Time Periodic Solutions of Some Non-linear Evolution Equations

By

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§ 1. Introduction

Considered in this paper are non-linear evolution equations of the form

\[
\frac{\partial^{x}u}{\partial t^{x}} + Au + B(u, \frac{\partial u}{\partial t}) = f(x, t) \quad \text{in} \ \Omega \times (-\infty, \infty)
\]

together with periodicity conditions

\[
u(x, t) = v(x, t + \tau), \quad u_t(x, t) = u_t(x, t + \tau)
\]

and Dirichlet boundary conditions

\[
D^\alpha u(x, t) = 0 \quad \text{on} \ \partial \Omega \quad \text{for} \ |\alpha| \leq m - 1.
\]

Each \(A\) in (1) is a non-linear elliptic operator of order \(2m\) in \(\Omega\), a fixed bounded domain in \(\mathbb{R}^N\) (which is similar to the one defined by F.E. Browder ([1])), and

\[
B(u, u_t) = \beta'(|u|^2)u_t - 4u_t, \quad \text{or more generally}
\]

\[
\sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^\alpha \beta'(D^\alpha u_t)^2 D^\alpha u_t
\]

where each \(\beta'(s^2)\) is a non-negative function on \(\mathbb{R}^1\), of polynomial growth. Each function \(f(x, t)\) on \(\Omega \times \mathbb{R}^1\) is periodic in \(t\) (of period \(\tau > 0\)) with values in an appropriate Sobolev space. The purpose of this paper is to prove...
an existence theorem of weak solutions for the equation (1) with conditions (2)–(3), subsequently to [3], where the theorem was proved for $A$, semi-linear elliptic operators, and $B(u, u_t) = (1 + \beta(|u|^2))u_t$. In case $A$ is a quasi-linear elliptic operator of the second order and $B(u, u_t) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} + u^p - 1$, initial-boundary value problems for (1) have been solved by M. Tsutsumi ([7]), J.C. Clements ([2]), also boundary value problems with periodic conditions for (1) by Clements ([2]). W.A. Strauss has obtained (in unpublished work) weak solutions periodic in $t$ of the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^{p-1} u + |u|^{q-1} \frac{\partial u}{\partial t} = f(x, t)$$

where $f(x, t)$ is a function periodic in $t$, $p \geq q \geq 1$ (cf. [6]).

An example of our theorem gives the existence of periodic solutions in $t$ for the equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \right) - \Delta \frac{\partial u}{\partial t} + |u|^{q-1} \frac{\partial u}{\partial t} = f(x, t)$$

where $f(x, t)$ is as above and $p/2 \geq q \geq 2$.

§ 2. Definitions and Main Theorem

Let $W^{m,r}(\Omega)$ be the Sobolev space

$$\{ u(x) | D^\alpha u(x) \in L^r(\Omega), |\alpha| \leq m \}$$

with norm

$$\|u\|_{m,r} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x)|^r \, dx \right)^{1/r}$$

where $D_j = \frac{\partial}{\partial x_j}$, $D_\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. $W^{m,r}(\Omega)$ is a separable Banach space for $1 < p < \infty$. $W_0^{m,r}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ (the space of all $C^\infty$-functions in $\Omega$ with compact support) in $W^{m,r}(\Omega)$. $C^1(\mathbb{R}^1)$ is defined as the set of all periodic functions in $C^1(\mathbb{R}^1)$ of

1) This result was informed to the author by Professor Strauss.
2) Throughout the paper we assume all functions considered are real valued.
period \( \tau \). By \( <u^*, u> \) we denote the value of \( u^* \in X^* \) at \( u \in X \) for a Banach space \( X \) and its dual \( X^* \). We denote by \( L^p(\tau; X) \) the Banach space of functions \( f \) which are in \( L^p \) over any \( I_\tau = [t, t+\tau] \) with values in \( X \) and

\[
f(t) = f(t+\tau) \quad \text{in } X \text{ for all } t \in \mathbb{R}^1,
\]

provided with the norm \( (1 \leq p < \infty) \)

\[
\|f\|_{L^p(\tau; X)} = \left\{ \int_I |f(t)|^p \, dt \right\}^{1/p}.
\]

As for \( L^\infty(\tau; X) \), the usual modification is needed. The \( L^p(\Omega) \) norm is denoted by \( \|\cdot\|_p \), especially by \( \|\cdot\| \) for \( p = 2 \).

**Assumption A.** Let \( A \) be a (non-linear) differential operator in \( \Omega \), given in divergence form

\[
Au(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^\alpha u)
\]

where \( D^\alpha u = \{D^\alpha u\}_{|\alpha| = k} \), and the following conditions are imposed on \( A_\alpha \):

i) each \( A_\alpha(x, \xi) \) is a continuous function of \( (x, \xi) \) (\( \xi \) is a real vector corresponding to \( \{D^\alpha u\}_{|\alpha| \leq m} \));

ii) there exists a continuous function \( g_0(s) \) on \( \mathbb{R}^1 \) such that

\[
|A_\alpha(x, u(x), \ldots, D^m u(x))| \leq g_0(\|u\|_{m,p}) \left\{ \sum_{|\beta| \leq m} |D^\beta u(x)|^{p-1} + 1 \right\}
\]

for all \( u \in W^{m,p}(\Omega) \) \( (p \geq 2) \), all \( \alpha \) with \( |\alpha| \leq m \) and almost all \( x \in \Omega \);

iii) the non-linear Dirichlet form on \( W \),

\[
a(u, v) = \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, u(x), \ldots, D^\alpha u(x)) D^\alpha v(x) \, dx
\]

satisfies, for a continuous function \( g_1(s) \geq 0 \) on \( \mathbb{R}^1 \),

\[
|a(u, v)| \leq g_1(\|u\|_{m,p}) \|v\|_{m,p}, \quad u, v \in \overline{W};
\]

iv) for \( u \in \overline{W} \),

\[
a(u, u) \geq c_0(\|u\|_{m,p}) + k_0 \|u\|^2
\]

\[1\) We denote \( W^{2,p}_\Omega \) \( (\Omega) \) by \( \overline{W} \) for simplicity.
where \( c_0(s) \) is a continuous function on \( \mathbb{R}^1 \) with \( \lim_{s \to \infty} c_0(s) = \infty \) and \( k_0 \) is a positive constant;

v) \( a(u, u-v) - a(v, u-v) \geq 0, u, v \in W; \)

vi) there exists a functional \( r(u) \) on \( W \) such that

\[
(8) \quad a(\phi(t), \phi'(t)) \geq -\frac{d}{dt} r(\phi(t))
\]

for any \( \phi(t) \), a finite sum of functions \( c(t)w \) for \( c(t) \in C^1(\tau) \) and \( w \in W \), and that for \( u \in W \)

\[
(9) \quad c_1(\|u\|_{m, p}) \leq r(u) \leq k_1 a(u, u) + k_2
\]

where \( c_1(s) \) is a continuous function on \( \mathbb{R}^1 \) with \( \lim_{s \to \infty} c_1(s) = \infty \) and \( k_1, k_2 \) are some constants.

**Assumption B.** Let \( \beta_0(s) \) be a twice differentiable function on \( \mathbb{R}^1 \) such that for \( |\alpha| \leq m-1 \)

\[
0 \leq \beta'_\alpha(s^2) \leq C |s|^{q-1}, \text{ in particular, } \varepsilon_0 \leq \beta'_\alpha(s^2) (|\alpha| = 1)
\]

\[
|\beta''_\alpha(s^2)| \leq C |s|^{q-3}
\]

where \( 2 \leq q \leq p/2, C, \varepsilon_0 \) are constants >0.

Now our theorem is stated as follows.

**Theorem.** Given \( f(t) \in L^2(\tau; W) \) (not zero), there exists a solution \( u \in L^\infty(\tau; W) \) of the equation (1):

\[
u_{tt} + A u + B(u, u_t) = f,
\]

such that \( u_t \in L^2(\tau; L^2(\Omega)), u_{tt} \in L^p(\tau; W^*) \) and that

\[
B(u, u_t) = \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^\alpha \beta'_\alpha(|D^\alpha u|^2) D^\alpha u_t
\]

where \( 1/p + 1/p' = 1 \), \( A \) and \( B \) satisfy Assumption A and Assumption B, respectively.

§3. **Proof of the Theorem**

We shall prove the theorem by means of Faedo-Galerkin's method
combined with the fixed point theorem and the compactness method. Since \( W \) is separable, there exists a countable basis \( \{w_n\} \) in \( W \) which is orthonormal in \( L^2(\Omega) \). Let \( W_n \) be the subspace of \( W \) spanned by \( w_1, \ldots, w_n \).

Consider the ordinary differential system in \( W_n \)

\[
(\mathbf{x}_n(t), \mathbf{y}_n(t)) = (f'(t), f''(t)) \quad (j=1, 2, \ldots, n)
\]

with periodic conditions

\[
\mathbf{u}_1(t) = \mathbf{u}_1(t + \tau), \quad \mathbf{u}_1(t) = \mathbf{u}_1(t + \tau)
\]

where

\[
b(\mathbf{u}_n(t), \mathbf{y}_n(t); \mathbf{w}_j) = (\mathbf{B}(\mathbf{u}_n(t), \mathbf{y}_n(t)), \mathbf{w}_j)
\]

\[
= (\beta'(\mathbf{u}_n(t)) u_n(t), \mathbf{w}_j) + (u_n(t), \mathbf{w}_j).
\]

The solutions will be of the form

\[
\mathbf{u}_n(t) = \sum_{k=1}^{n} c_{n,k}(t) w_k, \quad c_{n,k}(t) \in C^1(\tau)
\]

if they exist. Now the substitution of the \( \mathbf{u}_n(t) \) into (10), (11) gives the second order differential system of \( \mathbf{C}_n(t) = (c_{n1}(t), \ldots, c_{nn}(t))^* \),

\[
(\mathbf{C}_n(t) + \mathbf{F}(\mathbf{C}_n(t)) + \mathbf{H}(\mathbf{C}_n(t)), \quad \mathbf{C}_n(t)) = \mathbf{H}_0(t)
\]

and the periodic conditions

\[
\mathbf{C}_n(t) = \mathbf{C}_n(t + \tau), \quad \mathbf{C}_n(t) = \mathbf{C}_n(t + \tau),
\]

where

\[
\mathbf{F}(\mathbf{C}_n(t)) = (F_1(\mathbf{C}_n(t)), \ldots, F_n(\mathbf{C}_n(t)))^*,
\]

\[
F_j(\mathbf{C}_n(t)) = a(\mathbf{u}_n(t), \mathbf{w}_j);
\]

\[
\mathbf{H}(\mathbf{C}_n(t), \mathbf{C}_n(t)) = (H_1(\mathbf{C}_n(t)), \ldots, H_n(\mathbf{C}_n(t)))^*,
\]

1) * denotes the transpose operation of \( n \)-vector.
Lemma 1. There exists a solution $C_n(t)$ of (13), (14).

Proof. Consider a system of $\lambda$ dependence $(0 \leq \lambda \leq 1)$,

$$\begin{equation}
H_j(C_n(t), C'_n(t)) = b(u^a(t), u^a; w_j);
\end{equation}$$

$$\begin{equation}
\mathcal{H}_0(t) = (f_1(t), \ldots, f_n(t))^*, f_j(t) = (f(t), w_j) \quad (j = 1, 2, \ldots, n)
\end{equation}$$

(15) \quad \begin{align*}
C''_n(t) + \delta C'_n(t) + k C_n(t) \\
= \lambda \left\{ - F(C_n(t)) + k C_n(t) - \mathcal{H}(C_n(t), C'_n(t)) + \delta C'_n(t) \right\} \\
+ \mathcal{H}_0(t)
\end{align*}

\[
\text{together with (14). Here } \delta \text{ and } k \text{ are any fixed constants such that } 0 < \delta < \delta_0, 0 < k < k_0 \text{ where } \delta_0 \text{ is a constant satisfying } \delta_0 \|u\| \leq \|u\|_{1,2} \text{ for any } u \in H^1(\Omega), k_0 \text{ a constant in Assumption A-(iv). \ Let } G_n(t, s) \text{ be a unique Green's function of (15) for } \lambda = 0, \mathcal{H}_0 = 0 \text{ with conditions (14), and define the operator } T_n(\lambda) \text{ from a Banach space } X_n \text{ into itself:}
\end{align*}

$$\begin{equation}
T_n(\lambda) = \int_{t_r}^t \left\{ \lambda \left\{ - F(C_n(s)) + k C_n(s) - \mathcal{H}(C_n(s), C'_n(s)) + \delta C'_n(s) \right\} \\
+ \mathcal{H}_0(s) \right\} G_n(t, s) \, ds
\end{equation}$$

where

$$X_n = C^1(\tau) \times \cdots \times C^1(\tau) \quad (n \text{-copies})$$

with norm

$$\|C_n\|_{X_n} = \sup_{t_r} \{|C_n(t)| + |C'_n(t)|\} \quad (|\cdot|: \text{the length of } n\text{-vector}) \text{ for } C_n \in X_n.$$

To prove the lemma it suffices to show that the operator $T_n(1)$ has a fixed point in $X_n$. So we apply Leray-Schauder's theorem to the family of operators $T_n(\lambda) \ (0 \leq \lambda \leq 1)$ on the space $X_n$. We observe that

$$\begin{equation}
\|F(C_n^{(i)}) - F(C_n)\|_\infty \to 0
\end{equation}$$

and

$$\begin{equation}
\|H(C_n^{(i)}, C'_n^{(i)}) - H(C_n, C'_n)\| \to 0
\end{equation}$$
when $C_\nu \to C$ in $X$, as $\nu \to \infty$. In fact, (17) is a direct consequence of Assumption A on $A_\alpha$ by measure theoretical arguments. To show (18) we put

$$u^{(\nu)}(x, t) = \sum_{k=1}^\infty c_k^{(\nu)}(t) w_k(x)$$

$$u(x, t) = \sum_{k=1}^\infty c_k(t) w_k(x),$$

dropping the suffix $n$ for brevity in notation. Assumption B implies

$$|\beta'_\alpha(|u^{(\nu)}(x, t)|^2) - \beta'_\alpha(|u(x, t)|^2)|$$

$$\leq C |u^{(\nu)}(x, t) - u(x, t)||u^{(\nu)}(x, t)| + |u(x, t)||^{q-2}$$

for all $t$ and almost all $x \in \Omega$. Since $||C^{(\nu)}||_w, ||C^{(\nu)}||_m$ are bounded on $\nu$, we obtain by Hölder's inequality

$$|(u_t^{(\nu)}(x, t) - \beta'_\alpha(|u^{(\nu)}(x, t)|^2)) w_j)|$$

$$\leq C ||u_t^{(\nu)}(x, t)|| w_j$$

$$\leq C ||u^{(\nu)}(t)|| w_j$$

for some constant $K$. Similarly we have

$$|((u_t(x, t) - u_t^{(\nu)}(x, t)) \beta'(|u(x, t)|^2), w_j)| \leq K ||C^{(\nu)} - C||_{X_n}.$$  

Hence $(u_t^{(\nu)}(t) \beta'_\alpha(|u^{(\nu)}(t)|^2), w_j) \to (u_t(t) \beta'_\alpha(|u(t)|^2), w_j)$ uniformly on $t$ as $\nu \to \infty$, which implies (18). Thus the continuity of $T_n(\lambda)$ on $X$ follows immediately from (16), (17) and (18).

Next, let $S = \{C \in X_n \mid ||C|| \leq 1\}$. Then the properties of $G_n(t, s)$ imply that for each $\lambda$, $T_n(\lambda) S$ is bounded in $X$ and is a set of equi-continuous functions, and that

$$||(T_n(\lambda_2) - T_n(\lambda_1))/X|| \leq K |\lambda_2 - \lambda_1|$$

for a suitable constant $K$. Therefore each $T_n(\lambda)$ is a compact operator from $X$ into $X$ and the family \{ $T_n(\lambda) \mid 0 \leq \lambda \leq 1$ \} is homotopic. We note that the topological degree of $T_n(0)$ is $+1$ since the system (15) for $\lambda = 0$ has a unique solution in $X$. In order to see that the topological degree
of \( T_n(\lambda) \) is +1 (positive) it only remains to show that for each \( \lambda \)

\[
C(t) = T_n(\lambda)C(t) \Rightarrow \|C\|_{X_n} \leq L
\]

where \( L \) is a constant independent of \( \lambda \). The proof of (19) is a variant of that of the following lemma, and is omitted. Q.E.D.

**Lemma 2.** The solutions \( u^n \) of (10), (11) have the following estimates:

\[
\sum_{|\alpha| \leq 1} \int_{I_{\tau}} \|D^\alpha u^n(t)\|^2 dt \leq K_1, \tag{20}
\]

\[
\|u^n(t)\|, \|u^n(t)\|_{m,p} \leq K_2 \tag{21}
\]

where \( K_1, K_2 \) are constants independent of \( n \) and \( t \).

**Proof.** Since both sides of (10) are linear on \( w_j \), we have, replacing \( w_j \) by \( u^n \),

\[
(u^n(t), u^n(t)) + a(u^n(t), u^n(t)) + b(u^n(t), u^n(t)) = (f(t), u^n(t)).
\]

Integrating both sides over \( I_{\tau} \) with respect to \( t \) and using Assumptions A-(vi), B we obtain

\[
\sum_{|\alpha| \leq 1} \int_{I_{\tau}} \|D^\alpha u^n(t)\|^2 dt \leq \left( \int_{I_{\tau}} \|f(t)\|^2 dt \right)^{1/2} \left( \int_{I_{\tau}} \|u^n(t)\|^2 dt \right)^{1/2},
\]

from which the estimates (20) follows immediately.

Replacing \( w_j \) by \( u^n \) in (10) gives

\[
(u^n(t), u^n(t)) + a(u^n(t), u^n(t)) + b(u^n(t), u^n(t); u^n(t)) = (f(t), u^n(t)). \tag{22}
\]

We remark that

\[
\int_{I_{\tau}} b(u^n(t), u^n(t); u^n(t)) dt = 0
\]

because of the periodicity of \( u^n(t) \). Then, integrating (22) over \( I_{\tau} \) with respect to \( t \) and using Assumption A-(iv) we have
\[(23) \quad k_0 \int_{I_\varepsilon} \|u^n(t)\|^2 \, dt + \int_{I_\varepsilon} c_0(\|u^n(t)\|_{m,p}) \, dt \leq \int_{I_\varepsilon} \|u^n(t)\|^2 \, dt + \left( \int_{I_\varepsilon} \|f(t)\|^2 \, dt \right)^{1/2} \left( \int_{I_\varepsilon} \|u^n(t)\|^2 \, dt \right)^{1/2}.
\]

This yields that
\[(24) \quad \int_{I_\varepsilon} \|u^n(t)\|^2 \, dt, \quad \int_{I_\varepsilon} a(u^n(t), u^n(t)) \, dt
\]

have a bound independent of \(n\), because \(\int_{I_\varepsilon} c_0(\|u^n(t)\|_{m,p}) \, dt\) is bounded from below on \(n\).

Finally substituting \(u^n\) for \(w_j\) in (10) and integrating both sides from \(s\) to \(t\) (\(s < t\)), we obtain by Assumption A-(vi),
\[(25) \quad \frac{1}{2} \|u^n(t)\|^2 + c_1(\|u^n(t)\|_{m,p}) \leq K + \frac{1}{2} \|u^n(s)\|^2 + k_1 a(u^n(s), u^n(s)) + k_2
\]

where \(K\) is a constant dependent on \(f\) and \(K_1\) in (20). Further, integrating both sides of (25) with respect to \(s\) from \(t - \tau\) to \(t\) and noting that the right hand side is bounded by virtue of (24), we know that
\[\|u^n(t)\|, \quad c_1(\|u^n(t)\|_{m,p}) \leq K \quad \text{(independent of \(n\) and \(t\)).}
\]

Since \(\lim_{s \to \infty} c_1(s) = \infty\), this proves the lemma. Q.E.D.

Now we may infer that
\[\{u^n\} \text{ is bounded in } L^\infty(\tau; W),
\]
\[\{u^n\} \text{ is bounded in } L^2(\tau; H^1(\Omega)) \cap L^\infty(\tau; L^2(\Omega)).
\]

Then we may extract a subsequence \(\{u^\nu\}\) such that
\[u^\nu \to u \quad \text{(an element of } L^\infty(\tau; W)) \text{ weakly in } L^\infty(\tau; W),
\]
\[u^\nu \to u \quad \text{strongly in } L^2(\tau; L^2(\Omega));
\]
in addition,
$u^* \to u$ strongly in $L^p(\Omega \times I_\tau)$

where we have used that the injection mappings

\[ i: W^{k,p}(\Omega) \to W^{k-1,p}(\Omega) \]

\[ j: H^1(\Omega) \to L^2(\Omega) \]

are compact.

Making use of these results, we shall prove:

**Lemma 3.** For any $v \in L^2(\tau; W)$

\[
\int_{I_\tau} b(u^*, u_t^*; v) dt \to \int_{I_\tau} b(u, u_t; v) dt.
\]

**Proof.** By definition, we have

\[
b(u^*, u_t^*; v) - b(u, u_t; v)
\]

\[
= (u_t^* - u_t, \beta_0'(\|u\|^2)v) + (u_t^*, (\beta_0'(\|u^*\|^2) - \beta_0'(\|u\|^2))v)
\]

Assumption B implies

\[
\|\beta_0'(\|u(t)\|^2)v(t)\| \leq C\|u(t)\|^{q-1}|v(t)|.
\]

Therefore we obtain

\[
\int_\Omega |u(t)|^{2(q-1)}|v(t)|^2 dx \leq \|u(t)\|_{L^{2(q-1)}} \|v(t)\|_{L^q},
\]

which means

\[
\beta_0'(\|u(t)\|^2)v(t) \in L^2(\tau; L^2(\Omega)).
\]

As $u_t^* \to u_t$ strongly in $L^2(\tau; L^2(\Omega))$, we know that

\[
\int_{I_\tau} (u_t^* - u_t, \beta_0'(\|u\|^2)v) dt \to 0.
\]

For the second term of the right hand side of (26), since

\[
|\beta_0'(|u^*|^2) - \beta_0'(|u|^2)| \leq |u^* - u|(|u^*|^{q-2} + |u|^{q-2}),
\]

we obtain
\[
| (u_t^*, \beta_0'(|u|^2) - \beta_0(|u|^2))v |
\leq C \|u_t^*(t)\| \|u_t^*(t) - u(t)\|_{L^q} \left( \|u^*(t)\|_{L^q}^{2^{-1}} + \|u(t)\|_{L^q}^{2^{-1}} \right) \|v(t)\|_{L^q},
\]

taking boundedness of \( \|u^*(t)\|_{L^q}^{2^{-1}}, \|u_t^*(t)\| \) into consideration,
\[
\leq K \|u^*(t) - u(t)\|_{L^q} \|v(t)\|_{L^q},
\]
where \( K \) is a constant independent of \( n \) and \( t \). Therefore we have
\[
\left| \int_{I_r} (u_t^*, \beta_0'(|u|^2) - \beta_0(|u|^2))v \right| dt
\leq K \left( \int_{I_r} \|u^*(t) - u(t)\|_{L^q} \right)^{1/2} \left( \int_{I_r} \|v(t)\|_{L^q} \right)^{1/2}
\]
\[
\leq K_1 \|u^* - u\|_{L^q(I_r)} \|v\|_{L^q(I_r)},
\]
\( K_1 \) being another constant independent of \( n \). Since \( u^* \to u \) strongly in \( L^p(Q \times I_r) \), the last member in the above inequalities tends to zero. Thus we proved the lemma. Q.E.D.

Finally, to establish the remaining part of our theorem, we need the following assertion.

**Lemma 4.** There exists a subsequence \( \{\mu\} \) of \( \{v\} \) such that
\[
\int_{I_r} a(u^*(t), v(t)) dt \to \int_{I_r} a(u(t), v(t)) dt \quad (\mu \to \infty)
\]
for any \( v \in L^2(\tau; W) \).

**Proof.** Consider a linear form on \( L^2(\tau; W) \):
\[
v \to \int_{I_r} a(u(t), v(t)) dt.
\]
Since \( u \in L^\infty(\tau; W) \), Assumption A-(iii) implies that
\[
\left| \int_{I_r} a(u(t), v(t)) dt \right| \leq \int_{I_r} g_1(\|u(t)\|_{m,p}) \|v(t)\|_{m,p} dt
\]
\[
\leq K \|v\|_{L^2(\tau; W)}.
\]
Hence the linear form is continuous on \( L^2(\tau; W) \), so that there is an
element \( \mathcal{A} u \in L^2(\tau; W^*) \) such that
\[
\int_{I_t} a(u(t), v(t)) \, dt = \langle \mathcal{A} u, v \rangle.
\]

Here \( \langle \; , \; \rangle \) denotes the pairing between \( L^2(\tau; W) \) and \( L^2(\tau; W^*) \). The operator \( \mathcal{A} \) sending \( L^2(\tau; W) \) into \( L^2(\tau; W^*) \) satisfies
\[
\| \mathcal{A} u \|_{L^2(\tau; W)} = \sup_v \left| \int_{I_t} a(u^*(t), v(t)) \, dt \right| \leq K
\]
where \( v \) in the second member runs through the set \( \{ ||v||_{L^2(\tau; W)} : 1 \} \). Thus there exist a subsequence \( \{ u_j \} \subseteq \{ v \} \) and an element \( \xi \in L^2(\tau; W^*) \) such that for \( v \in L^2(\tau; W) \)
\[
\int_{I_t} a(u^*(t), v(t)) \, dt \rightarrow \langle \xi, v \rangle
\]
where \( \xi \) is an element of \( L^2(\tau; W^*) \). We assert \( \mathcal{A} u = \xi \). Take any \( \varphi \), a finite sum of \( c_k(t)w_k(x) \) where \( c_k \in C^1(\tau), w_k \in W \). Then for large \( n \) hold the equalities:
\[
- \int_{I_t} (u^*(t), \varphi(t)) \, dt + \int_{I_t} a(u^*(t), \varphi(t)) \, dt
\]
\[
+ \int_{I_t} b(u^*(t), u(t); \varphi(t)) \, dt = \int_{I_t} (f(t), \varphi(t)) \, dt.
\]

Letting \( \mu \rightarrow \infty \), we have
\[
(27) \quad - \int_{I_t} (u(t), \varphi(t)) \, dt + \langle \xi, \varphi \rangle + \int_{I_t} b(u(t), u(t); \varphi(t)) \, dt
\]
\[
= \int_{I_t} (f(t), \varphi(t)) \, dt.
\]

Let
\[
V(\tau; W) = \{ v \in L^2(\tau; W) \mid v_t \in L^2(\tau; L^2(\Omega)) \}
\]
with norm
\[
\| v \|_{V(\tau; W)} = \| v \|_{L^2(\tau; W)} + \| v_t \|_{L^2(\tau; L^2(\Omega))}.
\]
Since the set of the $\varphi$ defined above is dense in $V(\tau; W)$, (27) is valid for any $\varphi \in V(\tau; W)$, in particular, for $\varphi = u$. Hence,

$$-\int_{\tau} \|u_\tau(t)\|^2 \, dt + \langle \xi, u \rangle + \int_{\tau} b(u(t), u_\tau(t); u(t)) \, dt = \int_{\tau} (f(t), u(t)) \, dt.$$  

However, we observe that

$$\int_{\tau} (u, \beta_0^\tau(|u|^2), u) \, dt = \lim_{\tau \to 0} \int_{\tau} (\tau^\gamma \beta_0^\tau(|u|^2), u^\tau) \, dt = 0,$$

from which follows

$$-\int_{\tau} \|u_\tau\|^2 \, dt + \langle \xi, u \rangle = \int_{\tau} (f, u) \, dt. \quad (28)$$

Since

$$-\int_{\tau} \|u_\tau^\tau\|^2 \, dt + \int_{\tau} a(u^\tau, u^\tau) \, dt = \int_{\tau} (f, u^\tau) \, dt,$$

taking the limit inferior of both sides, and recalling that $u_\tau^\tau \to u_\tau$ strongly in $L^2(\tau; L^2(\Omega))$ we obtain that

$$-\int_{\tau} \|u_\tau\|^2 \, dt + \lim_{\tau \to 0} \langle \mathcal{A} u^\tau, u^\tau \rangle \leq \int_{\tau} (f, u) \, dt. \quad (29)$$

Comparing (29) with (28) yields

$$\langle \xi, u \rangle \geq \lim_{\tau \to 0} \langle \mathcal{A} u^\tau, u^\tau \rangle$$

from which we can conclude in the same way as in [5] that $\hat{\xi} = \mathcal{A} u$. Q.E.D.

Now we observe in the proof of Lemma 3 that $B(u, u_\tau) \in L^2(\tau; W^*)$, so that for any smooth function $\varphi(x, t)$ periodic in $t$, we have in the distributional sense that
\[
\int_{I_t} (u_{tt}, \varphi) dt + \int_{I_t} (A u, \varphi) dt + \int_{I_t} (B(u, u_t), \varphi) dt
\]
\[
= \int_{I_t} (f, \varphi) dt,
\]
\[
u_{tt} = -A u - B(u, u_t) + f \in L^2(\tau; \mathcal{W}^*),
\]
which completes the proof of our theorem, for 
\[
B(u, u_t) = \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^{\alpha} \beta'_\alpha(|D^{\alpha} u|^2) D^{\alpha} u_t,
\]
we need some modifications. Consider the system (15) for \(\varepsilon_0\delta\) instead of \(\delta\) and for
\[
b(u^n, u^n_t, w_j) = \sum_{|\alpha| \leq m-1} (\beta'_\alpha(|D^{\alpha} u^n|^2) D^{\alpha} u^n, D^{\alpha} w_j).
\]
Then we can obtain the estimates
\[
\sum_{|\alpha| \leq m-1} \int_{I_t} ||D^{\alpha} u^n||^2 dt \leq K_1
\]
as in the proof of Lemma 2. Also we know by (20)
\[
\sum_{|\alpha| \leq m-1} \int_{I_t} ||D^{\alpha} u^n||^2 dt \leq K_2.
\]
Therefore we may choose a further subsequence \(\{\sigma\}\) of \(\{\mu\}\) such that when \(\sigma \to \infty\),
\[
D^{\alpha} u^n \to D^{\alpha} u_t \quad \text{weakly in } L^2(\Omega \times I_t),
\]
and
\[
D^{\alpha} u^n \to D^{\alpha} u \quad \text{strongly in } L^p(\Omega \times I_t),
\]
both for \(|\alpha| \leq m-1\). Since, for \(v \in L^p(\tau; \mathcal{W})\),
\[
b(u^n, u^n_t; v) - b(u, u_t; v)
\]
\[
= \sum_{|\alpha| \leq m-1} (D^{\alpha} u^n t - D^{\alpha} u, \beta'_\alpha(|D^{\alpha} u|^2) D^{\alpha} v)
\]
\[
+ \sum_{|\alpha| \leq m-1} (D^{\alpha} u^n t (\beta'_\alpha(|D^{\alpha} u^n|^2) - \beta'_\alpha(|D^{\alpha} u|^2)), D^{\alpha} v),
\]
to prove Lemma 3 for \(v \in L^p(\tau; \mathcal{W})\) it is enough to show that for each \(\alpha\)
as \( \sigma \to \infty \).

The first assertion is obvious because of (32) and \( \beta_a'(\|D^a u\|^2)D^a v \in L^2(\Omega \times I_r) \).

For the second one, we can show as in the proof of Lemma 3 that

\[
\left| \int_{I_r} (D^a u_i^r(\beta_a'(\|D^a u\|^2) - \beta_a'(\|D^a u\|^2), D^a v) dt \right|
\]

\[
\leq C \|D^a u_i^r\|_{L^2(\Omega \times I_r)} \|D^a u\|_{L^2(\Omega \times I_r)}
\]

\[
\times \{ \|D^a u\|_{L^2(\Omega \times I_r)}^2 + \|D^a v\|_{L^2(\Omega \times I_r)}^2 \} \|D^a v\|_{L^2(\Omega \times I_r)}.
\]

from which (34) follows by virtue of (31), (33). Since Lemma 4 holds for \( v \in L^p(\tau; W) \) we have completed the proof of the theorem.

**Example 1.** Define \( A_\alpha, a(u, v) \) by

\[
A_\alpha(x, u, \ldots, D^m u) = |D^a u|^{p-2} D^a u
\]

and

\[
a(u, v) = \int_{\Omega} \sum_{|\alpha| \leq m} |D^a u|^{p-2} D^a u D^a v dx,
\]

respectively. It can be easily seen that \( A_\alpha \) and \( a(u, v) \), then, satisfy Assumption A. Hence an evolution equation

\[
\frac{\partial^2 u}{\partial t^2} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (|D^a u|^{p-2} D^a u) - \bar{A} \frac{\partial u}{\partial t} + |u|^{q-1} \frac{\partial u}{\partial t} = f(x, t)
\]

has a solution \( u(x, t) \) in \( L^\infty(\tau; W_{0}^{m, p}(\Omega)) \) provided \( 2 \leq q \leq p/2 \) and \( f(x, t) \in L^2(\tau; L^2(\Omega)) \).

**Example 2.** Let \( A \) be the operator defined in Example 1. Then an evolution equation

\[
\frac{\partial^2 u}{\partial t^2} + Au + \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} |D^a u|^{q-1} D^a \frac{\partial u}{\partial t} - \bar{A} \frac{\partial u}{\partial t} = f(x, t)
\]
has a solution $u(x, t)$ in $L^\infty(\tau; W^{q,p}_2(\mathcal{O}))$ provided $2 \leq q \leq p/2$ and $f(x, t) \in L^2(\tau; L^2(\mathcal{O}))$.

References