Propagation of the Irregularity of a Microdifferential System

By

Teresa Monteiro Fernandes

Abstract

We construct the functor of microlocal analytic irregularity $I\mu\text{hom}(\cdot,\mathcal{O}_X)$ which gives a natural third term of a distinguished triangle associated to the transformation $t\mu\text{hom}(\cdot,\mathcal{O}_X)\to\mu\text{hom}(\cdot,\mathcal{O}_X)$ of functors on the derived category of $\mathbb{R}$-constructible sheaves. When restricting to $\mathbb{C}$-constructible objects we prove that the microlocal irregularity of a microdifferential system propagates along non 1-microcharacteristic directions, as a consequence of the propagation for $t\mu\text{hom}(\cdot,\mathcal{O}_X)$ and $\mu\text{hom}(\cdot,\mathcal{O}_X)$.

Introduction

In this paper we treat a problem posed by P. Schapira: to show that the 1-microcharacteristic variety of a microdifferential system $\mathcal{M}$ along an involutive submanifold $V$ contains the microsupport of its solutions in the sheaves $t\mu\text{hom}(F,\mathcal{O}_X)$, $\mu\text{hom}(F,\mathcal{O}_X)$, whenever the microsupport of the $\mathbb{R}$-constructible complex $F$ is contained in $V$. Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on the complex manifold $X$. In other words, the microlocal $F$-irregularity of $\mathcal{M}$ propagates along the non 1-microcharacteristic directions. Let us recall that $\mu\text{hom}(\cdot,\mathcal{O}_X)$ and $t\mu\text{hom}(\cdot,\mathcal{O}_X)$ are the microlocalized of the functors $\mathbb{R}\text{Hom}(\cdot,\mathcal{O}_X)$ and $\mathcal{H}\text{om}(\cdot,\mathcal{O}_X)$, and were respectively introduced by Kashiwara-Schapira (cf. [K-S3]) and Andronikof (cf. [A]).


2000 Mathematics Subject Classification(s): 58J15, 58J47, 35A21, 35A27.

Key words: Microsupport, irregularity, 1-microcharacteristic variety, microdifferential systems.

Work supported by FCT, project PRAXIS/2/2.1/MAT/125/94, FEDER and PRAXIS XXI.

*Departamento de Matemática da Universidade de Lisboa Centro de Álgebra/UL, Complexo II 2, Av. Prof. Gama Pinto 1649-003 LISBOA Codex Portugal.

e-mail: tmf@ptmat.inesfc.ul.pt
At the present status of microlocal analysis, it is not clear if the sheaf \( \mathcal{E}_X \) of microdifferential operators acts on \( \mu \text{hom}(F, \mathcal{O}_X) \) and \( t\mu \text{hom}(F, \mathcal{O}_X) \) but only on its cohomology; however, the work developed by Kashiwara and Schapira ([K-Sâ]) seems to indicate a positive answer. In this paper, we will treat the \( \mathbb{C} \)-constructible case, and \( F \) is supposed to be perverse.

When \( F = \mathcal{C}_Y \) for a complex \( d \)-codimensional submanifold, one has
\[
\mathcal{C}_Y^\infty = \gamma^{-1} \gamma_* \mu \text{hom}(\mathcal{C}_Y, \mathcal{O}_X)[d] = \gamma^{-1} \gamma_* C^\infty_{Y/X},
\]
the holomorphic microfunctions along \( Y \), where \( \gamma \) is the projection of the cotangent bundle minus the zero section on the associated projective bundle.

Similarly,
\[
C_Y^\infty = \gamma^{-1} \gamma_* t\mu \text{hom}(\mathcal{C}_Y, \mathcal{O}_X)[d] = \gamma^{-1} \gamma_* C^\infty_{Y/X}
\]
is the sheaf of microfunctions of finite order. The propagation in \( C^\infty_{Y/X} \) was proved by Kashiwara-Schapira in [K-Sâ], and the propagation in \( C_Y^\infty \) was studied by the author in [MF2], Schapira in [S] and Laurent in [L]. When we want to prove the propagation theorem for \( t\mu \text{hom}(F, \mathcal{O}_X) \), some of the essential tools developed in the preceding works are no longer available.

However, in the \( \mathbb{C} \)-constructible case we can use the theory of regular holonomic \( \mathcal{D} \)-modules, and in particular Kashiwara’s theorem which asserts that \( t\mathcal{H}om(F, \mathcal{O}_X) \) has regular holonomic cohomology; moreover \( \mu \text{hom}(F, \mathcal{O}_X) \) and \( t\mu \text{hom}(F, \mathcal{O}_X) \) are obtained from \( t\mathcal{H}om(F, \mathcal{O}_X) \) tensorizing respectively by \( \mathcal{E}^\infty_X \), the sheaf of microlocal holomorphic operators, and by \( \mathcal{E}^\infty_{X/Y} \), the subsheaf of tempered microlocal operators.

Using the identification of \( X \) with the diagonal of \( X \times X \), we can reduce the problem to the propagation of the solutions of \( \mathcal{M} \) in \( C^\infty_{Y/X} \) and \( C^\infty_{Y/X} \) respectively, \( Y \) an arbitrary complex submanifold, that is, \( V = T^*_Y X \), the conormal bundle to \( Y \) minus the zero section.

The second essential tools are Bony’s results concerning the propagation for solutions in the sheaf of tempered microfunctions \( C^\infty \) for operators satisfying a Levi condition ([B]) together with a precise Cauchy-Kowalewski theorem of Kashiwara and Schapira for \( C^\infty_{Y/X} \) (cf. [K-Sâ]).

The paper is organized as follows: in the first section we construct the complex of sheaves of microlocal \( F \)-irregularity as the microlocalized of \( t\mathcal{H}om(F, \mathcal{O}_X) \) introduced in a previous work ([MF3]). Therefore, \( t\mu \text{hom}(F, \mathcal{O}_X) \) provides a natural third term to a distinguished triangle associated to the morphism
\[
t\mu \text{hom}(F, \mathcal{O}_X) \rightarrow \mu \text{hom}(F, \mathcal{O}_X)
\]
and represents the notion of \( F \)-irregularity: for example, when \( F = \mathbb{C}_Y \) for a submanifold \( Y \) of codimension \( d \), \( \mu \text{hom}(\mathcal{O}_Y(-d), \mathcal{O}_X) \) is nothing but the quotient \( \mathcal{C}^*_X(\mathcal{O}_X) \).

The second section is devoted to the propagation theorem in the \( \mathbb{C} \)-constructible perverse framework, and its successive reductions, for \( t\mu \text{hom}(F, \mathcal{O}_X) \), \( \mu \text{hom}(F, \mathcal{O}_X) \) and hence for \( I\mu \text{hom}(F, \mathcal{O}_X) \).

We are very happy to thank P. Schapira and M. Kashiwara for their useful suggestions, and V. Colin for her expertise on Andronikof’s work.

§1. Construction of \( I\mu \text{hom}(F, \mathcal{O}_X) \)

We shall recall some properties of the objects \( t\mu \text{hom}(F, \mathcal{O}_X) \) and \( \mu \text{hom}(F, \mathcal{O}_X) \), where \( X \) is an \( n \)-dimensional complex analytic manifold, \( \mathcal{O}_X \) is the sheaf of holomorphic functions and \( F \) is an object of \( \mathcal{D}_{\mathbb{C}}^b(X) \), that is a complex of sheaves of \( \mathbb{C} \)-vector spaces with bounded and \( \mathbb{R} \)-constructible cohomology. Recall that \( \mathcal{D}_{\mathbb{C}}^b(X) \) denotes the subcategory of \( \mathcal{D}_{\mathbb{C}}^b(X) \) whose objects are the \( \mathbb{C} \)-constructible complexes. The functor \( \mu \text{hom} \) was introduced in the 80\textsuperscript{ies} by Kashiwara and Schapira ([K-S3]), and the tempered version \( t\mu \text{hom} \) was introduced by Andronikof ([A]). We also recall some facts about the functor \( t\mathcal{H} \text{om} \) due to Kashiwara ([K]) which can be recovered by the restriction of \( t\mu \text{hom} \) to the base \( X \) of the cotangent bundle \( T^*X \to X \).

Let \( \mathcal{D}_X \) (resp. \( \mathcal{D}_X^\infty \)) be the sheaf on \( X \) of holomorphic differential operators of finite order (resp. infinite order), \( \mathcal{E}_X \) (resp. \( \mathcal{E}_X^\infty \)) the sheaf of microdifferential operators of finite (resp. infinite) order, \( \mathcal{E}_X^{\mathbb{R}^f} \) (resp. \( \mathcal{E}_X^\mathbb{R} \)) the sheaf of tempered microlocal operators (resp. microlocal operators).

\( \mathcal{E}_X \), \( \mathcal{E}_X^\infty \), as well as \( \mathcal{E}_X^{\mathbb{R}^f} \) and \( \mathcal{E}_X^\mathbb{R} \) are sheaves on \( T^*X \), satisfying

\[
\begin{align*}
\mathcal{E}_X^{\mathbb{R}^f} \mid_x &= \mathcal{E}_X \mid_x = \mathcal{D}_X \\
\mathcal{E}_X^\mathbb{R} \mid_x &= \mathcal{E}_X^\infty \mid_x = \mathcal{D}_X^\infty
\end{align*}
\]

and they are all particular cases of \( \mu \text{hom} \) and \( t\mu \text{hom} \). One denotes \( \mathcal{E}_X(m) \) (resp. \( \mathcal{D}_X(m) \)) the sheaf of operators of order at most \( m \).

For instance, when \( F = \mathbb{C}_Y \), for a smooth complex submanifold of \( X \) of codimension \( d \), \( \mu \text{hom}(\mathcal{C}_Y, \mathcal{O}_X) \) is the sheaf \( \mathcal{C}^\mathbb{R}_{Y/X}[-d] \) of [S-K-K] and \( t\mu \text{hom}(\mathcal{C}_Y, \mathcal{O}_X) \) is the sheaf \( \mathcal{C}^{\mathbb{R}^f}_{Y/X}[-d] \) defined in [A].

When \( F = (\mathbb{C}_M)^\dual \), the dual of \( \mathbb{C}_M \), for a real analytic submanifold \( M \) of \( X \) such that \( X \) is a complexified of \( M \), \( \mu \text{hom}((\mathbb{C}_M)^\dual, \mathcal{O}_X)[n] \) is isomorphic to the sheaf of microfunctions on \( M \) and \( t\mu \text{hom}((\mathbb{C}_M)^\dual, \mathcal{O}_X)[n] \) is isomorphic to the tempered microfunctions (cf. [A] and [B]).
When restricting to $X$ one obtains

$$\mu \text{hom}(F, O_X) |_{X} \cong \mathcal{E} \text{Hom}(F, O_X),$$

a complex of $\mathcal{D}_{X}^\infty$-modules, and

$$t\mu \text{hom}(F, O_X) |_{X} \cong t\mathcal{E} \text{Hom}(F, O_X),$$

a complex of $\mathcal{D}_{X}$-modules.

One also defines $I \mathcal{E} \text{Hom}(F, O_X)$, the sheaf of analytic local $F$-irregularity which appears to be a natural third term to a distinguished triangle

$$t\mathcal{E} \text{Hom}(F, O_X) \rightarrow \mathcal{E} \text{Hom}(F, O_X) \rightarrow I \mathcal{E} \text{Hom}(F, O_X) \rightarrow$$

where the left arrow is the usual one. In particular, when $F = (\mathbb{C}_{M})^\dagger$ one obtains

$$I \mathcal{E} \text{Hom}(F, O_X) = \frac{B_{M}}{Dh_{M}},$$

where $B_{M}$ is the sheaf of hyperfunctions on $M$ and $Dh_{M}$ is the sheaf of distributions on $M$; and when $F = \mathbb{C}_{Y}, Y$ a $d$-codimensional submanifold,

$$I \mathcal{E} \text{Hom}(\mathbb{C}_{Y}[-d], O_X) = \frac{B_{Y1X}^\infty}{B_{Y1X}}$$

where $B_{Y1X}^\infty$ (resp. $B_{Y1X}$) denotes the sheaf of holomorphic hyperfunctions (resp. holomorphic hyperfunctions with finite order) along $Y$.

Let us recall that $I \mathcal{E} \text{Hom}(F, O_X)$ has $\mathcal{D}_{X}$-module cohomology.

Let now $\Delta$ be the diagonal of $X \times X$ and $\tau : T_{\Delta}(X \times X) \rightarrow \Delta$ be the projection of the normal bundle to $\Delta$. Let $\bar{X}$ be the complex normal deformation of $X \times X$ along $\Delta$ ($\Delta$ identified with $X$ by the first projection $p_{1} : X \times X \rightarrow X$), let $p : \bar{X} \rightarrow X \times X$ be the deformation morphism, let $t : \bar{X} \rightarrow \mathbb{C}$ be the natural projection, let $\bar{\Omega} = t^{-1}(\mathbb{R}^+)$, and let $\Omega \hookrightarrow \bar{X}$ and $T_{\Delta}(X \times X) \rightarrow \bar{X}$ be the natural inclusions. (For more details we refer [A] (Prop. 3.2.1) and [K-S3].) Let $p_{2}$ denote the second projection of $X \times X$ on $X$. We have a distinguished triangle

$$t\mathcal{E} \text{Hom}((p_{1}^{!}p_{2}^{-1}F)_{\Omega}, O_{X}) \rightarrow \mathcal{E} \text{Hom}((p_{1}^{!}p_{2}^{-1}F)_{\Omega}, O_{X}) \rightarrow I \mathcal{E} \text{Hom}((p_{1}^{!}p_{2}^{-1}F)_{\Omega}, O_{\bar{X}}) \rightarrow$$

Following the constructions in ([A], Lemme 2.1.8) we apply the functor

$$s^{-1}\left(\mathcal{D}_{X \times \bar{X}} \otimes_{\mathcal{D}_{X}} \mathbb{C}_{\bar{X}}\right).$$
to (1) and set

\[ I\hom(F, \mathcal{O}_X) := \]

\[ D_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} s^{-1} \left( \mathcal{T}_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} \mathcal{G} \mathcal{H}(\mathcal{O}_X) \right) \]

\[ \nu\hom(F, \mathcal{O}_X) := \]

\[ D_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} s^{-1} \left( \mathcal{T}_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} \mathcal{G} \mathcal{H}(\mathcal{O}_X) \right) \]

\[ t\hom(F, \mathcal{O}_X) := \]

\[ D_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} s^{-1} \left( \mathcal{T}_{X \times X}^{\tau^{-1}} \mathcal{T}_{X \times X} \mathcal{G} \mathcal{H}(\mathcal{O}_X) \right) \]

Then

\[ t\hom(F, \mathcal{O}_X) \to \nu\hom(F, \mathcal{O}_X) \to I\hom(F, \mathcal{O}_X) \to \] is a distinguished triangle in \( D^b(\tau^{-1} \mathcal{T}_X) \), the derived category whose objects are complexes of \( \tau^{-1} \mathcal{T}_X \)-modules with bounded cohomology. Finally, denoting by \( \wedge \) the Fourier transform from \( D_{\mathbb{R}^+}^b(\mathcal{T}_X(X \times X)) \to D_{\mathbb{R}^+}^b(\mathcal{T}_X^*(X \times X)) \), we define \( I\mu\hom(F, \mathcal{O}_X) \) by

\[ I\mu\hom(F, \mathcal{O}_X) := I\nu\hom(F, \mathcal{O}_X)^\wedge. \]

Here \( D_{\mathbb{R}^+}^b(E) \), where \( E \) is a real vector bundle on \( X \), denotes the derived category of complexes of sheaves on \( E \) of \( \mathbb{C} \)-vector spaces with bounded and \( \mathbb{R}^+ \)-conic cohomology.

One easily deduces the isomorphisms

\[ I\mu\hom(F, \mathcal{O}_X) \mid_X \cong \mathbb{R}\pi, I\mu\hom(F, \mathcal{O}_X) \cong I\hom(F, \mathcal{O}_X) \]

from the analogous formulae for \( t\mu\hom \) and \( \mu\hom \). Moreover,

\[ \mathbb{R}\pi I\mu\hom(F, \mathcal{O}_X) = 0 \]

by (2.3.2) of [A].

Also, as pointed out above, the correspondence

\[ F \to I\mu\hom(F, \mathcal{O}_X) \]

defines a contravariant functor from \( D_{\mathbb{R}^+}^b(X) \) to \( D^b(\tau^{-1} \mathcal{T}_X) \).

On the other hand, since \( I\mu\hom(F, \mathcal{O}_X) \) is the third term of a distinguished triangle where the other two terms are supported by the microsupport of \( F \), \( SS(F) \), \( I\mu\hom(F, \mathcal{O}_X) \) is also supported by \( SS(F) \).
Finally, we observe that the cohomology of $I\mu \text{hom}(F, \mathcal{O}_X)$ is obviously provided of a canonical structure of $\mathcal{E}_X$-modules, induced by the structure on $t\mu \text{hom}(F, \mathcal{O}_X)$ and $\mu \text{hom}(F, \mathcal{O}_X)$ (cf. (TR4) of Proposition 1.4.4 of [K-S3]).

**Definition 1.1.** For any $F \in D^b_{X,c}(X)$, $I\mu \text{hom}(F, \mathcal{O}_X)$ is the complex of microlocal analytic $F$-irregularity.

§2. Statement of the Main Theorem and Reductions

In this section we start by briefly recalling the essential results on the 1-microcharacteristic variety of a coherent $\mathcal{E}_X$-module $\mathcal{M}$ along a smooth involutive manifold $V$ of $T^*X$. Here $T^*X$ denotes the complementary of the null section $X \hookrightarrow T^*X$. The 1-microcharacteristic variety $C^1_V(\mathcal{M})$ is a conic involutive analytic subset of $T_V(T^*X)$. To build it one needs to introduce the subring $\mathcal{E}_V$ of $\mathcal{E}_X$ generated by the operators of order at most one with a principal symbol vanishing on $V$; when $P \in \mathcal{E}_V(m) := \mathcal{E}_X(m) \cap \mathcal{E}_V$, modulo $\mathcal{E}_V(m - 1)$, the symbol $\sigma^1_V(P)$ is a homogeneous function on $T_V(T^*X)$ and when $\mathcal{S}$ is a coherent ideal of $\mathcal{E}_X$, $C^1_V(\mathcal{E}_X/\mathcal{S})$ is the subset of zeros of $\sigma^1_V(\mathcal{S} \cap \mathcal{E}_V)$. For further details see [MF1], [L] and [S].

Recall that if $\eta \in T_V(T^*X)$, one says that $\eta$ is non 1-microcharacteristic for $\mathcal{M}$ along $V$ if $\eta \notin C^1_V(\mathcal{M})$. Moreover, the normal cone $C_V(\text{Car} \mathcal{M})$ is contained in $C^1_V(\mathcal{M})$. When $\mathcal{M}$ is a coherent $\mathcal{D}_X$- or $\mathcal{E}_X$-module, $\text{Car} \mathcal{M}$ will denote its characteristic variety in $T^*X$. Let $F \in D^b_{X,c}(X)$, $SS(F)$ its microsupport in $T^*X$ (see [K-S3]); recall that $\text{supp}(F) = SS(F) \cap X$. Moreover, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, $\text{Car} \mathcal{M} = SS(\mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$. Let us recall that $C_V(\text{Car} \mathcal{M}) := C_V(\mathcal{M})$ was studied in [K-S1, K-S2]. It is called the microcharacteristic variety of $\mathcal{M}$ along $V$.

Let us denote by $D^b_{\mathcal{D}_X}(\mathcal{D}_X)$ the derived category whose objects are the complexes of left $\mathcal{D}_X$-modules with bounded regular holonomic cohomology. Recall that, as proved in [K], when $F \in D^b_{\mathcal{D}_X}(X)$, there exists a unique $\mathcal{N} \in D^b_{\mathcal{D}_X}(\mathcal{D}_X)$ such that $\mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \simeq F$ and that correspondence is an equivalence of categories. More precisely, $\mathcal{N} = t\mathcal{R}\text{Hom}(F, \mathcal{O}_X)$ and by ([L], Theorem 4.2.6)

$$\mathcal{E}^i_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{N} \simeq t\mu \text{hom}(F, \mathcal{O}_X)$$

as well as

$$\mathcal{E}^i_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{N} \simeq \mu \text{hom}(F, \mathcal{O}_X),$$

(cf. [K-S4]).
Let $V$ be a smooth submanifold of a manifold $X$. One denotes $\rho_V$ the projection

$$V \times TX \to T_X V.$$ 

In most cases studied in this paper, instead of $X$ we consider the cotangent bundle $T^* X$ and $V$ will be an involutive smooth submanifold; we then get

$$\rho_V : V \times T^* X \to T_V (T^* X).$$

Here we identify $T(T^* X)$ and $T^* (T^* X)$ by $-H$, where $H$ is the Hamiltonian isomorphism. Recall that, if $(x; \xi)$ denotes a system of local canonical coordinates on $T^* X$ in a neighborhood of $p \in T^* X$, if $(x, \xi; \zeta, \eta)$ denotes the associated canonical coordinates on $T^* (T^* X)$, $-H_p(\xi dx + \eta d\xi) = (\zeta \partial / \partial x - \eta \partial / \partial x) \in T_p^*(T^* X)$

Let $\Omega_X$ denote the sheaf of holomorphic differential $n$-forms on $X$.

**Theorem 2.1.** Let $F \in D^b_{\text{coh}}(X)$ be perverse and $M$ be a coherent $\mathcal{E}_X$-module. Let $V$ be smooth involutive in $T^* X$ such that $SS(F) \subset V$. Then one has the inclusions

\begin{align*}
(3) \quad a) & \quad \rho_V (SS(\mathbb{R} \text{Hom}_{\mathcal{E}_X}(M, \mu \text{hom}(F, \mathcal{O}_X))) \subset C_V (\mathcal{M}), \\
& \quad b) \quad \rho_V (SS(\mathbb{R} \text{Hom}_{\mathcal{E}_X}(M, \iota \mu \text{hom}(F, \mathcal{O}_X))) \subset C_V^1 (\mathcal{M}), \\
& \quad c) \quad \rho_V (SS(\mathbb{R} \text{Hom}_{\mathcal{E}_X}(M, \mu \text{hom}(F, \mathcal{O}_X))) \subset C_V^1 (\mathcal{M}).
\end{align*}

**Proof.** c) It derives from a) and b).

a) and b) We may assume that $\mathcal{N} = t \text{Hom}(F, \mathcal{O}_X)$ is concentrated in degree zero. Let us identify $T^*_X (X \times X)$ with $T^* X$ and denote by $j : T^* X \hookrightarrow T^*(X \times X)$ the associated inclusion. Denote by $s$ the duality functor

$$\mathbb{R} \text{Hom}_{\mathcal{E}_X}(\cdot, \mathcal{E}_X) \otimes_{\mathcal{O}_s} \pi^{-1} \Omega_X^0 [-n].$$

Hence it is sufficient to prove that

$$\rho_V \left( SS \left( j^{-1} \mathbb{R} \text{Hom}_{\mathcal{E}_X \times \mathcal{E}_X} \left( M \boxtimes \left( \mathcal{E}_X \otimes \pi^{-1} \mathcal{N} \right) \right) \right) \right) \subset C_V (\mathcal{M}),$$

as well as with $C^\mathbb{R}_{\Delta X \times \Delta X}$ replaced by $C^\mathbb{R}_{\Delta X \times \Delta X}$, and $C_V (\mathcal{M})$ replaced by $C_V^1 (\mathcal{M})$.

Since $C^\mathbb{R}_{\Delta X \times \Delta X}$ is supported by $T^*_X (X \times X)$, $SS(j^{-1} \mathbb{R} \text{Hom}_{\mathcal{E}_X \times \mathcal{E}_X} \left( M \boxtimes \tilde{\mathcal{N}} \right) \subset C^\mathbb{R}_{\Delta X \times \Delta X} (M \boxtimes \tilde{\mathcal{N}} \subset C^\mathbb{R}_{\Delta X \times \Delta X}(\mathcal{M} \boxtimes \tilde{\mathcal{N}} \subset C^\mathbb{R}_{\Delta X \times \Delta X} \subset C_V (\mathcal{M}))$
where $\tilde{N} = \mathcal{E}_X \otimes_{\pi^*D_{X}} \pi^{-1}N$, and the inclusion holds for $C_{\mathbb{D}_{X}X \times X}^{\mathbb{S}}$ as well. Here we identify $T_{T_{\mathbb{D}_{X}X}^{*}(X \times X)}(T^*(X \times X))$ with $T^*(T^*X)$ via the identification of $T_{T_{\mathbb{D}_{X}X}^{*}(X \times X)}(T^*(X \times X))$ with $T^*(T_{\mathbb{D}_{X}X}^{*}(X \times X))$ and the identification of the last one with $T^*(T^*X)$.

Suppose we know that for any $\mathcal{E}_X$-coherent module, for any smooth submanifold $Y$ of $X$,

\begin{align*}
\rho_{T_{\mathbb{D}_{X}X}^{*}}(SS(\mathbb{R}Hom_{\mathcal{E}_{X}}(\mathcal{M}, C_{\mathbb{D}_{X}X}^{\mathbb{S}}))) & \subset C_{T_{\mathbb{D}_{X}X}^{*}}(\mathcal{M}), \\
\rho_{T_{\mathbb{D}_{X}X}^{*}}(SS(\mathbb{R}Hom_{\mathcal{E}_{X}}(\mathcal{M}, C_{\mathbb{D}_{X}X}^{\mathbb{S}, f}))) & \subset C_{T_{\mathbb{D}_{X}X}^{*}}(\mathcal{M}),
\end{align*}

where $\rho_{T_{\mathbb{D}_{X}X}^{*}}$ is, as before, the projection

$$T_{\mathbb{D}_{X}X}^{*} \times_{X} T^*(T^*X) \to T_{\mathbb{D}_{X}X}^{*}(T^*X).$$

Then, replacing $\mathcal{M}$ by $\mathcal{M} \boxtimes \tilde{N}^*$ we get

\begin{align*}
\rho_{T_{\mathbb{D}_{X}X}^{*}}(X \times X) (SS(\mathbb{R}Hom_{\mathcal{E}_{X \times X}}(\mathcal{M} \boxtimes \tilde{N}^*, C_{\mathbb{D}_{X \times X}X}^{\mathbb{S}}))) & \subset C_{T_{\mathbb{D}_{X}X}^{*}(X \times X)}(\mathcal{M} \boxtimes \tilde{N}^*) := C(\mathcal{M}, \tilde{N}), \\
\rho_{T_{\mathbb{D}_{X}X}^{*}}(X \times X) (SS(\mathbb{R}Hom_{\mathcal{E}_{X \times X}}(\mathcal{M} \boxtimes \tilde{N}^*, C_{\mathbb{D}_{X \times X}X}^{\mathbb{S}, f}))) & \subset C_{T_{\mathbb{D}_{X}X}^{*}(X \times X)}(\mathcal{M} \boxtimes \tilde{N}^*) := C_{1}(\mathcal{M}, \tilde{N}),
\end{align*}

where we identify $T^*(X \times X)$ to $T^*X$ by the first projection.

Since $SS(F)$ equals the characteristic variety of $\mathcal{N}$, we know by [K-O] that $\mathcal{N}$, being regular holonomic, is regular along $V$. We shall use the following result, which is a slight improvement of the analogous in [MF1].

**Lemma 2.2.** Let $X$ be a complex analytic manifold. Let $\mathcal{N}$ be a coherent $\mathcal{E}_{X}$-module regular along a smooth involutive submanifold $V \subset T^*X$. Then, for any coherent $\mathcal{E}_{X}$-module $\mathcal{M}$,

\begin{align*}
\rho_{V}(C(\mathcal{M}, \mathcal{N})) & \subset C_{V}(\mathcal{M}), \\
\rho_{V}(C_{1}(\mathcal{M}, \mathcal{N})) & \subset C_{1,V}(\mathcal{M}).
\end{align*}

**Proof.** The first inclusion is obvious since $C(\mathcal{M}, \mathcal{N}) = C(\text{Car}(\mathcal{M}), \text{Car}(\mathcal{N}))$ and $\text{Car}(\mathcal{N}) \subset V$.

As for the second inclusion, let us start by assuming that $V$ is regular involutive, that is, the canonical 1-form never vanishes on $V$. By [K-O], we
know that $\mathcal{N}$ is locally a quotient of some power $N$ of a coherent $\mathcal{E}_{X}$-module $\mathcal{L}_V$ supported by $V$, with simple characteristics. Hence

$$C^1(\mathcal{M}, \mathcal{N}) \subset C^1(\mathcal{M}, \mathcal{L}_V^N)$$

and then we apply Theorem 1.4.2 in [MF1].

For the general case, we use the dummy variable trick, that is, we consider local canonical coordinates $(x, \xi)$ in $T^*X$ in a neighborhood of $V$, the regular involutive submanifold of $\hat{T}^*(X \times \mathbb{C})$,

$$\hat{V} = \{(x, t; \xi, \zeta) \in \hat{T}^*X \times \hat{T}^*\mathbb{C}; (x; \xi) \in V, \zeta \neq 0\}$$

and $\hat{\mathcal{M}} := \mathcal{M} \boxtimes \mathcal{E}_C$. Since

$$C^1(\hat{\mathcal{M}}) = C^1(\mathcal{M}) \times \hat{T}^*\mathbb{C},$$

we get

$$\rho_V(C^1(\hat{\mathcal{M}}, \hat{\mathcal{N}})) \subset C^1(\hat{\mathcal{M}}) = C^1(\mathcal{M}) \times \hat{T}^*\mathbb{C}.$$ 

On the other hand,

$$\rho_V(C^1(\hat{\mathcal{M}}, \hat{\mathcal{N}})) = \rho_V(C^1(\mathcal{M}, \mathcal{N})) \times \hat{T}^*\mathbb{C},$$

hence the result. $\square$

Therefore, a) and b) of Theorem 2.1 hold provided that we prove (4).

Let $f : \hat{X} \to X$ be a smooth morphism of complex manifolds, let $\Lambda$ be a smooth involutive submanifold of $\hat{T}^*X$, let $\omega : \hat{X} \times T^*X \to T^*X$ and $f' : \hat{X} \times X \to T^*X \to T^*\hat{X}$ be the canonical morphisms. Let $W := f' \circ \omega^{-1}(\Lambda)$. Then $W$ is a smooth involutive submanifold of $\hat{T}^*X$. In this situation, we have:

**Lemma 2.3.** Let $f : \hat{X} \to X$ be a smooth analytic morphism of finite dimensional complex analytic manifolds. Let $\Lambda$ be a smooth involutive submanifold of $T^*X$ and let $W = f' \circ \omega^{-1}(\Lambda) \subset T^*X$. Set $\omega^N : T_{\omega^{-1}(\Lambda)}(T^*X \times \hat{X}) \to T_{\Lambda}(T^*X)$ and $\pi^N : T_{\omega^{-1}(\Lambda)}(T^*X \times \hat{X}) \to T_W(T^*\hat{X})$, the canonical morphisms associated to $\omega$ and $f'$.

Then:

i) $\pi^N$ is injective.

ii) Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. We have the following estimation:

$$C^1(\mathcal{M}) = \pi^N(\omega^N)^{-1}C^1(\mathcal{M}).$$
Proof. Since the statement is of local nature, we may assume that $\tilde{X} \simeq X \times Y$, that $f : \tilde{X} \times Y \to X$ is the projection, and consider local coordinates on $X \times Y$, $(x, x')$, such that $x$ are local coordinates on $X$, $x'$ are local coordinates on $Y$ and $f(x, x') = x$. Consider the associated canonical coordinates $(x, x'; \xi, \xi')$ on $T^* \tilde{X}$. We get

$$\tilde{X} \times T^* X = Y \times T^* X,$$

$$W = \{(x', \xi') \in T^* Y, \xi' = 0\} \times \Lambda = Y \times \Lambda,$$

$$T_{\nu^{-1}(\Lambda)}(\tilde{X} \times T^* X) \simeq T_Y(T^* Y) \times T_{\nu}(T^* X),$$

and, for $x' \in Y$ and $p \in T_{\nu}(T^* X)$, $\pi^N(x', p) = ((x', 0), p)$. Here we identify $Y$ to the zero sections of $T Y$, of $T'' Y$ and of $T_Y(T^* Y)$. This proves i).

As for ii), it will be enough to consider the case where $\mathcal{M}$ is of the form

$$\mathcal{M} = \mathcal{E}_J^\xi$$

with $J$ a coherent ideal of $\mathcal{E}_X$. In that case, $f^* \mathcal{M}$ is isomorphic to $\mathcal{E}_J \mathcal{D}_{x'}$ where $\mathcal{E}_J \mathcal{D}_{x'}$ denotes the ideal generated by the derivations in the $x'$ variables. Hence,

$$C^1_{\Lambda}(f^* \mathcal{M}) = \{(q, p) \in T_Y(T^* Y) \times T_{\nu}(T^* X), q \in Y, p \in C^1_{\Lambda}(\mathcal{M})\},$$

hence the result. \hfill \square

We shall now return to (4).

Lemma 2.4. Let $Y$ be a smooth submanifold of $X$. Then, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$,

(7) \hspace{1cm} a') $\rho_{f^* X}(SS(\mathcal{E}_X(\mathcal{M}, C^0_{T^* X}(\mathcal{M}))) \subset C^1_{f^* X}(\mathcal{M}).$

b') $\rho_{f^* X}(SS(\mathcal{E}_X(\mathcal{M}, C^0_{T^* X}(\mathcal{M}))) \subset C^1_{f^* X}(\mathcal{M}).$

Proof. a') It is a form of Theorem 8.2.1 of [K-S1]. This proves a) of Theorem 2.1.

b') Since the statements are local and invariant by canonical transformation, we may assume that $Y$ is a hypersurface.

Let $(x, t) = (x_1, \ldots, x_{n-1}, t)$ be local coordinates on $X$ such that $t = 0$ defines $Y$. $(x, t; \xi, \zeta)$ the associated canonical coordinates in $T^* X$ and $\gamma$ be the section

$$Y = f^* X, \quad \gamma(y) = (y; 1).$$
in the neighborhood of $0 \in Y$.

We shall regard $T^*Y(\simeq T^*(\gamma(Y)))$ as a submanifold of $T^*(T_Y^*X)$ via the composition $\ell_Y$ of the morphisms:

$$T^*Y \hookrightarrow T_Y^*X \times T^*Y \hookrightarrow T^*(T_Y^*X)$$

where the left arrow derives from the section $\gamma$ and the right arrow is the immersion associated to $T_Y^*X \to Y$.

More precisely, if $(x;\xi) \in T^*Y$, $\ell_Y(x;\xi) = (x, 1; \xi, 0) \in T^*(T_Y^*X)$. Moreover, if $H$ denotes the Hamiltonian isomorphism, $-H$ induces an isomorphism:

$$T^*(T_Y^*X) \simeq T_{T_Y^*X}(T^*X)$$

(see (6.2.2) and (6.2.3) of [K-S 3]). We still denote by $-H$ this isomorphism for the sake of simplicity. Explicitely,

$$-H \circ \ell_Y(x;\xi) = -H(x, 1; \xi, 0) = (-\xi, 1; x, 0).$$

Therefore we may regard $T^*Y$ as a submanifold of $T_{T_Y^*X}(T^*X)$ by the immersion $\tilde{\rho}_Y := -H \circ \ell_Y$.

Let $V = \{\xi_1 = \cdots = \xi_{n-1} = 0; \; \xi \neq 0\}$. The composition of the natural morphism of vector bundles $s_Y: T_{T_Y^*X}(T^*X) \to T_Y(T^*X)$ with $\tilde{\rho}_Y$ is injective; more precisely, if $(x;\xi) \in T^*Y$ then

$$s_Y \tilde{\rho}_Y(x;\xi) = (-\xi, 0, 1; x) \in T_Y(T^*X).$$

We set $\phi_Y := s_Y \tilde{\rho}_Y$. By means of $\phi_Y$ we identify $T^*Y$ to a submanifold of $T_Y(T^*X)$. Moreover, if $\mathcal{M}$ is an arbitrary $\mathcal{E}_X$-coherent module we have an inclusion

$$C^1_Y(M) \cap T^*Y \subset s_Y \left(C^1_{T_Y^*X}(M) \cap T^*Y\right)$$

since the sheaf $\mathcal{E}_V$ is a subsheaf of $\mathcal{E}_{T_Y^*X}$ and, by the above identification, for any $P \in \mathcal{E}_V$, $\sigma^1_Y(P)|_{T^*Y} = \sigma^1_{T_Y^*X}(P)|_{T^*Y}$.

For any coherent $\mathcal{E}_X$-module $\mathcal{M}$ we regard $\mathcal{L} := \text{RHom}_X(M, C^0_{Y|X})$ as a complex on $T_Y^*X$. Then, to get $b^\ast$, it is enough to prove the inclusion

$$b^\ast H(SS(\mathcal{L})) \subset C^1_Y(M).$$

Assume that, for any coherent $\mathcal{E}_X$-module $\mathcal{M}$, the following inclusion holds:

$$(8) \quad SS(\mathcal{L}|_{\gamma(Y)}) \subset C^1_Y(M) \cap T^*Y.$$
Then, to get $b^*$, we may use (8) by adjunction of a new variable following a
suggestion of M. Kashiwara. Let $\tilde{X} = X \times \mathbb{C}$ with the coordinates $(x, t', s)$ and
let $f : \tilde{X} \to X$ be the smooth morphism
$$f(x, t', s) = (x, se^t).$$
Let $\tilde{\mathcal{M}} = f^*\mathcal{M}$ be the inverse image of $\mathcal{M}$, a coherent $\mathcal{E}_X$-module. Let $Y' \subset \tilde{X}$
be defined by $s = 0$ and let
$$V' = \{(x, t', s; \xi, \zeta'; \eta) ; \quad \xi = 0, \quad \zeta' = 0, \quad \eta \neq 0 \} \subset \tilde{T}^*\tilde{X}.$$ 
Let $f' : T\tilde{X} \to T.X \times \tilde{X}$, $i' : T^*X \times \tilde{X} \to T^*\tilde{X}$ and $\omega : T^*X \times \tilde{X} \to T^*X$ be the canonical morphisms:
$$\tilde{T}^*X \leftarrow T^*X \times \tilde{X} \to T^*\tilde{X}.$$ 
Explicitly
$$f'(x, t', s) \xi = (\xi', \zeta' se^t + \eta e^t),$$
and
$$i' f'(x, t', s) \xi = (\xi, \zeta se^t, \zeta' e^t).$$
Restricting to $\eta = 1$ and fixing a determination of $\log(1/\zeta)$ we get a section $h$ of $\omega$
$$h : \tilde{T}^*X \to \tilde{T}^*X \times \tilde{X}$$
and $\overline{h} := i' f' \circ h$ gives an immersion of the corresponding open subdomain of $\{(x, t; \xi, \zeta) \in \tilde{T}^*X ; \zeta \neq 0 \}$ in $\{(x, t, s; \xi', \zeta', \eta) \in \tilde{T}^*\tilde{X} ; \eta = 1 \}$. The image
$\overline{h}(\overline{T}^*_YX)$ is an open subset of $\overline{T}^*_Y \tilde{X} \cap \{\eta = 1 \}$. More precisely,
$$h(x, t; \xi, \zeta) = ((x, t; \xi, \zeta), \zeta t, \log(1/\zeta))$$
and
$$\overline{h}(x, t; \xi, \zeta) = (x, \log(1/\zeta), \zeta t; \xi, \zeta t, 1).$$

Remark that we can cover $\{p \in T^*X ; \zeta \neq 0 \}$ by two such open domains,
$\Omega_1$ and $\Omega_2$. Since the notion of microsupport is of local nature, it is enough
to prove $b^*$ for the restriction of $F$ to each of $\Omega_1 \cap \overline{T}^*_Y X$, $\Omega_2 \cap \overline{T}^*_Y X$. In that
situation, $\overline{h} : \tilde{T}^*X \to \tilde{T}^*\tilde{X}$ induces an analytic isomorphism
$$h_Y : \tilde{T}^*_Y \cong \tilde{T}^*_Y \tilde{X} \cap \{\eta = 1 \}.$$ 
More precisely, setting $i_Y : \tilde{T}^*_Y X \hookrightarrow \tilde{T}^*X$ and $i'_Y : \tilde{T}^*_Y \tilde{X} \cap \{\eta = 1 \} \hookrightarrow \tilde{T}^*\tilde{X}$,
we have
$$\overline{h} \circ i_Y = i'_Y \circ h_Y.$$
Let $h^\nu$ be the canonical isomorphism
\[ T^*(T^*_Y X) \to T^*(T^*_Y \tilde{X} \cap \{ \eta = 1 \}) (\simeq T^* Y') \]
induced by $h_Y$. Composing with the immersion
\[ \phi_{Y^\nu} : T^* Y' \hookrightarrow T_{V^\nu}(T^* \tilde{X}), \]
we get an isomorphism
\[ T^*(T^*_Y X) \simeq T_{V^\nu}(T^* \tilde{X}) |_{\nu=0, \eta=1}. \]

We set
\[ \tilde{h}^N := \phi_{Y^\nu} \circ h^\nu \circ (-H)^{-1} : T^*_Y X \to T_{V^\nu}(T^* \tilde{X}). \]

Remark that, by functoriality, $\tilde{h}^N$ is also the quotient morphism associated to $\tilde{h}$ and $\tilde{h} \circ i_Y$. On the other hand, the sequence of morphisms of vector bundles:
\[ T^* X \overset{h}{\to} T^* X \times \tilde{X} \overset{t^f}{\to} \hat{T}^* \tilde{X} \]
induces a sequence
\[ T^*_Y X \overset{e^\prime}{\to} T^*_Y X \times \tilde{X} \overset{\pi^\prime}{\to} V^\prime \]
and $\pi^\prime$ is injective since $t^f$ is injective. Setting $j_{Y^\nu} : T^*_Y \tilde{X} \cap \{ \eta = 1 \} \hookrightarrow V^\nu$, by construction
\[ j_{Y^\nu} \circ h_Y = \pi^\prime \circ e^\prime. \]

Set $\tilde{V}^\nu := T^*_Y X \times \tilde{X}$. Then
\[ \tilde{V}^\nu = T^* X \times \tilde{X} \cap \omega^{-1}(T^*_Y X), \]
\[ \pi^\prime(\tilde{V}^\nu) = T^*_Y \tilde{X}, \]
\[ \pi^\prime e^\prime(T^*_Y X) = h_Y(T^*_Y X) = T^*_Y \tilde{X} \cap \{ \eta = 1 \}. \]

Let
\[ e^N : T^*_Y X T^* X \to T_{V^\nu}(T^* X \times \tilde{X}), \]
\[ \omega^N : T_{V^\nu}(T^* X \times \tilde{X}) \to T^*_Y X T^* X, \]
\[ \pi^N : T_{V^\nu}(T^* X \times \tilde{X}) \to T_{V^\nu}(T^* \tilde{X}). \]
be the canonical morphisms respectively associated to \( h \) and \( e' \), to \( \omega \) and \( \omega |_{\mathbb{V}'} \), to \( f' \) and \( \pi' \). Consider the diagram of morphisms below:

\[
T^* (T^*_X - H) \cong \pi_N^* \cdot T^* X \to T_{\mathbb{V}} (T^* X \times X) \to T_{\mathbb{V}} (T^* \tilde{X}).
\]

(Here \(-H\) denotes the isomorphism induced by the Hamiltonian isomorphism as before.) Remark that \( \pi'^N \) is the composition of the morphisms:

\[
T_{\mathbb{V}} \left( T^* X \times X \right) \to \pi^* T_{\mathbb{V}} (T^* \tilde{X}) \to T_{\mathbb{V}} (T^* \tilde{X}).
\]

Furthermore, \( sy \big|_{\mathbb{V}'} \) is injective, \( \pi' \) is injective (cf. Lemma 2.3) and \( \pi'^N \circ e^N = h^N = \phi \circ h \circ (-H)^{-1} \). Since \( h \) is a section of \( \omega \), that is, \( \omega \circ h = \text{Id} \), by functoriality, \( e^N \) is a section of \( \omega^N \).

Recall we assumed (8), hence we have the inclusion

\[
\mathcal{S} (\mathcal{L} |_{\mathfrak{g} - 1}) \subset C^1_{\mathbb{V}} (\tilde{M} \cap T^* Y').
\]

where \( \mathcal{L} \) denotes

\[
\mathcal{E}_{\overline{\ell} X} (\tilde{M}, C_{Y' | \mathfrak{g} X}^f).
\]

On the other hand, for any coherent \( \mathcal{E}_X \)-module \( \mathcal{M} \),

\[
h^{-1} \mathcal{R} \mathcal{H} \mathcal{O} \mathcal{M}_X (\tilde{M}, C_{Y' | \mathfrak{g} X}^f) \mid_{\mathfrak{g} - 1} \cong \mathcal{R} \mathcal{H} \mathcal{O} \mathcal{M}_X (\mathcal{M}, C_{Y' | \mathfrak{g} X}^f) = \mathcal{L}.
\]

To prove this, it is enough to consider \( \mathcal{M} \cong \mathcal{E}_X \), hence

\[
\tilde{M} = \frac{\mathcal{E}_{\overline{\ell} X}}{\mathcal{E}_{\overline{\ell} X} (s D - D')}.
\]

in which case the isomorphism above is clear.

Therefore,

\[
\mathcal{S} (\mathcal{L} = (h^*)^{-1} (\mathcal{S} (\mathcal{L} |_{\mathfrak{g} - 1})))
\]

\[
\subset (h'^{-1} (C_{\overline{\ell} X}^1 (\tilde{M} \cap T^* Y'))
\]

\[
\subset (h'^{-1} (s Y' (C_{\overline{\ell} X}^1 (\tilde{M} \cap T^* Y'))).
\]

In order to get \( b'' \) we shall prove the inclusion

\[
-H (h^*)^{-1} (s Y' (C_{\overline{\ell} X}^1 (\tilde{M} \cap T^* Y'))) \subset C^1_{\overline{\ell} X} (\tilde{M}).
\]
Using Lemma 2.3 with $\Lambda = T^*_Y X$, we have

$$-H(h^c)^{-1} \left( s_Y \left( C^1_{T^*_Y X}(\mathcal{M}) \cap T^* Y' \right) \right)$$

$$= -H(h^c)^{-1} \phi_Y^{-1} \left( s_Y \left( \pi^N (\omega^N)^{-1} \left( C^1_{T^*_Y X}(\mathcal{M}) \cap T^* Y' \right) \right) \right)$$

$$= \phi_Y^{-1} \left( s_Y \left( \pi^N (\omega^N)^{-1} \left( C^1_{T^*_Y X}(\mathcal{M}) \cap T^* Y' \right) \right) \right)$$

$$= (e_Y)^{-1} (\pi^N)^{-1} \left( s_Y \left( \pi^N (\omega^N)^{-1} \left( C^1_{T^*_Y X}(\mathcal{M}) \cap T^* Y' \right) \right) \right)$$

$$= (e_Y)^{-1} (\pi^N)^{-1} \left( s_Y \left( \pi^N (\omega^N)^{-1} \left( C^1_{T^*_Y X}(\mathcal{M}) \cap T^* Y' \right) \right) \right)$$

$$\subset (e_Y)^{-1} (\omega^N)^{-1} \left( C^1_{T^*_Y X}(\mathcal{M}) \right)$$

$$\subset C^1_{T^*_Y X}(\mathcal{M}).$$

Therefore $b^+$ holds provided that we prove (8).

Let $\theta \in T_Y (T^* X)$ such that $\theta \notin C^1_Y (\mathcal{M})$. Considering the local coordinates in $T_Y (T^* X)$, $((x,t;\eta); \xi_1, \cdots, \xi_{n-1})$ and using the technique of [MF2] or [S], we may assume $\theta = (0,0;1;1, \cdots, 0)$. We shall identify $T^*_Y X \cap \{ \zeta = 1 \}$ to $Y$. Furthermore, we may assume by classical arguments that $\mathcal{M}$ is of the form $\mathcal{E}_X / \mathcal{E}_X P$ with

$$P(x,t, D_x, D_t) = D^m_{x_1} + \sum_{0 \leq j \leq m-1} A_j(x,t, D_x, D_t) D^j_{x_1},$$

where $D^m_{x_1} = (D_{x_2}, \cdots, D_{x_{n-1}})$ and $A_j \in \mathcal{E}_Y (m - j)$. We shall prove that $(0, dx_1) \notin SS(R\text{Hom}_{\mathcal{E}_X} (\mathcal{M}, C^{\mathbb{R}}_{Y^1 X}) )_{| \{\zeta = 1\}}$. For that purpose we need Lemmas 2.5, 2.6 and 2.7 below.

**Lemma 2.5.** The sheaf $C^{\mathbb{R}, f}_{Y^1 X}$ satisfies:

1) The analytic continuation principle: Let $\omega \subset \Omega$ be two open subsets of $Y$, $\Omega$ connected and $\omega \neq \phi$, and assume that $u \in \Gamma (\Omega, C^{\mathbb{R}, f}_{Y^1 X})_{| \{\zeta = 1\}}$ vanishes in $\omega$. Then $u \equiv 0$.

2) If $V$ is a conic subset of $T^*_Y X$ of the form $W \times \Gamma$, where $W$ is compact Stein in $Y$ and $\Gamma$ a convex cone in $\mathbb{C}^*$ containing 1, such that $\Gamma \cap S^1$ is closed, then

$$\forall j \geq 1, \quad H^j (V, C^{\mathbb{R}, f}_{Y^1 X}) = 0.$$

In particular, if $W$ is compact Stein in $Y$, $H^j (W, C^{\mathbb{R}, f}_{Y^1 X})_{| \{\zeta = 1\}} = 0, \forall j \geq 1$.

**Proof.** 1) Since $C^{\mathbb{R}, f}_{Y^1 X}$ is a subsheaf of $C^{\mathbb{R}}_{Y^1 X}$, it is enough to prove that $u$ vanishes as a section of $C^{\mathbb{R}}_{Y^1 X}$ but this is a consequence of the analytic
2) We have
\[ \forall j, \quad H^j(W \times \Gamma, C^\infty_{Y|X}) = \lim_{W' \to \Gamma'} \left( W' \times \Gamma', C^\infty_{Y|X} \right), \]
where \( W' \) runs through a neighborhood system of \( W \) formed by open Stein relatively compact subsets of \( Y \) and \( \Gamma' \) through a neighborhood system of \( \Gamma \) formed by convex open cones.

Let \( V' = W' \times \Gamma' \). Then
\[
H^j(V', C^\infty_{Y|X}) \simeq H^{j+1}_Y(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X))
\]
where \( \tau : T_Y X \to Y \) is the projection and \( \nu_Y(\mathcal{O}_X) \) denotes the tempered specialisation of the sheaf \( \mathcal{O}_X \) along \( Y \). We denote by \( \nu_Y(\mathcal{O}_X) \) the usual specialisation of \( (\mathcal{O}_X) \) along \( Y \). Let us study the long exact sequence
\[
\cdots \to H^j_Y(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) \to H^j(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) \to \\
\to H^j(\tau^{-1}\pi(V') \setminus V'^0, \nu_Y(\mathcal{O}_X)) \to H^{j+1}_Y(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) \to \cdots
\]

On one hand, we have, \( \forall i \geq 1 : \)
\[
H^i(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) = H^i(\tau^{-1}\pi(V'), \nu_Y(\mathcal{O}_X)) \\
= H^i(W', \mathcal{O}_X) \\
= 0.
\]

On the other hand, we have \( V'^0 = W' \times \Gamma'^0 \), therefore \( \tau^{-1}\pi(V') \setminus V'^0 = W' \times (\mathbb{C} \setminus \Gamma'^0) \).

Hence it remains to prove that, \( \forall i \geq 1, \)
\[
\lim_{W' \to \Gamma'} H^i(W' \times (\mathbb{C} \setminus \Gamma'^0), \nu_Y(\mathcal{O}_X)) = 0.
\]
This direct limit equals
\[
H^i(W \times H, \nu_Y(\mathcal{O}_X)) = 0,
\]
where \( H \) is the cone generated by \( \mathbb{C} \setminus \Gamma'^0 \cap S^1 \). Let now \( V \) denote \( W \times H \).

We have
\[
H^i(V, \nu_Y(\mathcal{O}_X)) = \lim_{V' \to V} H^i(\mathbb{R}\Gamma(X, t\mathbb{H}\text{om}(\mathbb{C}_U, \mathcal{O}_X)),
\]

Continuation principle for \( C^\infty_{Y|X} \) which is well known (cf. [K-S]).
with $U$ running through the open subanalytic sets in $X$ such that $V \cap \mathcal{C}_Y(X \setminus U) = \emptyset$, by (3.1.2) of [A].

By Proposition 4.1.3 of [K-S3], $U$ may be taken to range through the family $p(U' \cap t^{-1}(\mathbb{R}^+))$ where $U'$ ranges through a neighborhood system of $V$ in the real normal deformation of $X$ along $Y$, $\hat{X}_Y$, $t : \hat{X}_Y \to \mathbb{R}$ being the canonical projection and $p : \hat{X}_Y \to X$ the deformation morphism. Since $V = W \times H$ we may assume that $U$ is of the form $(W' \times \Gamma^n) \cap B$ where $W'$ is Stein subanalytic, $\Gamma^n$ is an open cone in $\mathbb{C}$ and $B$ is an open polydisc in $\mathbb{C}^n$. Hence $U$ is Stein subanalytic relatively compact and we may apply Hörmander’s results in [H]. More precisely, $t\mathcal{H}om(\mathcal{C}_U, \mathcal{O}_X)$ is concentrated in degree zero and $H^i(X, t\mathcal{H}om(\mathcal{C}_U, \mathcal{O}_X)) = 0$, $\forall i > 0$. ( Cf. also Lemma 2.6 and Lemma 2.16 of [Be].)

Proof of Lemma 2.4 (continued). We shall now use some concepts introduced by J.M. Bony and P. Schapira (cf. [B-S]).

Let $\Omega$ be an open convex subset of $Y$. Let us note $H_h$ and $Z$, respectively, the hyperplane of $\mathbb{C}^n$ of equation $x_1 = h$ and $x_1 = 0$, hence $Z = H_0$. Let $\delta$ be a real positive number. We say that $\Omega$ is $\delta$-flat if, whenever $x \in \Omega$ and $\bar{x} \in H_h$, satisfy

$$|x_1 - h| \geq \delta, |x_1 - \bar{x}_1| \leq \cdots, |x_{n-1} - \bar{x}_{n-1}|$$

$\delta$-flat entails $\bar{x} \in \Omega$.

If $\Omega$ is $\delta$-flat, then for any $\rho > 0$, $\rho \Omega$ is still $\delta$-flat, and for any $\omega = (\varepsilon, 0, \cdots, 0)$, $\Omega + \omega$ is still $\delta$-flat.

Let $P(x, t, D_x, D_t)$ be of the form (9), that is, $P$ is Weierstrass with respect to $D_{x_1}$, and belongs to $\mathcal{E}_Y$ in a suitable neighborhood of $0(1, \cdots, 0)$.

Lemma 2.6 (Precised Cauchy Problem). There exists an open neighborhood $\Omega_1$ of $0 \in Y$ and $\delta > 0$ such that, for any convex open subset $\Omega \subset \Omega_1$ which is $\delta$-flat, the Cauchy problem

$$P_t = g, \quad \gamma(f) = (h)$$

where $\gamma(f) = (f \mid_0, \cdots, D_m f \mid_0)$, $g \in C^{\mathcal{R}_f}_Y \mid_{(-1)}(\Omega)$ and

$$(h) \in C^{\mathcal{R}_f}_{Y \cap Z \mid 0} \mid_{(-1)}(\Omega \cap Z)^m$$

admits a unique solution

$$f \in C^{\mathcal{R}_f}_Y \mid_{(-1)}(\Omega).$$
Moreover, there exists $h_0 > 0$ depending only on $P$, $\Omega_1$ and $\delta$, such that, if $\Omega$ is $\delta$-\textit{H}$_h \cap Y$-flat, and the Cauchy data is given on $H_h$ for $| h | < h_0$, the same result holds in $\Omega$.

We shall not give here the detailed proof of this lemma since the unicity is an immediate consequence of Lemma 5.2 of [K-S2] and, as for the existence, one follows step by step the proof of this same lemma, using Theorem 2.4.3 of [B] to be sure that the unique solution is in fact in $C_{Y|X}^{\infty}$.

We give here a version, in our framework, of Zerner’s classical result on the propagation at the boundary for holomorphic solutions of partial differential equations ([Z]).

**Lemma 2.7.** Let $\varphi$ be a $C^{\infty}$ function in a neighborhood of $0 \in Y$ such that $\varphi(0) = 0$ and $d\varphi(0) = dx_1$. Let $\Omega = \{ x, \varphi(x) < 0 \}$. Let $u \in \Gamma(\Omega, C_{Y|X}^{\infty}|_{\{ -1 \}})$ and assume that $Pu$ extends as a section of $C_{Y|X}^{\infty}|_{\{ -1 \}}$ to a neighborhood of $0$. Then $u$ extends to a neighborhood of $0$.

**Proof.** Let $\Omega_1$, $h_0$ and $\delta$ be as in Lemma 2.6. We may assume $\varphi$ is defined in $\Omega_1$, and that $Pu$ extends to $\Omega_1$. We have $\varphi(x) = \text{Re} x_1 - \psi(\text{Im} x_1, x_2, \cdots, x_{n-1})$ with $d\psi(0) = 0$. Let $0 < \varepsilon \ll 1$ such that there exists $R > 0$ such that $\| x' \| < R$ entails $-\varepsilon < \varphi(0, x')$, that is, denoting as before $H_{-\varepsilon} = \{ x_1 = -\varepsilon \}$,

$$H_{-\varepsilon} \cap \{ (x, x') \in \Omega_1, | x' | < R \} \subset \Omega.$$

Since $\psi(0, x') = 0(\| x' \|)$, we may assume that $\varepsilon < \delta R$ and that the open polydisc centered in $(-\varepsilon, 0, 0)$ with radius $\max(R, \delta R)$ is contained in $\Omega_1$.

Then $W_\varepsilon = \{ (x_1, x'); x_1 + \varepsilon | < \delta (R - \| x' \|), \| x' \| < R \} \subset \Omega_1$ will be $\delta$-$H_{-\varepsilon} \cap Y$-flat and is a neighborhood of zero.

Let us now consider the Cauchy problem:

$$Pu = Pu, \quad \gamma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon(u),$$

where $\gamma_\varepsilon$ denotes the traces along $H_{-\varepsilon}$. By Lemma 2.6, the solution $u_\varepsilon = u$ is defined in $W_\varepsilon$, which achieves the proof.

**Conclusion of the proof of Lemma 2.4 and of Theorem 2.1.** We shall use Proposition 5.1.1. 3) of [K-S3] which gives a characterization of the microsupport of a complex of sheaves better adapted to our situation.

Let $\rho_0 = (0, dx_1) \in T^*Y$. Let $\Omega_1$, $\delta$ and $h_0$ be given by Lemma 2.6. Let $0 < \varepsilon \ll h_0$ and $0 < R \ll 1$ be small enough and satisfying:
i) \( \varepsilon < R \),

ii) the open polydisc \( B_R(-\varepsilon) \) centered in \((-\varepsilon, 0, \cdots, 0)\) and radius \( R \) is contained in \( \Omega_1 \).

Let \( U = B_R(-\varepsilon) \). Let \( \gamma \) be the proper closed convex cone of \( \mathbb{C}^{n-1} \) defined by

\[
\gamma = \{(x_1, \cdots, x_{n-1}), \text{Re} x_1 \leq -\delta(|x'| + |\text{Im} x'|)\}.
\]

In particular, \((1, 0, \cdots, 0) \in \text{int} \ \gamma'\text{so, } g\) denoting the antipodal map.

It is easy to check that \((U + \gamma) \cap H\), where \( H \) denotes the real half space \( \{(x_1, x')+ \text{Re} x_1 \geq -\varepsilon\} \), is bounded. Hence, for any \( x \in U \), \( H \cap (x + \gamma) \) is a compact.

Up to a suitable choice of \( \varepsilon, R \) and \( \delta \), we may assume that \((U + \gamma) \cap H \subset \Omega_1 \) and

\[
\{(U + \gamma) \cap H\} \times \text{Int} \ \gamma' \cap C^1 \left( \frac{\mathcal{E}_X}{\mathcal{E}_X \mathcal{F}} \right) = \phi.
\]

Let \( L = \{x, \text{Re} x_1 = -\varepsilon\} \). We shall prove that the natural morphism of complexes:

\[
\begin{align*}
R\Gamma(H \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}) & \xrightarrow{P} R\Gamma(H \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}) \\
\downarrow & \downarrow \\
R\Gamma(L \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}) & \xrightarrow{P} R\Gamma(L \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)})
\end{align*}
\]

is a quasi isomorphism.

Since \( H \cap (x + \gamma) \) and \( L \cap (x + \gamma) \) are compact convex subsets, hence Stein, by Lemma 2.5 it it remains to prove that \( P \) induces an isomorphism in the quotient

\[
\frac{\Gamma(L \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)})}{\Gamma(H \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)})}.
\]

Here, we used the analytic continuation principle to identify

\[
\Gamma(H \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}) \text{ with a submodule of } \Gamma(L \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}).
\]

To prove the surjectivity of \( P \), we apply the analogous of Lemma 2.4.7 of [MF1] or Lemma 3.1.5 of [S] which is proved by the same method thanks to Lemmas 2.6 and 2.7. Let \( v \in \Gamma(L \cap (x + \gamma), C^\infty_Y |_{(\zeta-1)}) \). We solve the equation \( Pu = v \) in a neighborhood of \( H_{\lambda} \cap (x + \gamma) \) using Lemma 2.6 and then extend \( u \) to a neighborhood of \( L \cap (x + \gamma) \) since any real hyperplane through \( x \), with a 1-microcharacteristic normal, which intersects \( L \cap (x + \gamma) \) intersects \( H_{\lambda} \cap (x + \gamma) \) (see for example page 152 of [S]).

As for the injectivity, we shall use the construction of [K-S3, Proposition 5.1.5].
For each $a \in U \cap (H \setminus L)$, we construct a family of open subsets $\{\Omega_t(a)\}_{t \in \mathbb{R}^+}$, such that:

i) $\Omega_t(a) \subset a + \text{ int } \gamma$,

ii) $\Omega_t(a) \cap L = (a + \text{ int } \gamma) \cap L$,

iii) $\Omega_t(a) = \bigcup_{x \leq 2} \Omega_t(x)$,

iv) $\delta \Omega_t(a)$ is smooth real analytic,

v) $Z_t(a) = \left( \bigcap_{t > t_0} \Omega_t(a) \right) \cap H \subset \delta \Omega_t(a)$, and the conormal of $\Omega_t(a)$ at $Z_t(a)$ is non $1$-microcharacteristic for the operator $P$ everywhere in $Z_t(a)$,

vi) $\left( \bigcup_{t > 0} \Omega_t(a) \right) \cap H = (a + \text{ int } \gamma) \cap H$,

vii) $\left( \bigcap_{t > 0} \Omega_t(a) \right) \cap H = (a + \text{ int } \gamma) \cap L$.

We recall that if $v = (1, 0, \cdots, 0)$, then the family

$$\{(x + \rho v + \text{ int } \gamma \cap H)_{\rho > 0}\}$$

forms a neighborhood system of $(x + \gamma) \cap H$, and the family $\{\Omega_t(x + \rho v) \cap H\}_{\rho > 0, t > 0}$ forms a neighborhood system of $(x + \gamma) \cap L$.

Let $u \in \Gamma(L \cap (x + \gamma), C^\infty_{\mathcal{M}}(\mathcal{O}_X \mid_{(x, \gamma)})$ such that $Pu = w$ extends to a neighborhood of $H \cap (x + \gamma)$.

Let $\rho > 0$ and $t_0 > 0$ such that $u$ is defined in $\Omega_{t_0}(x + \rho v) \cap H$ and $w$ in $(x + \rho v + \text{ int } \gamma) \cap H$. By v) and Lemma 2.7, $u$ extends to a neighborhood of $\delta \Omega_{t_0}(x + \rho v) \cap H$ hence to $\Omega_{t'}(x + \rho v) \cap H$ for some $t' > t_0$, and the definition of the family $\Omega_t$ entails that this procedure leads to an extension of $u$ in a neighborhood of $(x + \gamma) \cap H$.

Remark. By the functorial properties of $\mu \text{ hom}(\cdot, \mathcal{O}_X)$, assuming that $\mathcal{M}$ is generated by a coherent $\mathcal{D}_X$-module, one easily deduces that if $F$ is an object of $D^b_{\mathbb{R}^-}(X)$ such that $SS(F) \subset V$ as in Theorem 2.1,

$$SS(\mathbb{H}om_{\mathcal{M}}(\mathcal{M}, \mu \text{ hom}(F, \mathcal{O}_X))) \subset C_Y(\mathcal{M}),$$

and this inclusion may still be improved to the microdifferential framework using other tools.

Our conjecture is that Theorem 2.1 may be generalized to the case where $F \in D^b_{\mathbb{R}^-}(X)$.
References


