_1 = |\alpha_1| \exp(\theta i),
\end{equation}

and

\begin{equation}
\xi = |\xi| \exp(-\theta i/q).
\end{equation}

Then we have

\begin{equation}
\alpha_1 \xi q = |\alpha_1| \exp(\theta i) \exp(-\theta i) = |\alpha_1| |\xi|^\mu.
\end{equation}

Using Proposition 2.6, (2.22) becomes (2.26):

\begin{equation}
|\partial_t^{\mu} u(0, x_\xi) - \xi^{pm_1} \exp(|\alpha_1| |\xi|^{q-\tau})|
\leq C |\xi|^{pm_1 + q - \sigma} \exp(|\alpha_1| |\xi|^{q-\tau}) + R_1 + R_2,
\end{equation}

where $R_1$ and $R_2$ are the second and third terms of the right-hand side of (2.22) respectively. By Lemma 2.3, we have

\begin{equation}
R_1 = \sum_{s > |\xi|^\mu} |u_{m_1,s} x_\xi^{s}/s! \leq \sum_{s > |\xi|^\mu} C^s |\xi|^{pm_1 + q + \sigma} |\xi|^{-\tau}/s!.
\end{equation}

Here we recall Stirling's formula:

\begin{equation}
s! = \sqrt{2\pi s^{s+(1/2)}} \exp(-s + (\theta'/12s), \quad 0 \leq \theta' \leq 1.
\end{equation}
By (2.27), (2.28) and (2.21), we have the following for large \(|\xi|\):

\[
R_1 \leq \sum_{s \geq |\xi|} C^s |\xi|^{m_1 + p + (q-s)\varepsilon} s^s \leq \sum_{s \geq |\xi|} (\varepsilon C)^s |\xi|^{m_1 + p + (q-s)\varepsilon} s^s < \text{constant}/|\xi| \quad \text{(for large } |\xi| \text{)}.
\]

By the same way, we have

\[
R_2 \leq \text{constant}/|\xi|.
\]

Dividing (2.26) by \(|\xi|^{m_1} \exp(|\xi|^{q-s})\) we have

\[
|\xi|^{m_1} u(0, x_2) - \exp \left( -\frac{m_1 p\theta}{q} i \right) \leq C^s |\xi|^{s-1}
\]

for large \(|\xi|\), where \(\sigma > 0, C\); positive constant.

Now, we recall Goursat data (2.9'). Because of \(\xi^p = |\xi|^p \exp(-p\theta/q)\), if the real part of \(\exp(-p\theta/q) \geq 0 \ (or \ \leq 0)\) we consider the Goursat problem for \(t \leq 0 \ (or \ t \geq 0)\), then Goursat data have at most polynomial order of \(|\xi|\). So if we assume the \(\delta\)-wellposedness then the solution \(u(t, x)\) has at most polynomial order of \(|\xi|\). When \(|\xi| \to \infty\), (2.31) becomes \(1 \leq 0\) because of \(q - \varepsilon > 0\). This is a contradiction.

\[\text{q.e.d.}\]

§ 3. The Proof of Proposition 2.6

Recall (2.9), (2.10), (2.15) and (2.16):

\[
\begin{cases}
 u_{r,s} = 0, & 0 \leq s \leq \rho - 1, \\
 u_{r,s} = \alpha_s \xi^r \varphi^{r+qs}, & \rho \leq s \leq m_2 - 1, \quad r \geq 0, \\
 u_{r,s} = 0, & s \geq m_2, \quad 0 \leq r \leq m_2 - 1,
\end{cases}
\]

\[
u_{r,s} = \sum_{(i,j) \in \Omega} a_{ij} u_{r+i,s+j},
\]

\[
u_{r,s} = \sum_{k=0}^{pr+qs} t_r^{(k)} \varphi^{pr+qs-k},
\]

\[
u^{(0)}_{r,s} = \alpha_s \xi^r \quad \text{for} \quad r \geq m_1, \quad s \geq \rho.
\]

Because of (2.4) and (2.5), we rewrite (2.10):

\[
u_{r,s} = \sum_{d=0}^{pr+qs} \sum_{\frac{r+d+q}{d} \geq J} a_{ij} \nu_{r+i,s+j}.
\]

Putting (2.15) in (3.1) we have

\[
u_{r,s}^{(k)} = \sum_{d=0}^{pr+qs} \sum_{\frac{r+d+q}{d} \geq J} a_{ij} \nu_{r+i,s+j}^{(k)} \varphi^{pr+qs-J-d-k}.
\]
Comparing the coefficient of $\xi^{pr+qs+J-k}$, we have

$$u_{r+m_1,s+m_2}^{(k)} = \sum_{d=0}^{J} \sum_{pi+dj=J-d} a_{ij} u_{r+i,s+j}^{(k-d)}, \quad 0 \leq k \leq pr + qs + J.$$  

At first we notice that

**Lemma 3.1.** When $r \geq m_1(k + 1)$, $u_{r,s}^{(k)}$ is independent of $r$.

We prove this lemma by induction with respect to $k$ and $s$. Moreover we have the following lemma. This is the key lemma to prove Proposition 2.6.

**Lemma 3.2.** When $r \geq m_1(k + 1)$, $u_{r,s}^{(k)}$ has the following expression:

$$u_{r,s}^{(k)} = \sum_{h=1}^{N} \sum_{v=0}^{m(h,k)} z_h(v,k)(s - \rho)^v \alpha_h^v,$$

where $0^0 = 1$ and $(s - \rho)^v = 0$ for $s - \rho < 0$, $v \geq 0$.

Here $z_h(v,k)$ is independent of $r$ and has the following estimate:

$$|z_h(v,k)| \leq M_1^k M_2^{m(h,k)-v}/v!, \quad (M_1, M_2; \text{constant}).$$

**Concerning the definition of $n(h)$, refer to (2.7).**

The proof of Lemma 3.2 is fairly complicated. So we prove this later in §4.

In the proof of Proposition 2.6, we use the following:

**Corollary 3.3.** When $r \geq m_1(k + 1)$, $u_{r,s}^{(k)}$ has the following estimate:

$$|u_{r,s}^{(k)}| \leq C|\alpha_1|^s M^k s^{\rho^k},$$

where $\rho' = \max_{1 \leq h \leq N} n(h)$ and $C$, $M$ are constants.

**Proof.** By Lemma 3.2 we have

$$|u_{r,s}^{(k)}| \leq \sum_{h=1}^{N} \sum_{v=0}^{m(h,k)} |z_h(v,k)|(s - \rho)^v |\alpha_h|^s$$

$$\leq \sum_{h=1}^{N} \sum_{v=0}^{\rho^k} (M_1^{k} M_2^{\rho^k-v}/v!)(s - \rho)^v |\alpha_1|^s$$

$$\leq M |\alpha_1|^s M_2^{\rho^k} N \sum_{v=0}^{\rho^k} 1/(v! M_2^v)$$

$$= |\alpha_1|^s M_2^{\rho^k} \exp(1/M_2)$$

$$\leq C|\alpha_1|^s M^k s^{\rho^k}. \quad \text{q.e.d.}$$

In order to prove Proposition 2.6 we prepare some lemmas. By (2.10) we have
\[ (3.8) \quad u_{r,s} = \sum_{(i,j) \in \Omega} a_{ij} u_{r+i-m_1, s+j-m_2} \]

\[ = \sum_{i,j_1} a_{ij_1} \sum_{i,j_2} a_{ij_2} u_{r+i_1+i_2-2m_1, s+j_1+j_2-2m_2} \]

\[ = \sum_{i,j_1,i,j_2,\ldots,i,j_K} a_{i,j_1} a_{i,j_2} \cdots a_{i,j_K} u_{r+i_1+i_2+\cdots+i_K-Km_1, s+j_1+j_2+\cdots+j_K-Km_2} . \]

Especially, for \( r = m_1 \), it becomes

\[ (3.9) \quad u_{m_1,n} = \sum_{i,j_1,i,j_2,\ldots,i,j_K} a_{i,j_1} a_{i,j_2} \cdots a_{i,j_K} u_{m_1+i_1+i_2+\cdots+i_K-Km_1, n+j_1+j_2+\cdots+j_K-Km_2} . \]

In the right-hand side of (3.9), we replace \( u_{r,s} \) by lower order ones successively. Finally each term of the right-hand side of (3.9) arrives at the Goursat data.

By Lemma 2.3, \( u_{m_1,n} \) is the polynomial of \( \bar{\xi} \) with degree \( pm_1 + qn \). Let us pay attention to the coefficient of \( \bar{\xi}^{pm_1+qn-\lambda} \). We recall the Goursat data (2.9).

We are going to seek for Goursat data \( u_{r,s} \) which satisfy (3.10):

\[ (3.10) \quad pr + qs = pm_1 + qn - \lambda , \quad \rho \leq s \leq m_2 - 1 . \]

**Lemma 3.4.** The set of integers \( \{qn - \lambda - qk; \rho \leq k \leq \rho + p - 1\} \) is equal to the set of integers \( \{0, 1, 2, \ldots, p - 1\} \) modulo \( p \).

By (2.13) we have \( \rho = m_2 - hp, 1 \leq h \leq \lfloor m_2/p \rfloor \). So there exists one of Goursat data \( u_{r,s} \) which satisfies (3.10). Let it be \( u_{r_0,s_0} \), namely

\[ (3.11) \quad pr_0 + qs_0 = pm_1 + qn - \lambda , \quad \rho \leq s_0 \leq m_2 - 1 , \]

and

\[ (3.12) \quad \begin{cases} m + i_1 + i_2 + \cdots + i_K - Km_1 = r_0 = (pm_1 + qn - \lambda - s_0 q)/p , \\ n + j_1 + j_2 + \cdots + j_K - Km_2 = s_0 . \end{cases} \]

We are going to estimate the coefficient of \( u_{r_0,s_0} \) in the right-hand side of (3.9). For this, we want to estimate \( K \) in (3.12). By (3.12) we have

\[ (3.13) \quad \begin{cases} pm_1 + p(i_1 + i_2 + \cdots + i_K) - pKm_1 = pm_1 + qn - \lambda - s_0 q , \\ qn + q(j_1 + j_2 + \cdots + j_K) - qKm_2 = qs_0 . \end{cases} \]

Therefore the following holds:

\[ (3.14) \quad p(i_1 + i_2 + \cdots + i_K) + q(j_1 + j_2 + \cdots + j_K) - K(pm_1 + qm_2) = -\lambda . \]

Considering \( pm_1 + qm_2 = J \), (3.14) becomes the following:

\[ (3.15) \quad \sum_{k=1}^{K} (pi_k + qj_k - J) = -\lambda . \]

Let the number of \( (i_k, j_k) \in \Gamma' = (\Omega - \Gamma(J)) \) in (3.15) be \( K' \). The number of
\((i_k, j_k) \in \Gamma(J)\) becomes \(K - K'\). According to the definition of \(\Gamma(k)\) it holds that

\[
\begin{cases}
  p_k + q_k - J = 0, & \text{for } (i_k, j_k) \in \Gamma(J), \\
  p_k + q_k - J \leq -1, & \text{for } (i_k, j_k) \in \Gamma'.
\end{cases}
\]

(3.16)

By (3.15) and (3.16), we obtain

(3.17) \[ K' \leq \lambda. \]

Recall the first equation in (3.12). The following holds:

(3.18) \[ \sum_{k=1}^{K} (i_k - m_1) = (q_n - \lambda - s_0 q)/p. \]

According to the definition of \(\Gamma(J)\) it holds that

(3.19) \[ i_k - m_1 \geq q \geq 1 \quad \text{for } (i_k, j_k) \in \Gamma(J). \]

By (3.18), (3.19) and (3.17), it holds that

(3.20) \[ 1(K - K') - m_1 K' \leq (q_n - \lambda - s_0 q)/p, \]

that is,

\[ K \leq (q/p)(n - s_0) + (m_1 + 1) K' - (\lambda/p). \]

Thus we arrive at

\[ K \leq (q/p)n + (m_1 + 1)\lambda. \]

Finally we have

(3.21) \[ K < n + (m_1 + 1)\lambda. \]

Therefore the coefficient of \(u_{m_1,n}\) is estimated by \((AN_1)^{n+(m_1+1)\lambda} \cdot \). Here \(N_1\) is the number of the elements of \(\Omega\) and \(A\) is a constant satisfying \(|a_{ij}| \leq A\) for \((i, j) \in \Omega\). On the other hand, the number of \(u_{r,s}\)'s which satisfy (3.10) is at most \(m_2\). Then we obtain the following lemma.

**Lemma 3.5.** Let

(3.22) \[ u_{m_1,n} = \sum_{\lambda=0}^{pm_1+qn} u_{m_1,n}^{(\lambda)} \cdot e^{pm_1+qn-\lambda}. \]

It holds that

(3.23) \[ |u_{m_1,n}^{(\lambda)}| \leq C_1(C_2)^{n+(m_1+1)\lambda}, \]

where \(C_1, C_2\) are constants independent of \(n\) and \(\lambda\).

This lemma is a very rough estimate. We use this lemma for large \(\lambda\). For small \(\lambda\), we need more delicate estimate. To obtain this, we use Corollary...
3.3. In the right-hand side of (3.9), the Goursat data \( u_{r,s} \) with \( r > m_1(\lambda + 1) + m_2 \) must pass through \( u_{r,s} \) with \( m_1(\lambda + 1) \leq r \leq m_1(\lambda + 1) + m_2 \). Let us notice \( u_{r,s} \) with \( m_1(\lambda + 1) \leq r \leq m_1(\lambda + 1) + m_2 \). Let one of them be \( u_{r_0,s_0} \):

\[
(3.24) \quad m_1(\lambda + 1) \leq r_0 \leq m_1(\lambda + 1) + m_2 ,
\]

\[
(3.25) \quad pm_1 + qn - \lambda \leq pr_0 + qs_0 \leq pm_1 + qn ,
\]

\[
(3.26) \quad \begin{cases} m_1 + i_1 + i_2 + \cdots + i_k - Km_1 = r_0 , \\ n + j_1 + j_2 + \cdots + j_k - Km_2 = s_0 . \end{cases}
\]

Let us estimate \( K \) which satisfies (3.24), (3.25) and (3.26). By (3.26) we have

\[
(3.27) \quad \sum_{k=1}^{K} (pi_k + qj_k - J) = pr_0 + qs_0 - (pm_1 + qn) .
\]

Let the number of \( (i_k, j_k) \in \Gamma'' \) in (3.27) be \( K'' \). Then the number of \( (i_k, j_k) \in \Gamma(J) \) becomes \( K - K'' \). By (3.16), (3.25) and (3.27) we have

\[
(3.28) \quad K'' \leq \lambda .
\]

The first equation of (3.26) is

\[
(3.29) \quad \sum_{k=1}^{K} (i_k - m_1) = r_0 - m_1 .
\]

By (3.19) and (3.29), it holds that

\[
(3.30) \quad (K - K'') - m_1 K'' \leq r_0 - m_1 .
\]

Therefore we have

\[
(3.31) \quad K \leq (m_1 + 1)K'' + r_0 - m_1 \leq (m_1 + 1)\lambda + m_1(\lambda + 1) + m_2 - m_1 \\
= \lambda(2m_1 + 1) + m_2 .
\]

Then the coefficient of \( u_{r_0,s_0} \) in the right-hand side of (3.9) is estimated by

\[
(3.32) \quad (AN_1)^{\lambda(2m_1 + 1) + m_2} .
\]

\( u_{r_0,s_0} \) is the polynomial of \( \xi \) with degree \( pr_0 + qs_0 \). We want to estimate the coefficient of degree \( pm_1 + qn - \lambda \) of \( u_{r_0,s_0} \). Putting

\[
(3.33) \quad pr_0 + qs_0 - \lambda' = pm_1 + qn - \lambda ,
\]

\( \lambda' \) satisfies

\[
\lambda' = \lambda + pr_0 + qs_0 - (pm_1 + qn)
\]

and

\[
(3.34) \quad 0 \leq \lambda' \leq \lambda .
\]
According to Corollary 3.3, we have
\[ |u_{o_{1}, n}^{(i)}| \leq C|\alpha_{1}|^{n}M^{\lambda_{n}}s_{0}^{\lambda_{n}} \leq C|\alpha_{1}|^{n}M^{\lambda}n^{\rho_{n}}. \]

By (3.10) and \( r > m_{1}(\lambda + 1) + m_{2} \), it holds that
\[ \lambda < (q_{n} - q_{s} - p_{m_{2}})/ (p_{m_{1}} + 1), \quad \rho \leq s \leq m_{2} - 1. \]

After all we have the following:

**Lemma 3.6.** When
\[ \lambda < (q_{n} - q_{m_{2}} - p_{m_{2}} + q)/(p_{m_{1}} + 1) \]

it holds that
\[ |u_{m_{1}, n}^{(i)}| \leq (AN_{1})^{(2m_{1} + 1)+m_{2}}C|\alpha_{1}|^{n}M^{\lambda}n^{\rho_{n}} \lambda_{m_{2}} + 1) \]
\[ \leq C_{1}C_{2}^{\lambda}|\alpha_{1}|^{n^{\rho_{n}} \lambda}, \quad \text{where } C_{1}, C_{2} \text{ are constants}. \]

Finally let us prove Proposition 2.6. Putting
\[ q/(p_{m_{1}} + 1) = \omega_{1} \quad \text{and} \quad (q_{m_{2}} + p_{m_{2}} - q)/(p_{m_{1}} + 1) = \omega_{2}, \]
\[ (3.37) \]

becomes (3.40):
\[ \lambda < \omega_{1} n - \omega_{2}. \]

Recall
\[ u_{m_{1}, n} = \sum_{\lambda \geq 1} u_{m_{1}, n_{\xi}}^{(i)} \xi^{pm_{1} + qn - \lambda}, \]
and let
\[ Q = \sum_{\lambda \geq 1} u_{m_{1}, n_{\xi}}^{(i)} \xi^{-\lambda}. \]

By Lemma 2.5, (2.20) is equivalent to (3.42).
\[ |Q| < C|\alpha_{1}|^{n}|\xi|^{-\sigma}, \quad \sigma > 0, \quad \text{for } \rho \leq n \leq |\xi|. \]

Hereafter we are going to prove (3.42). We decompose \( Q \) as follows:
\[ Q = \sum_{1 \leq \lambda < \omega_{1} n - \omega_{2}} u_{m_{1}, n_{\xi}}^{(i)} \xi^{-\lambda} + \sum_{\lambda \geq \omega_{1} n - \omega_{2}} u_{m_{1}, n_{\xi}}^{(i)} \xi^{-\lambda} \]
\[ = Q_{1} + Q_{2}. \]

First we consider \( Q_{1} \). According to Lemma 3.6 and assumption of Prop. 2.6 we have
\[ |Q_{1}| \leq \sum_{1 \leq \lambda < \omega_{1} n - \omega_{2}} |u_{m_{1}, n_{\xi}}^{(i)}| \xi^{-\lambda} \]
\[ \leq C_{1} \sum_{1 \leq \lambda < \omega_{1} n - \omega_{2}} C_{2}^{\lambda}|\alpha_{1}|^{n^{\rho_{n}} \lambda} |\xi|^{-\lambda} \]
\[ \leq C_1 |x_1|^n \sum_{1 \leq \lambda} C_2^2 (|\xi|^n)^{\mu\rho'} |\xi|^{-\lambda} \]
\[ = C_1 |x_1|^n \sum_{1 \leq \lambda} (C_2 |\xi|^{\mu\rho'-1})^\lambda \]
\[ \leq C_3 |x_1|^n |\xi|^{\mu\rho'-1} \quad \text{for } |\xi| \text{ large,} \]

where \( C_3 \) is constant and \( \mu\rho' < 1 \).

Next we consider \( Q_2 \). By Lemma 3.5 we have

\[ |Q_2| \leq \sum_{\lambda \geq \omega_1 n - \omega_2} |u_{m_1, n}^{(\lambda)}| |\xi|^{-\lambda} \]
\[ \leq C_1 \sum_{\lambda \geq \omega_1 n - \omega_2} (C_2)^{n+(m_1+1)\lambda} |\xi|^{-\lambda} \]
\[ = C_1 |x_1|^n \sum_{\lambda \geq \omega_1 n - \omega_2} (C_2/|x_1|^m)\lambda C_2^{(m_1+1)\lambda} |\xi|^{-\lambda}, \]
\[ \lambda \geq \omega_1 n - \omega_2, \quad \text{that is, } n \leq (\lambda/\omega_1) + (\omega_2/\omega_1), \]

and

\[ |Q_2| \leq C_1 \sum_{\lambda \geq 1} (C_2)^{\lambda+\omega_2+(m_1+1)\lambda} |\xi|^{-\lambda} \]
\[ \leq \text{constant } \sum_{\lambda \geq 1} (C_2^{(m_1+2)}/|\xi|)^\lambda \leq \text{constant } |\xi|^{-1} \quad \text{for large } |\xi|. \]

Thus we have proved (3.42).

\section*{§ 4. The Proof of Lemma 3.2}

Putting (3.4) into (3.3), we have

\[ \sum_{h=1}^{N} \sum_{v=0}^{n(h)k} z_h(v, k)(s + m_2 - \rho)\alpha_h^{n}m_2 \]
\[ = \sum_{d=0}^{J} \sum_{i, j \in T(J-d)} a_{ij} \sum_{h=1}^{N} \sum_{v=0}^{n(h)(k-d)} z_h(v, k-d)(s + j - \rho)\alpha_h^{n+j}. \]

Using the following equality;

\[ (s + m_2 - \rho)^r = [(s - \rho) + m_2]_v = \sum_{n=0}^{r} \begin{pmatrix} v \\ n \end{pmatrix} (s - \rho)^m m_2^{v-n}, \]

we rewrite (4.1) and obtain (4.1');

\[ \sum_{h=1}^{N} \sum_{v=0}^{n(h)k} \sum_{n=0}^{v} z_h(v, k)\begin{pmatrix} v \\ n \end{pmatrix} (s - \rho)^m m_2^{v-n}\alpha_h^{n+m_2} \]
\[ = \sum_{d=0}^{J} \sum_{i, j \in T(J-d)} a_{ij} \sum_{h=1}^{N} \sum_{v=0}^{n(h)(k-d)} \sum_{n=0}^{v} z_h(v, k-d)\begin{pmatrix} v \\ n \end{pmatrix} (s - \rho)^m j^{v-n}\alpha_h^{n+j}. \]
We take the coefficients of \((s - \rho)^n \alpha^k_h\) of both sides of (4.1') equal. We have

\[
(4.2) \quad \sum_{v=n}^{kn(h)} z_h(v, k) \binom{v}{n} m_2^{-n} \alpha^{m_2}_h - \sum_{(i,j) \in I(J)} \sum_{v=n}^{kn(h)} a_{ij} z_h(v, k) \binom{v}{n} j^{v-n} \alpha^i_h = \sum_{d=1}^{j} \sum_{(i,j) \in I(J-d)} a_{ij} \sum_{v=n}^{k-dn(h)} z_h(v, k - d) \binom{v}{n} j^{v-n} \alpha^i_h,
\]

where \(0 \leq n \leq kn(h), \ 1 \leq h \leq N\).

When \((k - 1)n(h) < n \leq kn(h)\), the right-hand side of (4.2) vanishes and the left-hand side is the following:

\[
(4.3) \quad \sum_{v=n}^{kn(h)} z_h(v, k) \binom{v}{n} \left\{ \frac{m_2^{-n} \alpha^{m_2}_h}{n} - \sum_{(i,j) \in I(J)} a_{ij} j^{v-n} \alpha^i_h \right\}.
\]

According to the equality:

\[
(4.4) \quad y^t = \sum_{g=1}^{t} b_g y(y - 1)(y - 2) \ldots (y - g + 1)
\]

it holds that

\[
(4.5) \quad m_2 \alpha^{m_2}_h - \sum_{(i,j) \in I(J)} a_{ij} j \alpha^i_h = \sum_{g=1}^{t} b_g \left\{ m_2(m_2 - 1) \ldots (m_2 - g + 1) \alpha^{m_2}_h - \sum_{(i,j) \in I(J)} a_{ij} j(j - 1) \ldots (j - g + 1) \alpha^i_h \right\} = \sum_{g=1}^{t} b_g \alpha^g_h \left( d/d\zeta \right)^g L_J(1, \alpha_h).
\]

As \(\alpha_h\) is the \(n(h)\)-tuple root of \(L_J(1, \zeta) = 0\), the following holds:

\[
(4.6) \quad \left( d/d\zeta \right)^g L_J(1, \alpha_h) = 0, \quad 0 \leq g \leq n(h) - 1.
\]

By (4.5) and (4.6), we have (4.3) = 0. Therefore (4.2) holds for \((k - 1)n(h) < n \leq kn(h)\) and for any \(\{z_h(v, k): (k - 1)n(h) < v \leq kn(h)\}\). Next we consider (4.2) for \(n \leq (k - 1)n(h)\). By (4.6) we can rewrite (4.2) as follows:

\[
(4.7) \quad z_h(n + n(h), k) \binom{n + n(h)}{k} \alpha^{n(h)}_h \left( d/d\zeta \right)^{n(h)} L_J(1, \alpha_h)
\]

\[
= - \sum_{v=n+n(h)+1}^{kn(h)} z_h(v, k) \binom{v}{n} \left\{ \frac{m_2^{-n} \alpha^{m_2}_h}{n} - \sum_{(i,j) \in I(J)} a_{ij} j^{v-n} \alpha^i_h \right\}
\]

\[
+ \sum_{d=1}^{j} \sum_{(i,j) \in I(J-d)} a_{ij} \sum_{v=n}^{k-dn(h)} z_h(v, k - d) \binom{v}{n} j^{v-n} \alpha^i_h,
\]

\(0 \leq n \leq (k - 1)n(h)\).
As \( x_h \) is the \( n(h) \)-tuple root of \( L_J(1, \zeta) = 0 \), we have

\[
(4.8) \quad \frac{d}{d\zeta} y^{(h)} L_J(1, x_h) \neq 0.
\]

Therefore if we assume that the right-hand side of (4.7) is given then \( z_h(v, k) \) \((n(h) \leq v \leq kn(h))\) is determined by (4.7). However \( \{z_h(v, k); 0 \leq v \leq n(h) - 1\} \) are not determined by (4.7). These are determined by the following way.

Recall (3.4):

\[
(3.4) \quad u_{r,s}^{(k)} = \sum_{h=1}^{N} \sum_{v=0}^{n(h)k} z_h(v, k)(s - \rho)^v x_h^s.
\]

By (2.9) and (2.15), we have

\[
(4.9) \quad u_{r,s}^{(k)} = 0, \quad (k \geq 1, 0 \leq s \leq m_2 - 1).
\]

In the case \( 0 \leq s \leq \rho - 1 \), it holds that \( s - \rho < 0 \). Then the right-hand side of (3.4) vanishes. We consider the case \( \rho \leq s \leq m_2 - 1 \), and set the right-hand side of (3.4) 0. We have

\[
(4.10) \quad \sum_{h=1}^{N} \sum_{v=0}^{n(h)k} z_h(v, k)(s - \rho)^v x_h^s = 0, \quad \rho \leq s \leq m_2 - 1.
\]

More precisely, (4.10) becomes

\[
(4.10') \quad \begin{cases}
\sum_{h=1}^{N} z_h(0, k) = 0, \quad (s = \rho), \\
\sum_{h=1}^{N} \sum_{v=0}^{n(h)-1} z_h(v, k)x_h = - \sum_{h=1}^{N} \sum_{v=n(h)}^{n(h)k} z_h(v, k)x_h, \quad (s = \rho + 1), \\
\sum_{h=1}^{N} \sum_{v=0}^{n(h)-1} z_h(v, k)2^v x_h^2 = - \sum_{h=1}^{N} \sum_{v=n(h)}^{n(h)k} z_h(v, k)2^v x_h^2, \quad (s = \rho + 2), \\
\vdots \\
\sum_{h=1}^{N} \sum_{v=0}^{n(h)-1} z_h(v, k)(m_2' - 1)^v x_h^{m_2'-1} = - \sum_{h=1}^{N} \sum_{v=n(h)}^{n(h)k} z_h(v, k)(m_2' - 1)^v x_h^{m_2'-1}, \quad (s = m_2 - 1 = m_2' + \rho - 1 \text{ and } m_2' = m_2 - \rho).
\end{cases}
\]

We consider that (4.10') is a system of equations and its unknowns are \( \{z_h(v, k); 0 \leq v \leq n(h) - 1, 1 \leq h \leq N\} \). Let the coefficient matrix of (4.10') be \( A \). We have

\[
(4.11) \quad |\det A| = \left| \prod_{i<j} (x_i - x_j)^{n(i)n(j)} \prod_{i=1}^{N} x_i^{n(i)(n(i)-1)} \right| \neq 0.
\]

Therefore if we assume that the right-hand side of (4.10') is given, \( \{z_h(v, k); 0 \leq v \leq n(h) - 1, 1 \leq h \leq N\} \) are determined uniquely.
Now we determine \( \{z_h(v, k)\} \) in the following way. First, by (2.16) and (3.4) we have

\[
(4.12) \quad z_1(0, 0) = 1, \quad z_h(0, 0) = 0 \quad 1 < h \leq N.
\]

Assuming that \( \{z_h(v, k'); k' < k\} \) are already determined, we determine \( z_h(kn(h), k) \) by (4.7). Next, we determine \( z_h(kn(h) - 1, k) \) by (4.7), and so on. At last we determine \( z_h(n(h), k) \) by (4.7). Finally we determine \( \{z_h(v, k); 0 \leq v < n(h), 1 \leq h \leq N\} \) by solving the system of equations (4.10'). We can prove (3.5) by induction. Thus we complete the proof of Lemma 3.2.

**Remark 4.1.** The coefficients \( \{z_h(v, k); 0 \leq v \leq kn(h), 1 \leq h \leq N\} \) which satisfy (4.2) and (4.12) are determined uniquely. (4.2) is a sufficient condition for (4.1). However (4.2) is not a necessary condition for (4.1), therefore the expression in the right-hand side of (3.4) is not unique.

§5. General Case

At last we consider the general case where the operator \( \mathcal{L} \) includes \( \partial_y \);

\[
(5.1) \quad \mathcal{L}(\partial_t, \partial_x, \partial_y)u(t, x, y) = 0.
\]

Putting

\[
(5.2) \quad u(t, x, y) = \exp(\imath \eta y)v(t, x), \quad \eta \in \mathbb{R}^n,
\]

(5.1) becomes (5.3):

\[
(5.3) \quad \mathcal{L}(\partial_t, \partial_x, \imath \eta)v(t, x) = 0.
\]

If \( \mathcal{L}(\partial_t, \partial_x, \partial_y) \) is not N-type, \( \mathcal{L}(\partial_t, \partial_x, \imath \eta) \) is not N-type with respect to some \( \eta \), too. So we can reduce the general case to the case of §2.

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References