Actions on Invariant Spheres around Isolated Fixed Points of Actions of Cyclic Groups

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§ 1. Introduction

Fix a prime number \( p \) and let \( \mathbb{Z}_p \) be a cyclic group of order \( p \). We consider a pair \((M, \phi)\) consisting of a compact simply connected almost complex manifold \( M \) without boundary and a smooth \( \mathbb{Z}_p \)-action \( \phi: \mathbb{Z}_p \times M \rightarrow M \) preserving the almost complex structure of \( M \). We suppose that \( M \) is given an invariant Riemannian metric. If \( a \in M \) is an isolated fixed point, then the induced action of \( \mathbb{Z}_p \) on the tangent space at \( a \) gives a complex \( \mathbb{Z}_p \)-module \( V_a \) which has no trivial irreducible factor. Let \( \xi: E\mathbb{Z}_p \rightarrow \mathbb{BZ}_p \) be a universal principal \( \mathbb{Z}_p \)-bundle and let \( \xi(V_a): E\mathbb{Z}_p \times_{\mathbb{Z}_p} V_a \rightarrow \mathbb{BZ}_p \) be the \( V_a \)-bundle associated with \( \xi \). If \( a \) and \( b \) are isolated fixed points, we compare the cobordism Euler classes \( e(\xi(V_a)) \) and \( e(\xi(V_b)) \) which belong to the complex cobordism group \( \text{MU}^*(\mathbb{BZ}_p) \) of the classifying space \( \mathbb{BZ}_p \) of \( \mathbb{Z}_p \). Let \( F_U \) be the universal formal group law over \( \text{MU}^* \), and write

\[ x \circ_f y = F_U(x, y). \]

For a positive integer \( n \), \( [n]_f(x) \) is inductively defined by

\[ [1]_f(x) = x \]

and

\[ [n]_f(x) = [n-1]_f(x) +_f x. \]

It is known that the cobordism ring \( \text{MU}^*(\mathbb{BZ}_p) \) is formal power series algebra \( \text{MU}^*[[x]] \) over \( \text{MU}^* \) modulo an ideal generated by \([p]_f(x)\)

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Let us write

\[ [p]_F(x) = px + a_1^{(p)} x^2 + a_2^{(p)} x^3 + \cdots, \]

where \( a_i^{(p)} \in MU^{-i} \), and

\[ \langle p \rangle_F(x) = p + a_1^{(p)} x + a_2^{(p)} x^2 + \cdots. \]

Let \( S \) denote the multiplicative set in \( MU^*(BZ_p) \) consisting of cobordism Euler classes \( e(\xi(V)) \), \( V \) the nontrivial complex \( Z_p \)-module, and let \( \lambda: MU^*(BZ_p) \to S^{-1}MU^*(BZ_p) \) be the canonical map [9]. In this paper we show the following

**Theorem A.** Assume that \( H^i(BZ_p; \{\pi_i(M)\}) \cong 0 \) for \( 1 \leq i \leq 2n - 1 \) (cf. [4, p. 355]), and \( \lambda(\alpha) = e(\xi(V_a))/e(\xi(V_b)) \). Then for any Landweber-Novikov operation \( S_\alpha^w \), \( \omega \neq (0) \) [14], [17], \( S_\alpha^w(\alpha) \) belongs to an ideal generated by \( x^e \) and \( \langle p \rangle_F(x) \) in \( MU^*(BZ_p) \), where \( x = e(\xi(L)) \) and \( L \) is the canonical one dimensional complex \( Z_p \)-module with an action of \( Z_p \) given by multiplication by \( \rho = \exp(2\pi i/p) \) on \( C^1 \).

The action of \( Z_p \) on \( M \) induces a natural action on a unit sphere \( S(V_a) \) in a tangent space \( V_a \) at an isolated fixed point \( a \) which is equivalent to the action of \( Z_p \) on a sphere around the fixed point. The action \( \phi_a: Z_p \times S(V_a) \to S(V_a) \) determines a weakly complex bordism class \([S(V_a), \phi_a] \) of the bordism group \( MU_*(Z_p) \) of fixed point free \( Z_p \) actions preserving a weakly complex structure, which is generated as an \( MU_* \)-module by the set of \( Z_p \)-manifolds \( \{[S^{2n+1}, \widehat{\phi}]\} \), where the action \( \widehat{\phi} \) of \( Z_p \) on a sphere \( S^{2n+1} \subset C^{n+1} \) is defined by \( \widehat{\phi}(g, z) = g \cdot z \), \( g \) a generator of \( Z_p \) [6], [11]. Kasparov in [13] showed that the weakly complex bordism class \([S(V_a), \phi_a] \) is computable. By making use the Kasparov theorem and Theorem A, we obtain the following

**Theorem B.** Assume that \( H^i(BZ_p; \{\pi_i(M)\}) \cong 0 \) for \( 1 \leq i \leq 2n - 1 \).

If \( V_a = L^1 \oplus \cdots \oplus L^k \) and \( V_b = L^m_1 \oplus \cdots \oplus L^m_k \), then

\[
\ell_1 \cdots \ell_k [S(V_a), \phi_a] - m_1 \cdots m_k [S(V_b), \phi_b] = \bar{\beta}_1 [S^{2n-3}, \widehat{\phi}] + \bar{\beta}_2 [S^{2n-5}, \widehat{\phi}] + \cdots + \bar{\beta}_{n-1} [S^1, \widehat{\phi}],
\]

where \( \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{n-1} \) belong to an ideal generated by \( p, a_1^{(p)}, a_2^{(p)}, \ldots, \).
In Section 2 we investigate $S^1$-actions on a product space $S^{2m+1} \times S^{2n+1}$ of spheres and equivariant maps between the $S^1$-spaces. In Section 3 the Umkehr homomorphism of some map between the orbit spaces $(S^{2m+1} \times S^{2n+1})/S^1$ is computed to give a slightly different proof of the Kasparov theorem [13] in Section 4. In Section 5 we discuss about relations among cobordism characteristic classes [7] of $\xi(V_0)$ and $\xi(V_B)$ and give a proof of Theorem A. Section 6 is devoted to prove Theorem B. In Section 7 we study the isolated fixed point set of $Z_r$-actions.

Bredon in Section 10 of Chapter VI of [4] compared representations at two fixed points of a smooth action, by using equivariant $K$-theory.

§ 2. On Orbit Spaces of $S^{2m+1} \times S^{2n+1}$ with Respect to $S^1$

We define $\phi(l_0, l_1, \cdots, l_n) : S^1 \times S^{2m+1} \times S^{2n+1} \rightarrow S^{2m+1} \times S^{2n+1}$ by

$$\phi(l_0, l_1, \cdots, l_n)(z, (u_0, u_1, \cdots, u_m), (v_0, v_1, \cdots, v_n)) = ((zu_0, zu_1, \cdots, zu_m), (z^1v_0, z^1v_1, \cdots, z^1v_n)).$$

This is differentiable and the orbit space $(S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n)$ is an orientable smooth manifold. Let $S^1$ act on $S^{2m+1} \times C^1$ by

$$z \cdot ((u_0, \cdots, u_m), v) = ((zu_0, \cdots, zu_m), zv).$$

The orbit space induces a complex line bundle over the complex projective space

$$\pi : S^{2m+1} \times S^1 \times S^{2n+1} / S^1 = CP^m,$$

which is denoted by $\eta$. The total space $S(\eta^1 \oplus \cdots \oplus \eta^r)$ of the sphere bundle associated with $\eta^1 \oplus \cdots \oplus \eta^r$ is diffeomorphic to $(S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n)$. The structure of the integral cohomology group $H^*(S(\eta^1 \oplus \cdots \oplus \eta^r))$ is determined as follows in [18].

**Proposition 2.1.** (1) If $m \leq n$, then $H^j(S(\eta^1 \oplus \cdots \oplus \eta^r)) \cong H^j(CP^m)$ and $H^{j-1}(S(\eta^1 \oplus \cdots \oplus \eta^r)) \cong H^{j-1}(CP^m)$.

(2) If $m > n$, then
The map $f: S^{2m+1} \times S^{2n-1} \to S^{2m+1} \times S^{2n+1}$ defined by

$$f((u_0, \ldots, u_m), (v_0, \ldots, v_n)) = \left( (u_0, \ldots, u_m), \frac{1}{r} (v_0^1, \ldots, v_n^1) \right),$$

induces a map of the orbit spaces

$$\tilde{f}: \left( S^{2m+1} \times S^{2n+1} \right) / \phi (1, \ldots, 1) \to \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n).$$

Denote by $[M]$ the fundamental class of a compact orientable manifold $M$. Then we have

**Proposition 2.2.** $\tilde{f}_* \left[ \left( S^{2m+1} \times S^{2n+1} \right) / \phi (1, \ldots, 1) \right] = l_0 l_1 \cdots l_n [ \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n) ].$

**Proof.** $\tilde{f}$ is a fiber preserving map of sphere bundles $S((n+1)\eta)$ and $S(\eta^1 \oplus \cdots \oplus \eta^n)$, as $\eta^1 \oplus \cdots \oplus \eta^n$ is isomorphic to a bundle of an orbit space of an $S^1$-action on $S^{2m+1} \times C^{n+1}$ defined by

$$z \cdot (u, (v_0, \ldots, v_n)) = (z^1 u, (z^2 v_0, \ldots, z^n v_n)).$$

Let $f_1$ be a fiber preserving map from $(n+1)\eta$ to $\eta^1 \oplus \cdots \oplus \eta^n$ defined by

$$f_1 (u, (v_0, \ldots, v_n)) = (u, (v_0^1, \ldots, v_n^1))$$

which induces a map between the Thom complexes

$$\tilde{f}_1: T(1, \ldots, 1) \to T(l_0, \ldots, l_n),$$

where $T(l_0, \ldots, l_n) = E(l_0, \ldots, l_n) / \{E(l_0, \ldots, l_n) - \text{the zero section}\}$, and $E(l_0, \ldots, l_n)$ is the total space of $\eta^1 \oplus \cdots \oplus \eta^n$. $S(\eta^1 \oplus \cdots \oplus \eta^n)$ and $E(l_0, \ldots, l_n) - \{\text{the zero section}\}$ are of the same homotopy type, and the following diagram is homotopy commutative
$E(1, \ldots, 1) \rightarrow \{ \text{the zero section} \} \xrightarrow{f_1} E(l_0, \ldots, l_n) \rightarrow \{ \text{the zero section} \}$

$S((n+1)\eta) \xrightarrow{\tilde{f}} S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n})$.

Let $t(l_0, \ldots, l_n)$ be the Thom class of $\eta^{l_0} \oplus \cdots \oplus \eta^{l_n}$. Then we have $\tilde{f}_1^* (t(l_0, \ldots, l_n)) = l_0 l_1 \cdots l_n t(1, \ldots, 1)$. Since the coboundary homomorphism $\delta: H^{2m+2n+1}(S(\eta^{l_0} \oplus \cdots \oplus \eta^{l_n})) \rightarrow \tilde{H}^{2m+2n+2}(T(l_0, \ldots, l_n))$ is isomorphic, the fundamental class of $(S^{2m+1} \times S^{2n+1}) / \phi(l_0, \ldots, l_n)$ is the dual class of $\delta^{-1}\{ \pi^*([CP^m]*) \cup t(l_0, \ldots, l_n) \}$, where $\pi: E(l_0, \ldots, l_n) \rightarrow CP^m$ is the projection and $[CP^m]*$ is the dual of $[CP^m]$. Then the assertion follows.

Suppose that $M^m$ and $N^n$ are orientable manifolds. A continuous map $h: M^m \rightarrow N^n$ determines the Umkehr homomorphism

$$D \xrightarrow{h_*} D^{-1}$$

where $D$ is the Poincaré duality.

**Proposition 2.3.** Assume that $g$ is an embedding of $(S^{2m+1} \times S^{2n+1}) / \phi(1, \ldots, 1) \rightarrow S^N$ for a large $N$. Then the Umkehr homomorphism of $F = f \times g: (S^{2m+1} \times S^{2n+1}) / \phi(1, \ldots, 1) \rightarrow (S^{2m+1} \times S^{2n+1}) / \phi(l_0, \ldots, l_n) \times S^y$, $\tilde{f} \times g(x) = (\tilde{f}(x), g(x))$, satisfies

$F_1(\tilde{f}^*(y)) = l_0 \cdots l_n y \times [S^y] *$

where $[S^y]*$ is the dual of $[S^y]$.

**Proof.** The Umkehr homomorphism satisfies $F_1(F^*(a) \cup b) = a \cup F_1(b)$ [8]. We calculate using Proposition 2.2,

$F_1(\tilde{f}^*(y))$

$= (y \times 1) \cup F_1(1)$

$= (y \times 1) \cup D^{-1}(\tilde{f} \times g)_* [(S^{2m+1} \times S^{2n+1}) / \phi(1, \ldots, 1)]$

$= (y \times 1) \cup D^{-1}((l_0 \cdots l_n) [(S^{2m+1} \times S^{2n+1}) / \phi(l_0, \ldots, l_n)] \times 1)$

$= (y \times 1) \cup l_0 \cdots l_n (1 \times [S^y]*).$  Q.E.D.
If $m \leq n$, then we get a short exact sequence

$$0 \to MU^*(CP^n) \xrightarrow{\pi^*} MU^*(S(\eta^1 \oplus \cdots \oplus \eta^{in}))$$

$$\delta \to \tilde{MU}^*(T(l_0, \cdots, l_n)) \to 0$$

and $\delta : MU^{2n+1}(S(\eta^1 \oplus \cdots \oplus \eta^{is})) \to \tilde{MU}^{2n+1}(T(l_0, \cdots, l_n))$ is isomorphic. In this case we may determine the ring structure of $MU^*((S^{2m+1} \times S^{2n+1}) / \phi(l_0, \cdots, l_n))$ (cf. [18]).

**Proposition 2.4.** If $m \leq n$, then $MU^*((S^{2m+1} \times S^{2n+1}) / \phi(l_0, \cdots, l_n))$ is $MU^*[x, y]/(x^{m+1}, y^2)$ where $x$ is the first cobordism Chern class $c_b(\pi^*\gamma)$ and $y$ is an element of $MU^{2n+1}(S(\eta^1 \oplus \cdots \oplus \eta^{is}))$ such that $dy$ is the Thom class of $\eta^1 \oplus \cdots \oplus \eta^{is}$.

**Proof.** $MU^*(S(\eta^1 \oplus \cdots \oplus \eta^{is}))$ is isomorphic to the direct sum of $MU^*(CP^n)$ and $\tilde{MU}^*(T(l_0, \cdots, l_n))$. We have

$$( -1)^{deg a} \delta (\pi^* a \cup b) = \pi^* a \cup \delta b$$

(cf. Chapter 13 of [20]), and $MU^*(S(\eta^1 \oplus \cdots \oplus \eta^{is}))$ is a free $MU^*$-module generated by $\{(\pi^* x)^i, i = 1, 2, \cdots, m\}$ and $\{(\pi^* x)^i \cup y, i = 1, 2, \cdots, m\}$. It follows from Proposition 2.1 that $MU^{2n+1}(S(\eta^1 \oplus \cdots \oplus \eta^{is}))$ is zero. Q.E.D.

§ 3. On the Umkehr Homomorphism of $\tilde{f}$ with the $MU^*$-Orientation

For any set $\omega = (i_1, \cdots, i_r)$ of positive integers, let $\sum t_1^{i_1} \cdots t_r^{i_r}$ be the symmetric polynomial of variable $t_i, 1 \leq i \leq n$ to be the smallest symmetric polynomial containing the monomial $t_1^{i_1} \cdots t_r^{i_r}$, which is expressible uniquely as a polynomial with integral coefficients in the elementary symmetric polynomials $E_1, E_2, \cdots, E_n$ of the $t$'s and write

$$P_\omega(E_1, E_2, \cdots, E_n) = \sum t_1^{i_1} \cdots t_r^{i_r}.$$

For an $n$-dimensional complex vector bundle $\zeta$ over $X$, we define

$$c^H_\omega(\zeta) = P_\omega(c^H_1(\zeta), c^H_2(\zeta), \cdots, c^H_n(\zeta))$$

and $c^H_{(0, \cdots, 0)}(\zeta) = 1$, where $c^H_i(\zeta)$ are the ordinary cohomology Chern classes.
Suppose that \( x \in MU^k(X) \) is represented by
\[
g : S_2^{2N} X^+ \to MU(N).
\]
We define
\[
S^H_\omega(x) = \sigma^{k-2N} \Phi \sigma^H(\gamma_N),
\]
where \( \Phi : H^*(BU(N)) \to \tilde{H}^*(MU(N)) \) is the Thom isomorphism, \( \sigma^{k-2N} \) denotes \((k-2N)\)-fold iterated suspension isomorphism and \( \gamma_N \) is the \( N \)-dimensional universal complex vector bundle. The ring \( H^*_o(MU) \) is isomorphic to \( \mathbb{Z}[t_1, t_2, \ldots] \). Let
\[
\omega = (1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k)
\]
and we define
\[
|\omega| = i_1 + 2i_2 + \cdots + ki_k
\]
and
\[
t^\omega = t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k}.
\]
There exists a multiplicative natural transformation
\[
\beta^H : MU^*(X) \to (H \wedge MU)^*(X) = H^*(X) [[t_1, t_2, \ldots]]
\]
defined by
\[
\beta^H(x) = \sum_\omega s^H_\omega(x) \ t^\omega
\]
which is called Boardman map (cf. [1]). \( \beta^H : MU^*(S^0) \to H^*_o(MU) \) is the Hurewicz homomorphism which is injective [16]. Given \( x \in MU^*(X) \) with \( x = [g : S_2^{2N} X^+ \to MU(N)] \), the Thom homomorphism \( \mu : MU^*(X) \to H^*(X) \) is defined by \( \mu(x) = \sigma^{k-2N} g \Phi(1) = S^H_{(0, \ldots, 0)}(x) \).

**Proposition 3.1.** Suppose that a finite CW-complex \( X \) has no torsion in its integral cohomology, then the Boardman map \( \beta^H \) is injective.

**Proof.** Since the cohomology of \( X \) has no torsion, the Thom homomorphism is surjective. Suppose that \( \gamma_1^{(p)}, \gamma_2^{(p)}, \ldots, \gamma_k^{(p)} \) are the basis of \( H^*(X) \), then we can take \( u_{ij}^{(p)} \) with \( \mu(u_{ij}^{(p)}) = \gamma_i^{(p)} \). The correspondence
An isomorphism \( H^*(X) \otimes MU^* \cong MU^*(X) \) (cf. [5]). We see

\[
\beta_H(\sum b_j^{(n)} u_j^{(n)}) = \sum \beta_H(b_j^{(n)}) \{y_j^{(n)} + \sum_{s>0} S^H_s(u_j^{(n)}) t^s\}.
\]

Let \( \beta_H(\sum b_j^{(n)} u_j^{(n)}) = 0 \), and we can derive inductively that \( \beta_H(b_j^{(n)}) = 0 \) and \( b_j^{(n)} = 0 \). Q.E.D.

For an \( n \)-dimensional complex vector bundle \( \zeta \) over \( X \), consider a formal power series of \( t \)'s:

\[
c^H_t(\zeta) = \sum a c^H_a(\zeta) t^a.
\]

This satisfies the naturality and \( c^H(\zeta_1 \oplus \zeta_2) = c^H(\zeta_1) c^H(\zeta_2) \). Suppose that \( X \) and \( M \) are weakly almost complex manifolds. An embedding \( h: M \to X \) with the normal vector bundle \( \nu \) equipped with the complex structure induces the Umkehr homomorphisms:

\[
h_!: MU^*(M) \to MU^*(X),
\]

and

\[
h^H_!: H^*(M) [[t_1, t_2, \ldots]] \to H^*(X) [[t_1, t_2, \ldots]].
\]

Now we recall the following (cf. [19])

**Theorem 3.2.** \( \beta_H(h_!(1)) = h^H_!(c^H_t(\nu)) \).

**Proof.** A composition of a collapsing map \( c \) of the Thom construction and a classifying map \( g_\nu \) for \( \nu \)

\[
\bar{g}_\nu: X \xrightarrow{c} T(\nu) \xrightarrow{g_\nu} MU(k)
\]

represents \( h_!(1) \in MU^*(X) \). By making use of the following commutative diagram:

\[
\begin{array}{ccc}
D & \cong & 0 \\
\downarrow h_! & & \downarrow \cong \\
H^*(X) & \xleftarrow{\cong} & H^*(X) \xleftarrow{c^*} H^*(T(\nu)) \\
\end{array}
\]

\[
\begin{array}{ccc}
D & \cong & 0 \\
\downarrow h_! & & \downarrow \cong \\
H^*(M) & \xleftarrow{\cong} & H^*(M)
\end{array}
\]
we calculate
\[ \beta_H(h_i(1)) = \sum_n S_n^R[g_c] t^n \]
\[ = \sum_n c^* \phi H c^H(\nu) t^n \]
\[ = h_i^H \left( \sum_n c^H(\nu) t^n \right). \]
Q.E.D.

\( MU^*(BU(1)) \) is isomorphic to \( MU^*[[x_{MU}]] \), \( x_{MU} = c^V(\nu) \). The first cobordism Chern class \( c_V(\nu) \) of the \( k \)-fold tensor product of \( \nu \) is described as
\[ c_V(\nu^k) = [k] \cdot (x_{MU}) \]
\[ = kx_{MU} + a^{(k)}_1 x_{MU}^2 + \cdots. \]

Let \( g: X \to BU(1) \) be a classifying map for a complex line bundle \( \zeta \) over \( X \). We see
\[ \langle k \rangle \cdot (c_V(\zeta)) = g^* \{ k + a^{(k)}_1 x_{MU} + a^{(k)}_2 x_{MU}^2 + \cdots \}. \]

The map \( \tilde{f}: (S^{2m+1} \times S^{2m+1})/\phi(1, \cdots, 1) \to (S^{2m+1} \times S^{2m+1})/\phi(l_0, \cdots, l_n) \) defined by
\[ \tilde{f}([u_0, \cdots, u_m], [v_0, \cdots, v_n]) \]
\[ = \left[ (u_0, \cdots, u_m), \frac{1}{r} (v_0^1, \cdots, v_n^m) \right], \]
\[ r = \sqrt{|u_0|^{2m} + \cdots + |u_n|^{2m}}, \]
and an embedding \( h: (S^{2m+1} \times S^{2m+1})/\phi(1, \cdots, 1) \to S^{2k} \) for a large \( N \) determine a bordism class \([ (S^{2m+1} \times S^{2m+1})/\phi(1, \cdots, 1), \tilde{f} \times h ] \) of \( MU_4((S^{2m+1} \times S^{2m+1})/\phi(l_0, \cdots, l_n) \times S^{2k}) \). The projection \( \pi: (S^{2m+1} \times S^{2m+1})/\phi(l_0, \cdots, l_n) \to CP^m \) is defined by \( \pi[u, v] = [u] \). Then we have

**Theorem 3.3.** Suppose that \( m \leq n \). Then it follows that
\[ [(S^{2m+1} \times S^{2m+1})/\phi(1, \cdots, 1), \tilde{f} \times h] \]
\[ = D_{MU}(\langle \nu \rangle \cdot (c_V(\eta)) \langle \nu \rangle \cdot (c_V(\eta)) \cdots \langle \nu \rangle \cdot (c_V(\eta))) \times [P \subset S^{2k}] \]
where \( P = \{ \text{a point} \} \) and \( D_{MU} \) is the Atiyah-Poincare isomorphism \([3]\).
Proof. If \( m \leq n \), then \( H^* \left( \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n) \right) \) has no torsion from Propositions 2.1 and 3.1 implies that

\[
\beta_H : MU^* \left( \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n) \times S^{2n} \right) \to H^* \left( \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n) \times S^{2n} \right) [t_1, t_2, \ldots]
\]

is injective. The tangent bundle of \( \left( S^{2m+1} \times S^{2n+1} \right) / \phi (l_0, \ldots, l_n) \) is stably isomorphic to \( \pi' (\tau (CP^m) \oplus \eta^2 \oplus \cdots \oplus \eta^4) \) where \( \eta \) is the Hopf bundle over \( CP^m \) and \( \tau (M) \) denotes the tangent bundle of \( M \) [18]. The normal vector bundle \( \nu \) for \( \bar{f} \times h \) satisfies that \( \nu \oplus \nu \left( \left( S^{2m+1} \times S^{2n+1} \right) / \phi (1, \ldots, 1) \right) \)

is isomorphic to \( \pi' \left( \left( S^{m+1} \times S^{n+1} \right) / \phi (l_0, \ldots, l_n) \right) \oplus 2N \), where \( \varepsilon \) is a trivial real line bundle. It follows directly from the definition that

\[
c^H_i (\eta) = 1 + xt_1 + x^2 t_2 + \cdots + x^m t_m, \quad x = c^H_i (\eta)
\]

and

\[
c^H_i (\nu) = \pi^* \left\{ \frac{c^H_i (\eta^k) \cdots c^H_i (\eta^n)}{\left\{ c^H_i (\eta) \right\}^{n+1}} \right\},
\]

since the following diagram is commutative

\[
\begin{array}{ccc}
(S^{2m+1} \times S^{2n+1}) / \phi (1, \ldots, 1) & \xrightarrow{\bar{f} \times h} & (S^{2m+1} \times S^{2n+1}) / \phi (l_0, \ldots, l_n) \\
\Downarrow & & \Downarrow \\
CP^m & & CP^m
\end{array}
\]

By using Theorem 3.2 and Proposition 2.3 we have

\[
\beta_H (\left( \bar{f} \times h \right) (1)) = (\bar{f} \times h)^H c^H_i (\nu)
\]

\[
= \pi^* \left\{ \frac{l_0 \cdots l_n c^H_i (\eta^k) \cdots c^H_i (\eta^n)}{\left\{ c^H_i (\eta) \right\}^{n+1}} \right\} \times [S^{2n}]^*.
\]

On the other hand, we see that

\[
\beta_H (c^H_i (\eta^k)) = c^H_i (\eta^k) c^H_i (\eta^k) = kc^H_i (\eta) c^H_i (\eta^k)
\]

and

\[
\beta_H (\langle k \rangle_F (c^H_i (\eta))) = \beta_H (\langle k \rangle_F (c^H_i (\eta))) \cdot c^H_i (\eta) = \beta_H (\langle k \rangle_F (c^H_i (\eta))) \beta_H (c^H_i (\eta)).
\]

Therefore we have

\[
\beta_H (\langle k \rangle_F (c^H_i (\eta))) = \frac{kc^H_i (\eta)}{c^H_i (\eta)}.
\]
Noting that $\beta_H$ maps $D_{\mathbb{R}^2}([P \subset S^{2n}])$ to $[S^{2n}]^*$, we obtain
\[
\beta_H(\pi^*\langle c_{v}(\eta) \rangle \cdots \langle c_{v}(\eta) \rangle) \times D_{\mathbb{R}^2}([P \subset S^{2n}]) = \beta_H((\bar{f} \times h); 1)).
\]
This completes the proof.

§ 4. Another Proof of the Kasparov Theorem

Let $l_0, l_1, \ldots, l_n$ be integers prime to $p$. An action of $Z_p$ on $S^{2m+1} \times S^{2n+1}$ is defined by
\[
\phi_p(l_0, \ldots, l_n) (g, ((u_0, \ldots, u_m), (v_0, \ldots, v_n))) = ((\rho u_0, \ldots, \rho u_m), (\rho^j v_0, \ldots, \rho^j v_n)),
\]
where $\rho = \exp(2\pi i/p)$ and $g$ is a generator of $Z_p$. The map $f: S^{2m+1} \times S^{2n+1} \to S^{2m+1} \times S^{2n+1}$ with
\[
f((u_0, \ldots, u_m), (v_0, \ldots, v_n)) = (u_0, \ldots, u_m), \frac{1}{r}(v_0^l, \ldots, v_n^l),
\]
induces a map of orbit spaces:
\[
\bar{f}_p: (S^{2m+1} \times S^{2n+1})/\phi_p(1, \ldots, 1) \to (S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \ldots, l_n).
\]
Let $\pi: (S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \ldots, l_n) \to (S^{2m+1} \times S^{2n+1})/\phi(l_0, \ldots, l_n)$ be the natural projection. We take up a differentiable embedding
\[
h: (S^{2m+1} \times S^{2n+1})/\phi(1, \ldots, 1) \to S^{2n}
\]
for a sufficiently large $N$.

**Proposition 4.1.** In the following commutative diagram
\[
(S^{2m+1} \times S^{2n+1})/\phi_p(1, \ldots, 1) \xrightarrow{\pi} (S^{2m+1} \times S^{2n+1})/\phi(1, \ldots, 1) \\
\downarrow f_p \times h \pi \quad \quad \quad \downarrow \bar{f} \times h \\
(S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \ldots, l_n) \times S^{2n} \xrightarrow{\pi \times id} (S^{2m+1} \times S^{2n+1})/\phi(l_0, \ldots, l_n) \times S^{2n}
\]
(1) $\bar{f}_p \times h \pi$ and $\bar{f} \times h$ are embeddings
(2) $\pi \times id$ is transverse regular to $(\bar{f} \times h)((S^{2m+1} \times S^{2n+1})/\phi(1,
\[ (\pi \times \text{id})^{-1}(\tilde{f} \times h) \left( (S^{2m+1} \times S^{2n+1})/\phi(1, \cdots, 1) \right) = (\tilde{f}_p \times h\pi) \left( (S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1) \right). \]

Proof. A tangent vector at a point of \((S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1)\) is described as \(\tilde{v} + \tilde{w}\) with \(\tilde{v} \in \{\text{the tangent space along the base space of the smooth fiber bundle } \pi: (S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1) \to (S^{2m+1} \times S^{2n+1})/\phi(1, \cdots, 1)\}\) and \(\tilde{w} \in \{\text{the tangent space along the fiber}\}\). Let \(d(\tilde{f}_p \times h\pi)(\tilde{v} + \tilde{w}) = 0\), then \(d(\tilde{f} \times h)(\tilde{v}) = 0\). Since \(\tilde{f} \times h\) is an embedding, \(\tilde{v} = 0\). On the other hand, \(d\tilde{f}_p\) is injective on each tangent space along the fiber, and \(\tilde{w} = 0\). This implies that \(\tilde{f}_p \times h\pi\) is embedding, because \(\tilde{f}_p \times h\pi\) is injective. The differentiable fibration \(\pi \times \text{id}\) is transversely regular to any submanifold of \((S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n) \times S^{2n}\). Q.E.D.

Considering the geometric interpretation of the cobordism group [19], we can see that Proposition 4.1 implies

**Proposition 4.2.** The induced homomorphism \((\pi \times \text{id})^*\): \(MU^*((S^{2m+1} \times S^{2n+1})/\phi(l_0, \cdots, l_n) \times S^{2n}) \to MU^*((S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \times S^{2n})\) sends \(D^{\nu}_W[(S^{2m+1} \times S^{2n+1})/\phi(1, \cdots, 1), \tilde{f} \times h]\) to \(D^{\nu}_W[(S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1), \tilde{f}_p \times h\pi]\).

Let \(\psi_p(l_0, \cdots, l_n) : Z_p \times S^{2n+1} \to S^{2n+1}\) be an action of \(Z_p\) on \(S^{2n+1}\) defined by

\[ \psi_p(l_0, \cdots, l_n) (g, (v_0, \cdots, v_n)) = (\rho^{l_0}v_0, \cdots, \rho^{l_n}v_n). \]

We have a complex line bundle \(\tilde{\xi}(L) : S^{2n+1} \times \mathbb{C}^1 \to S^{2n+1}/\psi_p(l_0, \cdots, l_n)\) by taking the orbit space of an action of \(Z_p\) on \(S^{2n+1} \times \mathbb{C}^1\)

\[ g \cdot ((u_0, \cdots, u_n), z) = ((\rho^{l_0}u_0, \cdots, \rho^{l_n}u_n), \rho z) \]

where \(g\) is a generator of \(Z_p\). Denote by

\(\tilde{\xi}(L) : S^{2n+1} \times \mathbb{C}^1 \to S^{2n+1}/Z_p\)

a line bundle over a standard lens space which is the orbit space of an action of \(Z_p\) on \(S^{2n+1} \times \mathbb{C}^1\) defined by \(g \cdot ((u_0, \cdots, u_n), z) = ((\rho u_0, \cdots, \rho u_n), \rho z)\). The bordism class of \(\tilde{f}_p \times h : (S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1) \to (S^{2m+1} \times S^{2n+1})/\phi_p(1, \cdots, 1)\).
$S^{2n+1})/\phi_p(l_0, \cdots, l_n) \times S^{2N}$ with the embedding $\tilde{h}$ for a large $N$ is described as follows.

**Proposition 4.3.** Suppose that $m \leq n$. Then

$$\left( S^{2m+1} \times S^{2n+1} \right)/\phi_p(1, \cdots, 1), \tilde{f}_p \times \tilde{h}$$

$$= D_{hN} \{ \pi^* \langle l_0 \rangle_F (c_B (\xi (L))) \cdots \langle l_n \rangle_F (c_B (\xi (L))) \} \times \{ P \subset S^{2N} \},$$

in $MU_* \left( (S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \right)$, where $P = \{ \text{a point} \}$ and $\pi$: $(S^{2m+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \rightarrow S^{2m+1}/\psi_p(1, \cdots, 1)$ is the natural projection.

**Proof.** Theorem 3.3 and Proposition 4.2 imply that

$$\left( S^{2m+1} \times S^{2n+1} \right)/\phi_p(1, \cdots, 1), \tilde{f}_p \times h \pi$$

$$= D_{hN} \{ \pi^* \langle l_0 \rangle_F (c_B (\xi (L))) \cdots \langle l_n \rangle_F (c_B (\xi (L))) \} \times \{ P \subset S^{2N} \}.$$

But $h \pi$ is homotopic to $\tilde{h}$, and the bordism class is homotopy invariant, and hence the proposition follows.

The map $f: S^{2n+1} \rightarrow S^{2n+1}$ with $f(v_0, \cdots, v_n) = \frac{1}{r}(v_0^r, \cdots, v_n^r)$, $r$ the norm of $(v_0^r, \cdots, v_n^r)$, induces a map of orbit spaces

$$\tilde{f}_p: S^{2n+1}/\psi_p(1, \cdots, 1) \rightarrow S^{2n+1}/\psi_p(l_0, \cdots, l_n).$$

**Theorem 4.4.** In $MU_* \left( S^{2n+1}/\psi_p(l_0, \cdots, l_n) \right)$, $[S^{2n+1}/\psi_p(1, \cdots, 1), \tilde{f}_p]$ $= D_{hN} \langle l_0 \rangle_F (c_B (\xi (L))) \cdots \langle l_n \rangle_F (c_B (\xi (L))) \rangle.$

**Proof.** Define $\pi_2$: $(S^{2n+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \rightarrow S^{2n+1}/\psi_p(l_0, \cdots, l_n)$ by $\pi_2[u, v] = [v]$ and take a differentiable embedding $h: S^{2n+1}/\psi_p(l_0, \cdots, l_n) \rightarrow S^{2N}$ for a sufficiently large $N$. In the commutative diagram

\[
\begin{array}{ccc}
(S^{2n+1} \times S^{2n+1})/\phi_p(1, \cdots, 1) & \xrightarrow{\pi_2} & S^{2n+1}/\psi_p(1, \cdots, 1) \\
\tilde{f}_p \times h \pi_2 & \downarrow & \tilde{f}_p \times h \\
(S^{2n+1} \times S^{2n+1})/\phi_p(l_0, \cdots, l_n) \times S^{2N} & \xrightarrow{\pi_2 \times id} & S^{2n+1}/\psi_p(l_0, \cdots, l_n) \times S^{2N} \\
\tilde{f}_p \times h \pi_2 is an embedding and $\pi_2 \times id$ is transverse regular to $(\tilde{f}_p \times h)$ $(S^{2n+1}/\psi_p(1, \cdots, 1))$. Thus it follows that
We now note that the induced bundle $\pi_2^!\xi(L)$ by the projection $\pi: (S^{2n+1} \times S^{2n+1})/\phi_p(l_0, \ldots, l_n) \to S^{2n+1}/\phi_p(1, \ldots, 1)$ is isomorphic to the induced bundle $\pi_2^!\xi(L)$ by the natural projection $\pi_2: (S^{2n+1} \times S^{2n+1})/\phi_p(l_0, \ldots, l_n) \to S^{2n+1}/\phi_p(l_0, \ldots, l_n)$. Proposition 4.3 implies that

$$(\pi_2 \times id)^*D_{\mathbb{H}}[S^{2n+1}/\phi_p(1, \ldots, 1), \tilde{f}_p \times h] = D_{\mathbb{H}}[\phi_p(1, \ldots, 1), \tilde{f}_p \times h] = \phi_p(1, \ldots, 1), \tilde{f}_p \times h] .$$

Since $(\pi_2 \times id)^*$ is injective, it follows that

$$[S^{2n+1}/\phi_p(1, \ldots, 1), \tilde{f}_p \times h] = D_{\mathbb{H}}[\phi_p(1, \ldots, 1), \tilde{f}_p \times h] \subset S^{2n+1}/\phi_p(p, \ldots, 4).$$

Applying the homomorphism $MU_* (S^{2n+1}/\phi_p(l_0, \ldots, l_n) \times S^{2n}) \to MU_* (S^{2n+1}/\phi_p(l_0, \ldots, l_n))$ induced by the projection, we obtain the assertion.

**Theorem 4.5.** Let $\tilde{g}_p: S^{2n+1}/\phi_p(l_0, \ldots, l_n) \to S^{2n+1}/\phi_p(1, \ldots, 1)$ be the map of orbit spaces defined by

$$\tilde{g}_p[v_{l_0}, \ldots, v_{l_n}] = \left[ \frac{1}{r} (v_{l_0}^{l'_0}, \ldots, v_{l_n}^{l'_n}) \right]$$

where $l_jl'_j=1$ modulo $p$ and $r$ is the norm of $(v_{l_0}^{l'_0}, \ldots, v_{l_n}^{l'_n})$. Then

$$D_{\mathbb{H}}[S^{2n+1}/\phi_p(l_0, \ldots, l_n), \tilde{g}_p] = \langle l'_0 \rangle \ast \langle l'_1 \rangle \ast \cdots \ast \langle l'_n \rangle = \langle p \rangle \ast \langle x \rangle \mod (\langle p \rangle \ast \langle x \rangle)$$

where $\langle p \rangle \ast \langle x \rangle \in MU_* (S^{2n+1}/\phi_p(1, \ldots, 1))$ and $x = c_{l_0}^! (\xi(L))$.

**Proof.** Consider the natural injection $j: S^{2n+1}/\phi_p(1, \ldots, 1) \to S^{2n+2}/\phi_p(1, \ldots, 1)$. We can see that $j\tilde{g}_p \tilde{f}_p = j$ and $\tilde{g}_p^! (\xi(L)) = \xi(L)$. We note that the Atiyah-Poincare isomorphism $D_{\mathbb{H}}: MU_* (X) \to MU_* (X), X$ a weakly almost complex manifold, is given by

$$D_{\mathbb{H}}(z) = z \cap [X, identity].$$

We put $U = [S^{2n+1}/\phi_p(1, \ldots, 1), identity] \in MU_{2n+1}(S^{2n+1}/\phi_p(1, \ldots, 1))$ and $\tilde{U} = [S^{2n+1}/\phi_p(l_0, \ldots, l_n), identity] \in MU_{2n+1}(S^{2n+1}/\phi_p(l_0, \ldots, l_n))$. Let
us compute with Theorem 4.4

\[ j_\#(U) = j_\# \tilde{p}_* \tilde{f}_\#(U) \]

\[ = j_\# \tilde{p}_* \{ [S^{n+1}/\psi_p(1, \ldots, 1), \tilde{f}_\#] \} \]

\[ = j_\# \tilde{p}_* \{ \tilde{g}_\# \{ \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \} \cap \tilde{U} \} \]

\[ = j_\# \{ \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \cap \tilde{g}_\#(\tilde{U}) \} \]

\[ = j_\# \{ \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \cup D_{MU} (\tilde{g}_\#(\tilde{U})) \} \cap U \} \]

Hence \( \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \cup D_{MU} (\tilde{g}_\#(\tilde{U})) \) \(-1\) belongs to \( D_{MU} (j_\#^{-1}(0)) \).

We recall the following commutative diagram:

\[
\begin{array}{ccc}
MU_*(S^{n+1}/\psi_p(1, \ldots, 1)) & \xrightarrow{D_{MU}} & MU_*(S^{n+1}/\psi_p(1, \ldots, 1)) \\
\uparrow j_\# & & \uparrow \phi_U \\
MU_*(S^{n+1}/\psi_p(1, \ldots, 1)) & \xrightarrow{c^*} & \widetilde{MU}*(T(\xi(L)))
\end{array}
\]

where \( \phi_U \) is the Thom isomorphism and \( c \) is the canonical collapsing map. Since \( \phi_U^{-1}c^{-1}(0) \) is generated by \( \langle p \rangle_F(x) \) (cf. [12]), \( \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \cup D_{MU} (\tilde{g}_\#(\tilde{U})) \) \(-1\) belongs to the ideal generated by \( \langle p \rangle_F(x) \) in \( MU_*(S^{n+1}/\psi_p(1, \ldots, 1)) \). On the other hand, since \( \{ \xi(L)^{1j} \} \tilde{\xi}(L) \), we get

\[ \langle [l_j]_F(x) \rangle \langle [l_j]_F(x) \rangle = x \]

and it follows from Lemma 5 of [9] that \( \{ \langle l_j \rangle_F([l_j]_F(x)) \} \langle l_j \rangle_F(x) \) \(-1\) belongs to an ideal generated by \( \langle p \rangle_F(x) \). Then we have

\[ D_{MU} \tilde{g}_\#(\tilde{U}) \]

\[ = \{ \langle l_0 \rangle_F(x) \cdots \langle l_n \rangle_F(x) \} \langle [l_0]_F(x) \rangle \cdots \]

\[ \langle [l_n]_F(x) \rangle \cup D_{MU} (\tilde{g}_\#(\tilde{U})) \} \] modulo \( \langle p \rangle_F(x) \),

and

\[ D_{MU} (\tilde{g}_\#(\tilde{U})) \equiv \langle [l_0]_F(x) \rangle \cdots \langle [l_n]_F(x) \rangle \] modulo \( \langle p \rangle_F(x) \).

Q.E.D.

Let us consider the composite
where $i_*$ is the $MU_*$-homomorphism induced from the natural injection and $\vartheta$ is the natural isomorphism given in [5]. Now we shall prove the Kasparov theorem.

**Theorem 4.6.** Assume that $ljl' = 1$ modulo $p$. Then

$$[S^{2n+1}, \varphi_p(l_0, \ldots, l_n)]$$

$$= j_*(D_{MU} \{\langle l'_0 \rangle_F ([l_0]_F (x)) \cdots \langle l'_n \rangle_F ([l_n]_F (x)) \},$$

where $x = c_2 (\xi (L)) \in MU^3 (S^{2n+1}/\varphi_p(1, \ldots, 1))$.

**Proof.** From Theorem 4.5 there exists $h (x) \in MU^3 (S^{2n+1}/\varphi_p(1, \ldots, 1))$ such that

$$D_{MU}^{-1} [S^{2n+1}/\varphi_p(l_0, \ldots, l_n), \vartheta]$$

$$= \{\langle l'_0 \rangle_F ([l_0]_F (x)) \cdots \langle l'_n \rangle_F ([l_n]_F (x)) + \langle p \rangle_F (x) h (x) \} \cap U$$

and

$$[S^{2n+1}/\varphi_p(l_0, \ldots, l_n), \vartheta]$$

$$= \{\langle l'_0 \rangle_F ([l_0]_F (x)) \cdots \langle l'_n \rangle_F ([l_n]_F (x)) + \langle p \rangle_F (x) h (x) \} \cap i_* (U)$$

where $U = [S^{2n+1}/\varphi_p(1, \ldots, 1), identity]$. Let $\bar{x}$ be the first cobordism Chern class of the canonical line bundle $\xi (L)$ over $S^{2n+1}/\varphi_p(1, \ldots, 1)$ and let

$$\bar{U} = [S^{2n+1}/\varphi_p(1, \ldots, 1), identity]$$

which belongs to $MU_{2n+2} (S^{2n+1}/\varphi_p(1, \ldots, 1))$. Then we have

$$\bar{x} \cap \bar{U} = i_* U \quad (cf. [11]).$$

Noting that $[p]_F (\bar{x}) = 0$, we calculate

$$i_* [S^{2n+1}/\varphi_p(l_0, \ldots, l_n), \vartheta]$$

$$= i_* \{i_* \{\langle l'_0 \rangle_F ([l_0]_F (\bar{x})) \cdots \langle l'_n \rangle_F ([l_n]_F (\bar{x})) + \langle p \rangle_F (\bar{x}) h (\bar{x}) \} \cap i_* (U) \}

$$= \{\langle l'_0 \rangle_F ([l_0]_F (\bar{x})) \cdots \langle l'_n \rangle_F ([l_n]_F (\bar{x})) + \langle p \rangle_F (\bar{x}) h (\bar{x}) \} \cap i_* (U)

$$= \langle l'_0 \rangle_F ([l_0]_F (\bar{x})) \cdots \langle l'_n \rangle_F ([l_n]_F (\bar{x})) \cap i_* (U)
§ 5. Characteristic Classes of $\xi(V_a)$

The product space $I \times X$ of a $Z_p$-space $X$ and an interval $I = [0, 1]$ has a $Z_p$-action with $g \cdot (t, x) = (t, g \cdot x)$, and we have $Z_p$-spaces

$S(X):$ the usual suspension of $X$

$C^+(X) = X \times [1/2, 1] / X \times \{1\}$

$C^-(X) = X \times [0, 1/2] / X \times \{0\}.$

Denote by $p_0$ and $p_1$ vertices obtained by the identification of $X \times 0$ and $X \times 1$ in these spaces. A map $\varepsilon_i: EZ_p \times_{Z_p} \{p\} \to EZ_p \times_{Z_p} S(X)$ is defined to be $\varepsilon_i(x, p) = (x, p_i)$, and a map $\pi: EZ_p \times_{Z_p} X \to EZ_p \times_{Z_p} \{p\} = BZ_p$ is defined to be $\pi(y, x) = (y, p)$. We can derive the following propositions after the fashion of Proposition 10.1 and Theorem 10.2 of [4].

**Proposition 5.1.** Suppose that $X$ is a compact $Z_p$-space. Then there exists an exact sequence:

$$MU^*(EZ_p \times_{Z_p} S(X)) \xrightarrow{\varepsilon^*_0 - \varepsilon^*_1} MU^*(BZ_p) \xrightarrow{\pi^*} MU^*(EZ_p \times_{Z_p} X).$$

**Proof.** $\tilde{MU}^*((EZ_p)^+ \wedge_{Z_p} -)$ is an equivariant cohomology theory described in [10]. Consider the Mayer-Vietoris exact sequence for a triple $(\{S(X)\}^+; \{C^+(X)\}^+, \{C^-(X)\}^+)$

$$\longrightarrow MU^*(EZ_p \times_{Z_p} S(X)) \xrightarrow{j^*} MU^*(EZ_p \times_{Z_p} C^+(X)) \oplus MU^*(EZ_p \times_{Z_p} C^-(X)) \xrightarrow{k^*} MU^*(EZ_p \times_{Z_p} X) \longrightarrow$$

where $j^*(x) = (j^*_0(x), j^*_1(x))$ and $k^*(x_1, x_0) = i^*_1(x_1) - i^*_0(x_0)$, and $j_i$ and $i_i$ are natural inclusions. The isomorphisms $MU^*(EZ_p \times_{Z_p} C^+(X)) \cong MU^*(BZ_p)$ and $MU^*(EZ_p \times_{Z_p} C^-(X)) \cong MU^*(BZ_p)$ yield the proposition.
Let $\Psi : \text{Vect}_c(-) \to MU^*(-)$ be a natural transformation assigning a complex vector bundle over $X$ to an element of $MU^*(X)$ which satisfies

$$\Psi(f^*\zeta) = f^*\Psi(\zeta).$$

Consider complex vector bundles

$$\xi(V_a) : EZ_p \times_{Z_p} V_a \to BZ_p$$

where $V_a$ is the complex $Z_p$-module obtained by the tangent space at an isolated fixed point $a$ of an almost complex $Z_p$-manifold $M$. Then we have

**Proposition 5.2.** Suppose that $a$ and $b$ are isolated fixed points of a simply connected almost complex $Z_p$-manifold. If $H^i(BZ_p; \{\pi_1(M)\}) = 0$ for $1 \leq i \leq 2n-1$, then $\Psi(\xi(V_a)) - \Psi(\xi(V_b))$ belongs to an ideal generated by $x^n$ in $MU^*(BZ_p) \cong MU^*[x]/([p^r(x)])$, where $x = c_1(\xi(L))$, $L$ the canonical one dimensional complex $Z_p$-module.

**Proof.** The $(2n-1)$-skeleton of $EZ_p$ can be taken to be $S^{2n-1}$ with the action given by the complex $n$-dimensional $Z_p$-module $nL$. We take an invariant subspace $EZ_p \times \{0, 1\}$ is a $Z_p$-space $EZ_p \times I$ with $g \cdot (e, t) = (g \cdot e, t)$. Consider the constant maps

$$h_0 : EZ_p \to \{b\} \quad \text{and} \quad h_1 : EZ_p \to \{a\}$$

which induce maps

$$\tilde{h}_0 : S^{2n-1} \subset EZ_p \to \{b\} \quad \text{and} \quad \tilde{h}_1 : S^{2n-1} \subset EZ_p \to \{a\}.$$  

We can construct an equivariant homotopy $h : S^{2n-1} \times I \to M$ between $\tilde{h}_0$ and $\tilde{h}_1$, by using the condition for the cohomology $H^i(BZ_p; \{\pi_1(M)\})$, and an equivariant map $\tilde{h} : S(S^{2n-1}) \to M$ (cf. [4, p. 355]). Since

$$\xi(V_a) = c_1(id \times_{Z_p} \tilde{h})^* \xi \quad \text{and} \quad \xi(V_b) = c_1(id \times_{Z_p} \tilde{h})^* \xi,$$

where $\xi$ denotes a vector bundle $EZ_p \times_{Z_p} E(\pi(M)) \to EZ_p \times_{Z_p} M$, it follows from Proposition 5.1 that $\pi^* (\Psi(\xi(V_a)) - \Psi(\xi(V_b))) = 0$. By using the Gysin exact sequence

$$MU^*(BZ_p) \xrightarrow{x^n} MU^{*+2n}(BZ_p) \xrightarrow{\pi^*} MU^{*+2n}(EZ_p \times_{Z_p} S^{2n-1}) \to$$

we complete the proof.
We consider the symmetric polynomial $P_\omega (\xi_1, \ldots, \xi_n)$ discussed in Section 3, and put $c^\omega_i (\tau_n) = P_\omega (c^\omega (\tau_n), \ldots, c^\omega (\tau_n))$, where $c^\omega_i (\tau_n)$ is the $i$-th cobordism Chern class [7]. The Landweber-Novikov operation

$$S^\omega_u : MU^* (X) \to MU^{*+2\omega} (X)$$

is defined as follows: for $x = [f]$, $f : S^{n+k} \to MU(n)$,

$$S^\omega_u (x) = \sigma^{n+k} f^* \Phi_u (c^\omega_i (\tau_n)) \quad (\text{cf. } [14], [17]).$$

The Boardman map $\beta_v : MU^* (X) \to (MU \wedge MU)^* (X) \cong MU^* (X) [[t_1, t_2, \ldots]]$ is defined by

$$\beta_v (x) = \sum_S S^\omega_u (x) t^e \quad (\text{cf. } [2], [19]),$$

which is natural and multiplicative. Let $J(G)$ be the set of isomorphism classes of non trivial irreducible complex $\mathbb{Z}_p$-modules, and let $G^V = \{ V_1^k \oplus \cdots \oplus V_k^k : |V_i| \in J(G) \text{ and } k \text{'s are non negative integers} \}$. We consider the multiplicative system $S$ consisting of cobordism Euler classes $\{ e(EZ_p \times \mathbb{Z}_p V) \mid V \in G^V \}$ in $MU^* (BZ_p)$. For a $Z_p$-space $X$, $MU^* (EZ_p \times \mathbb{Z}_p X)$ is a $MU^* (BZ_p)$-module by a map $EZ_p \times \mathbb{Z}_p X \to BZ_p \times (EZ_p \times \mathbb{Z}_p X)$ sending $[e, x]$ to $([e], [e, x])$. The localized module $S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X)$ is the $MU^* (BZ_p)$-module $MU^* (EZ_p \times \mathbb{Z}_p X)$ consists of all fractions $\{ x/e : x \in MU^* (EZ_p \times \mathbb{Z}_p X), e \in S \}$. For a complex vector bundle $\zeta$ over $X$, we put

$$c^\omega_i (\zeta) = 1 + \sum u c^\omega_u (\zeta) t^u$$

which is an invertible element of $MU^* [[t_1, t_2, \ldots]]$. We define $\tilde{\beta}_v : S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X) \to S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X) [[t_1, t_2, \ldots]]$ by

$$\tilde{\beta}_v (y/e (\xi (V))) = \left( \beta_v (y) \cdot \frac{1}{c^\omega_i (\xi (V))} \right) e (\xi (V))$$

which is multiplicative and natural. Moreover, we define

$$\tilde{S}^\omega_u : S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X) \to S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X)$$

by $\tilde{\beta}_v (x/e) = \sum_S \tilde{S}^\omega_u (x/e) t^e$.

**Proposition 5.3.** The operation $\tilde{S}^\omega_u$ on $S^{-1}MU^* (EZ_p \times \mathbb{Z}_p X)$ have the following properties:

1. $\tilde{S}^\omega_u$ is natural.
2. $\tilde{S}^\omega_u ((x_1/e_1) \cdot (x_2/e_2)) = \sum_{u=(w,w')} \tilde{S}^\omega_u (x_1/e_1) \tilde{S}^\omega_u (x_2/e_2)$, where for
\(\omega' = (j'_1, \ldots, j'_r)\) and \(\omega^* = (j''_1, \ldots, j''_r)\), \((\omega' \omega^*)\) denotes \((j'_1, \ldots, j'_r, j''_1, \ldots, j''_r)\).

(3) \(\tilde{S}^\omega_u(x/1) = S^\omega_u(x)/1\), where \(S^\omega_u\) is the ordinary Landweber-Novikov operation, i.e. \(\lambda S^\omega_u = \tilde{S}^\omega_u \lambda,\) where \(\lambda: MU^*(EZ_p \times Z_p) \rightarrow S^{-1}MU^*(EZ_p \times Z_p)\) is the canonical map.

(4) For \(\omega = \left(\frac{1}{i_1}, \frac{1}{i_2}, \ldots, \frac{1}{i_k}\right),\)

\[
\tilde{S}^\omega_u(1/e(\xi(L))) = (-1)^{i_1 + \cdots + i_k} \left\{ \frac{(i_1 + \cdots + i_k)!}{i_1! \cdots i_k!} e(\xi(L))^{i_1 + \cdots + i_k}\right\}/1.
\]

Proof. By making use of the multiplicativity and the naturality of \(\beta^u\), we derive (1) and (2). For a zero dimensional complex \(Z_p\)-module 0, we have \(e(\xi(0)) = 1\) and \(c_i^u(\xi(0)) = 1\), and

\[
\tilde{\beta}^u(x/1) = \frac{\beta^u(x) \cdot 1}{c_i^u(\xi(0))} e(\xi(0))
\]

which implies (3). To prove (4), we calculate

\[
\tilde{\beta}^u(1/e(\xi(L)))
= \left\{ \frac{1}{1 + e(\xi(L)) t_1 + e(\xi(L)) t_2 + \cdots} \right\}/e(\xi(L))
= \{ \sum (-1)^{t_1 + \cdots + t_2 + \cdots} \}/e(\xi(L)).
\]

This completes the proof.

We see easily the following

Proposition 5.4. \(S^\omega_u(e(\xi(V))) = e(\xi(V)) c_i^u(\xi(V))\).

Taking two complex \(Z_p\)-modules \(V_a\) and \(V_b\) obtained from tangent spaces at isolated fixed points \(a\) and \(b\) of an almost complex \(Z_p\)-manifold, a fraction \(e(\xi(V_a))/e(\xi(V_b))\) is an integral element from the following
Proposition 5.5. Suppose that $L$ is a canonical complex one dimensional $\mathbb{Z}_p$-module. Take $k_i$ and $l_j$ such that $(k_i, p) = 1$ and $(l_j, p) = 1$. Then for $n \geq m$, $e(\xi (L^{k_1} \oplus \cdots \oplus L^{k_n})) / e(\xi (L^{l_1} \oplus \cdots \oplus L^{l_m}))$ belongs to the image of $\lambda: MU^* (BZ_p) \to S^{-1}MU^* (BZ_p)$ which sends $x$ to $x/1$.

Proof. For $x = c_b(\xi (L))$, 

$$e(\xi (L^k)) = \lfloor k \rfloor_F (x) = kx + a^{(k)} x^3 + a^{(k)} x^5 + \cdots$$

and

$$e(\xi (L^k)) / x = \langle k \rangle_F (x) / 1.$$

Assume that $(l, p) = 1$, then there is an integer $l'$ such that $l' l \equiv 1$ modulo $p$ and

$$x = \langle l' \rangle_F ([l]_F (x)) \cdot [l]_F (x).$$

Therefore we have

$$\frac{e(\xi (L^{k_1} \oplus \cdots \oplus L^{k_n}))}{e(\xi (L^{l_1} \oplus \cdots \oplus L^{l_m}))} = \langle l'_1 \rangle_F ([l_1]_F (x)) \cdots \langle l'_m \rangle_F ([l_m]_F (x)) \langle k_i \rangle_F (x) \cdots \langle k_n \rangle_F (x) [k_{m+1}]_F (x) \cdots [k_n]_F (x) / 1.$$ 

where $l' l \equiv 1$ modulo $p$. Q.E.D.

Proof of Theorem A. For brevity, we put $e_a = e(\xi (V_a))$ and $e_b = e(\xi (V_b))$. We show by induction with respect to the length of the partition $\omega$ that

$$\bar{S}_\omega \left( \frac{e_a}{e_b} \right) = e_a \cdot \frac{a_x (x) \cdot x^n}{e_b}$$

where $a_x (x) \in MU^* (BZ_p)$. By using (2) of Proposition 5.3 we obtain

$$\bar{S}_{(t)} \left( \frac{e_a}{1} \right) = \bar{S}_{(t)} \left( \frac{e_a}{e_b} \cdot \frac{e_b}{1} + \frac{e_a}{e_b} \cdot \bar{S}_{(t)} \left( \frac{e_b}{1} \right) \right).$$

Hence it follows from (3) of Propositions 5.3 and 5.4 that
Proposition 5.2 implies that there is an element \( h(x) \in MU^*(BZ_p) \) such that \( c^U_0(\xi) = h(x) x^n \), and

\[
\bar{S}^U_0 \left( \begin{array}{c} e_a \\ e_b \\ \end{array} \right) = \frac{e_a \cdot c^U_0(\xi)}{e_b}. 
\]

Suppose the result is proved for \( \omega' \) whose length is less than the length of \( \omega \). By using (2) of Proposition 5.3 with the inductive hypothesis we calculate

\[
\bar{S}^U_0 \left( \begin{array}{c} e_a \\ e_b \\ \end{array} \right) = \frac{e_a \cdot h(x) x^n}{1}.
\]

where \( h(x) \in MU^*(BZ_p) \). Moreover it follows from Propositions 5.4 and 5.2 that there exists an element \( h'_0(x) \in MU^*(BZ_p) \) such that

\[
\bar{S}^U_0 \left( \begin{array}{c} e_a \\ e_b \\ \end{array} \right) = \frac{e_a h'_0(x) x^n}{1} - \sum_{\omega \in \omega' \omega''} \frac{e_a}{e_b} \{ h'_0(x) x^n c^U_0(\xi) \}/1,
\]

and there is an element \( h(x) \in MU^*(BZ_p) \) such that

\[
\bar{S}^U_0 \left( \begin{array}{c} e_a \\ e_b \\ \end{array} \right) = \frac{e_a h(x) x^n}{1}.
\]

It is pointed out by [9] that the canonical map \( \lambda: MU^*(BZ_p) \to S^1 MU^*(BZ_p) \) with \( \lambda(x) = x/1 \) has the kernel which is an ideal generated by \( \langle p \rangle \) for \( x \). We then complete the proof.

\section*{§ 6. On the Bordism Classes of Actions on Invariant Spheres around the Isolated Fixed Points}

The Thom homomorphism \( \mu: MU^*(-) \to H^*(-) \) is the multiplicative natural transformation with the following properties.

\textbf{Proposition 6.1.} Let \( \xi \) be a complex vector bundle over \( X \).

Then
(1) \( \mu c_w^n(\xi) = c^n_w(\xi) \)

(2) \( \Phi_f(x) = \Phi(\mu(x)) \), where \( \Phi_f : MU^*(X) \to MU^*(T(\xi)) \) and \( \Phi : H^*(X) \to H^*(T(\xi)) \) are the Thom homomorphisms.

Recall the following property of the Umkehr homomorphism [8].

**Proposition 6.2.** \( g_1(g^*(x) \cup y) = x \cup g_1(y) \).

We observe \( S^n_w : MU^*(X) \to H^*(X) \) for a weakly complex manifold \( X \).

**Proposition 6.3.** Take an element \( x = [M \to X] \in MU_*(X) \), where \( X \) is a weakly complex manifold and \( g \) is a differentiable map. Then,

\[ S^n_w D_{MU}^1(x) = \sum_{w = (w', w'')} c^n_w(\bar{\tau}(X)) g_1(c^n_w(\nu)) \]

where \( \nu \) is the normal bundle of \( M \) in a Euclidean space with the complex structure and \( \bar{\tau}(X) \) is the Whitney sum of \( \tau(X) \) and some trivial bundle which is a complex bundle.

**Proof.** Let \( \tilde{g} : M \to X \times R^i \) be an embedding with the normal bundle \( \bar{\nu} \) equipped with a complex structure and \( \tilde{g} = g \cdot D_{MU}^1(x) \) is represented by the composition

\[ S^i \wedge X^* \xrightarrow{c} T(\bar{\nu}) \xrightarrow{\tilde{g}} MU(k) \]

which \( c \) is the collapsing map and \( \tilde{g} \) is the map induced by the classifying map for \( \nu \). The Whitney sum \( \bar{\nu} \circlearrowleft \tau(M) \) is stably equivalent to \( g^* \tau(X) \) and

\[ c^n_t(\bar{\nu}) \cdot c^n_t(\bar{\tau}(M)) = g^* c^n_t(\bar{\tau}(X)) \cdot c^n_t(\nu) . \]

Hence we have that \( c^n_t(\bar{\nu}) = g^* c^n_t(\bar{\tau}(X)) \cdot c^n_t(\nu) \). We calculate with Propositions 6.1 and 6.2

\[ S^n_w D_{MU}^1(x) = \mu S^n_w D_{MU}^1(x) = \sigma^{-1} c^* \{ \Phi(c^n_w(\bar{\nu})) \} \]

\[ = g_1(c^n_w(\bar{\nu})) \]

\[ = g_1 \left( \sum_{w = (w', w'')} g^* (c^n_w(\bar{\tau}(X)) c^n_w(\nu)) \right) \]
MU^k is isomorphic to MU_{-k} and a bordism class [M] of a weakly almost complex manifold can be regarded to be in MU^*. Directly Proposition 6.3 implies

**Corollary 6.4.** \( \mu S^v_\ast [M] = \langle c^H_{\omega} (\nu), [M] \rangle \), where \( \nu \) is the normal vector bundle of \( M \) in a Euclidean space which is equipped with the complex structure, where \( c^H_{(i_1, \ldots, i_l)} \) is the Chern class for \( \sum t_1^i \cdots t_l^i \).

We consider the ideal \( \mathcal{J}_p \) in \( MU^* \) which is generated by \( p, a_{i}^{(p)}, a_{i}^{(p)}, \ldots, a_{i}^{(p)}, \ldots \) which are coefficients of

\[
[p]_F (x) = px + a_{i}^{(p)} x^i + a_{i}^{(p)} x^i + \cdots.
\]

We recall the following property of \( \mathcal{J}_p \).

**Proposition 6.5** (cf. [9]). \([M] \) belongs to \( \mathcal{J}_p \) if and only if \( c^H_\omega [M] = \langle c^H_\omega (\tau(M)), [M] \rangle \equiv 0 \) modulo \( p \), for any \( \omega \), where \( p \) is prime.

**Proof.** Let \( y = c^H_0 (\gamma) \) be the cobordism first Chern class of the Hopf bundle \( \gamma \) over \( CP^n \). It is known (cf. [14], [17]) that

\[
S^v_\ast ([p]_F(y)) = \begin{cases} 
\{ [p]_F (y) \}^{i+1} & \text{if } \omega = (i) \\
0 & \text{otherwise.}
\end{cases}
\]

We see \( S^v_\ast ([p]_F(y)) \equiv 0 \) modulo \( p \), and

\[
S^v_\ast (py + a_{i}^{(p)} y^i + a_{i}^{(p)} y^i + \cdots) \equiv 0 \pmod{p}.
\]

Then we can deduce that \( S^v_\ast (a_i^{(p)}) \equiv 0 \pmod{p} \). Therefore we have that the Chern numbers of \( [N] \) are zero modulo \( p \) if \( [N] \) belongs to \( \mathcal{J}_p \). The Hopf bundle \( \tilde{\gamma} \) over \( CP^n \) satisfies that

\[
D_{MU} (c^H_0 (\tilde{\gamma})) = q [CP^{n-1} \subset CP^n] + a_{i}^{(p)} [CP^{n-2} \subset CP^n] + \cdots + a_{n-1}^{(p)} [P \subset CP^n], \text{ in } MU_\ast (CP^n).
\]

Let \( D_{MU} (c^H_0 (\tilde{\gamma})) = [V_{0}^{n-1} \subset CP^n] \), then

\[
(V_{0}^{n-1}) = q [CP^{n-1}] + a_{i}^{(p)} [CP^{n-2}] + \cdots + a_{n-1}^{(p)}.
\]

We note that \( V_{0}^{n-1} \) is a \( U \)-submanifold dual to \( c^H_0 (\tilde{\gamma}) \) (cf. [7, p. 81]),

\[
= \sum_{\omega = (x^s)} c^H_\omega (\tau(X)) \tilde{g}_\omega (c^H_\omega (\nu)).
\]

Q.E.D.
and the fundamental classes of $V_{\otimes}^{n-1}$ and $CP^n$ satisfy that $i_*[V_{\otimes}^{n-1}] = c^f(\bar{\eta}) \cap [CP^n]$, $i: V_{\otimes}^{n-1} \subset CP^n$. Noting that the normal bundle $\nu$ of $V_{\otimes}^{n-1}$ in $CP^n$ is isomorphic to $c^f(\bar{\eta})$, we have that $c^f_{k(n-1)}(\tau(V_{\otimes}^{n-1})) = i^* \{(n+1) - q^n\}$ $\bar{\eta}^{n-1}$, where $\bar{\eta} = c^f(\bar{\eta})$. Therefore it follows that the Chern number $c^f_{k(n-1)}[V_{\otimes}^{n-1}] = q(n+1) - q^n$. Using (*) and $c^f_{k(n-1)}[CP^n] = n$, we have $c^f_{k(n-1)}[a_{\otimes}^{n-1}] = q - q^n$. For prime $q$, we take

$$[W_{q^{k-1}}] = a_{\otimes}^{q-1} + q^b[CP^n], \ b = q^k - k \ \text{and} \ u = q^k - 1$$

whose Chern number $c^f_{(q-1)}[W_{q^{k-1}}]$ equals to $q$. Take a $2i$-dimensional weakly almost complex manifold $W_i$, $i \neq q^k - 1$ for any prime $q$, such that $c^f_{q}[W_i] = 1$. According to [16], $MU_* = \mathbb{Z}[[W_i], [w_2], \cdots]$. Assume that $c^f_{q}[M] \equiv 0$ modulo $p$ for any $\omega$ and

$$[M] = \sum a_{i-t} [W_i]^{t_1} \cdots [W_n]^{t_n}.$$  

Noting that

$$S_{t_1, \cdots, t_n} \left[\frac{c^f_{q}[W_i]}{t_1} \frac{c^f_{q}[W_2]}{t_2} \cdots \frac{c^f_{q}[W_n]}{t_n}\right]^{t_1}$$

$$= (c^f_{q}[W_i])^{t_1} (c^f_{q}[W_2])^{t_2} \cdots (c^f_{q}[W_n])^{t_n},$$

we inductively deduce that if $i_s = 0$ for $s = p^k - 1$, then $a_{t_i, t_2, \cdots, t_n} \equiv 0$ modulo $p$, and $[M] \in \mathcal{F}_p$. Q.E.D.

We now go back to consider the cobordism Euler class of complex vector bundle $\xi(V_a): EZ_p \times_{Z_p} V_a \to BZ_p$, $V_a$ the complex $Z_p$-module given by the tangent space at the isolated fixed points of a $Z_p$-manifold.

**Proposition 6.6.** Suppose that $V_a$ and $V_b$ are complex $Z_p$-modules given by tangent spaces at isolated fixed points $a$ and $b$ of a simply connected almost complex $Z_p$-manifold $M$, and $\lambda(\alpha) = e(\xi(V_a)) / e(\xi(V_b))$, where $\lambda: MU^*(BZ_p) \to S^{-1}MU^*(BZ_p)$ is the canonical homomorphism. If $H^i(BZ_p; \{\pi_i(M)\}) \cong 0$ for $1 \leq i \leq 2n-1$, then

$$\alpha = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots$$

where $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ belong to $\mathcal{F}_p$.

**Proof.** Suppose that $|\omega| = 2i$, $1 \leq i \leq n-1$. Then $S_{\omega}^i \lambda \in MU^{2i-2k}$.  

Note that \( \mu: MU^k(P) \to H^k(P), \ P = \{ \text{a point} \} \), is the zero homomorphism for \( k > 0 \), and \( S_\ell^0(\lambda_0) = 0 \) if \( \omega \neq (0) \). Suppose that \( \lambda, j = 1, 2, \ldots, i-1 \), belong to \( J_\ell \). Then

\[
\mu S_\ell^0(\alpha) = \mu S_\ell^0(\lambda_0) \cdot x_H^j = c^H_\alpha[\lambda_i] x_H^j,
\]

where \( x_H^j = c^H_\alpha(\xi(L)) \). Since \( S_\ell^0(\alpha) \) belongs to an ideal generated by \( x^n \) and \( \langle p \rangle_F(c^F_\alpha(\xi(L))) \) from Theorem A, \( c^H_\alpha[\lambda_i] x_H^j = 0 \) in \( H^*(BZ_p) \). Proposition 6.5 implies that \( \lambda_i \in J_\ell \). Q.E.D.

**Proof of Theorem B.** Let \( \xi(V) \) be a complex vector bundle \( S^{2k-1} \times_{Z_p} V \to S^{2k-1}/Z_p \), where \( V \) is a complex \( Z_p \)-module and \( S^{2k-1} \) has the \( Z_p \)-action \( \psi_p(1, \ldots, 1) \). Let \( i: S^{2k-1}/\psi_p(1, \ldots, 1) \to BZ_p \) be the natural injection. Put \( x = c^F_\alpha(\xi(L)) \) and \( \bar{x} = c^F_\alpha(\xi(L)). \) Then, \( i_\ell \xi(L) \cong \xi(L) \).

We see that in \( S^{-1}MU^*(BZ_p) \),

\[
\begin{align*}
& l_1 \cdots l_k \frac{x^k}{e(\xi(V_a))} - m_1 \cdots m_k \frac{x^k}{e(\xi(V_b))} \\
= & l_1 \cdots l_k \frac{x^k}{e(\xi(V_a))} - m_1 \cdots m_k \frac{x^k}{e(\xi(V_a))} \cdot \frac{e(\xi(V_a))}{e(\xi(V_b))},
\end{align*}
\]

On the other hand it follows from Proposition 6.6 that

\[
m_1 \cdots m_k \langle l_1 \rangle_F(x) \langle m'_1 \rangle_F([m_1]_F(x)) \cdots \langle l_k \rangle_F(x) \langle m'_k \rangle_F([m_k]_F(x)) \equiv l_1 \cdots l_k + h(x) x^n \text{ modulo } J_\ell
\]

where \( m_i m'_i \equiv 1 \) modulo \( p \). Therefore we get

\[
l_1 \cdots l_k \langle l'_1 \rangle_F([l_1]_F(x)) \cdots \langle l'_k \rangle_F([l_k]_F(x)) \\
- m_1 \cdots m_k \langle m'_1 \rangle_F([m_1]_F(x)) \cdots \langle m'_k \rangle_F([m_k]_F(x)) \\
\equiv \tilde{h}(x) x^n \text{ modulo } J_\ell, \ l_1 l'_1 \equiv 1 \text{ modulo } p, \text{ where } \tilde{h}(x) \in MU^*(BZ_p).
\]

Applying \( i^* \) to the above, we have

\[
l_1 \cdots l_k \langle l'_1 \rangle_F([l_1]_F(x)) \cdots \langle l'_k \rangle_F([l_k]_F(x)) \\
- m_1 \cdots m_k \langle m'_1 \rangle_F([m_1]_F(x)) \cdots \langle m'_k \rangle_F([m_k]_F(x)) \\
\equiv \tilde{h}(x) x^n \text{ modulo } J_\ell \ (\text{cf. [12]}).
\]

Since \( j_\ast D_{MU} x^n = [S^{2k-1}, \phi] \) (cf. [11]), Theorems 4.5 and 4.6 imply the theorem.
§ 7. The Isolated Fixed Points of $Z_3$-Actions

In this section, we will consider a complex structure preserving smooth $Z_3$-action $(M^{2k}, \theta)$ on a simply connected closed almost complex manifold $M^{2k}$. Let $a$ and $b$ be isolated fixed points. We describe the induced actions of $Z_3$ on the tangent spaces at $a$ and $b$ as complex $Z_3$-modules

$$V_a = sL^3 \oplus (k-s)L$$

and

$$V_b = (s+t) L^3 \oplus (k-s-t)L.$$

Recall that

$$\langle 2 \rangle_F(x) = a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2 + \cdots, \ a_0^{(2)} \in MU^{-2i}$$

and

$$c_{(n)}^H(a_n^{(2)}) = 2 - 2^{n+1}.$$

In this situation, we shall first indicate a lemma which is derived as proof of Theorem B.

**Lemma 7.1.** Suppose that $H^i(BZ_3; \pi_1(M^{2k})) = 0$ for $1 \leq i \leq 2n-1$. Then for $1 \leq j \leq n-1$

$$\sum_{l_1 + \cdots + l_j = j} a_1^{(j)} \cdots a_j^{(j)} \in J_3.$$

**Proof:** In $S^{-1}MU^*(BZ_3)$, $MU^*(BZ_3) \cong MU^*[[x]]/[3]_F(x)$, we have

$$\frac{e(V_a)}{e(V_b)} = \mu_0 + \mu_1 x + \cdots + \mu_k x^k + \cdots, \ \mu_1, \cdots, \mu_{n-1} \in J_3,$$

from Proposition 6.6 and

$$\frac{2^r x^k}{e(V_a)} - \frac{2^{r+1} x^k}{e(V_b)} = \overline{\mu}_1 x + \overline{\mu}_2 x^2 + \cdots + \overline{\mu}_k x^k + \cdots,$$

$$\overline{\mu}_1, \cdots, \overline{\mu}_{n-1} \in J_3.$$

Noting the fact that the kernel of the canonical map $\lambda: MU^*(BZ_3) \to$
$S^{-1}MU^*(BZ_0)$ is the ideal generated by $\langle 3 \rangle_F(x)$, we obtain

$$2^i x^i e(V_0) - 2^{i+1} x^i e(V_a)$$

$$= e(V_a) e(V_b) \{ \bar{\mu}_1 x + \bar{\mu}_2 x^2 + \cdots + \bar{\mu}_k x^k + \cdots \}$$

and

$$2^i (\langle 2 \rangle_F(x))^i - 2^i$$

$$= \bar{\mu}_1 x + \bar{\mu}_2 x^2 + \cdots + \bar{\mu}_k x^k + \cdots, \bar{\mu}_1, \cdots, \bar{\mu}_{n-i} \in \mathcal{J}_1. \quad \text{Q.E.D.}$$

Then we obtain the following

**Lemma 7.2.** Suppose that $H^i(BZ_0; \{ \pi_i(M^n) \}) \cong 0$ for $1 \leq i \leq 2n - 1$. Then, for $1 \leq m \leq n - 1$ the binomial coefficients $\binom{t}{m}$ are divisible by 3.

**Proof.** We take a partition

$$\omega = (k, \ldots, k, \ldots, 2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$$

of $k$, where

$$|\omega| = 1 \cdot j_1 + 2 \cdot j_2 + \cdots + k \cdot j_k = k$$

and

$$j_0 + j_1 + \cdots + j_k = t.$$ 

We define now

$$a_\omega^{(2)} = \{ a_k^{(2)} \} j_1 \cdots \{ a_1^{(2)} \} j_1 \{ a_0^{(2)} \} j_0$$

and

$$\lambda_\omega = \frac{t!}{j_k! \cdots j_2! j_1! j_0!}.$$ 

Then we have the following

$$\sum_{t_1 + \cdots + t_k = j} a_1^{(2)} \cdots a_k^{(2)} = \sum_{|\omega| = j} \lambda_\omega a_\omega^{(2)}.$$ 

We take up the case $k = 1$. Since from Lemma 7.1 $2^{i-1} \cdot a_1^{(2)} = \sum_{t_1 + \cdots + t_k = i}$
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\( a_i^{(2)} \cdots a_i^{(2)} \) belongs to \( F_3 \), and \( c(0)(a_i^{(2)}) = -2 \). \( t \) is divisible by 3. Assume that \( m < n \) and \( \binom{t}{j} \), \( j = 1, \ldots, m-1 \), are divisible by 3. From Lemma 7.1 \( \sum_{\omega = m} \omega a_i^{(2)} \) belongs to \( F_3 \), and for \( \| \omega \| \leq m-1 \)
\[
\lambda_\omega = \frac{\| \omega \| !}{j_k! \cdots j_1! \cdots j_1!} \left( \binom{t}{j} \right) \equiv 0 \pmod{3}.
\]
By induction we complete the proof.

We shall give some information on isolated fixed points of \( Z_3 \)-actions.

**Theorem 7.3.** Let \( a \) and \( b \) be isolated fixed points of a complex structure preserving smooth action of \( Z_3 \) on the simply connected closed almost complex manifold \( M^{2k} \). Suppose that
\[
k = \lambda_u 3^n + \lambda_{u-1} 3^{n-1} + \cdots + \lambda_3 3 + \lambda_0, \quad 0 \leq \lambda_j \leq 2 \quad \text{and} \quad \lambda_u \neq 0
\]
and
\[
H_i^k(BZ_3; \{ \pi_i(L^{2k}) \}) \cong 0 \quad \text{for} \quad 1 \leq i \leq 2 \cdot 3^n + 1.
\]
Then \( V_a \) is equivalent to \( V_b \).

**Proof:** Let \( V_a = sL^2 \oplus (k-s) \) \( \) \( L \) and \( V_b = (s+t) L^2 \oplus (k-s-t) L \).
Suppose that \( t = \lambda_u 3^n + \lambda_{u-1} 3^{n-1} + \cdots + \lambda_3 3 + \lambda_0 \leq k \).
It follows from Lemma 7.2 that
\[
\lambda'_t = \binom{t}{3} \equiv 0 \pmod{3}.
\]
Hence \( \lambda'_t = 0 \) and \( t = 0 \).

**Q.E.D.**

**Corollary 7.4.** Suppose that \( Z_3 \) acts on a simply connected almost complex closed \( 2k \)-dimensional manifold \( M \) as a complex structure preserving diffeomorphism with isolated fixed points only. Let
\[
k = \lambda_u 3^n + \cdots + \lambda_3 3 + \lambda_0, \quad 0 \leq \lambda_j \leq 2, \quad \text{and} \quad \lambda_u \neq 0.
\]
If \( H_i^k(BZ_3; \{ \pi_i(L^{2k}) \}) \cong 0 \) for \( 1 \leq i \leq 2 \cdot 3^n + 1 \), then the number of fixed points is divisible by \( 2^{(n+1)/2} - 1 \).

**Proof:** Let \( n \) be the number of the fixed points. Theorem 7.3
implies that
\[ n[S(V_a), \phi_a] = 0 \text{ in } MU_*(Z) \]
where \( V_a = sL^2 + (k-s)L \). The Kasparov theorem (Theorem 4.6) implies that
\[ n(l + 3m)[S^{2k-1}, \tilde{\phi}] + \mu_l[S^{2k-3}, \tilde{\phi}] + \cdots + \mu_{l-1}[S^1, \tilde{\phi}] = 0 \]
where \( l \equiv 0 \) modulo 3 and \( \mu_l \in \Gamma(3), \Gamma(3)[[CP^2]] = MU_* \) (cf. [6], [11]).

From the result of [6] and [11] we can derive the assertion.

References


