The Tan $2\Theta$ Theorem for indefinite quadratic forms

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Abstract. A version of the Davis–Kahan Tan $2\Theta$ theorem [3] for not necessarily semibounded linear operators defined by quadratic forms is proven. This theorem generalizes a result by Motovilov and Selin [13].

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1. Introduction

In the 1970 paper [3], Davis and Kahan studied the rotation of spectral subspaces for $2 \times 2$ operator matrices under off-diagonal perturbations. In particular, they proved the following result, the celebrated

**Tan $2\Theta$ Theorem.** Let $A_{\pm}$ be strictly positive bounded operators in Hilbert spaces $\mathcal{H}_{\pm}$, respectively, and $W$ a bounded operator from $\mathcal{H}_-$ to $\mathcal{H}_+$. Denote by

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & -A_- \end{pmatrix} \quad \text{and} \quad B = A + V = \begin{pmatrix} A_+ & W \\ W^* & -A_- \end{pmatrix}$$

the block operator matrices with respect to the orthogonal decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Then

(i) the open interval $(\min \text{spec}(A_+), \max \text{spec}(-A_-))$ belongs to the resolvent set of the operator $B$;

(ii) the operator angle $\Theta$ between the subspaces $\text{Ran} E_A(\mathbb{R}_+)$ and $\text{Ran} E_B(\mathbb{R}_+)$ admits the bound

$$\| \tan 2\Theta \| \leq \frac{2 \| V \|}{d}, \quad \text{spec}(\Theta) \subset [0, \pi/4), \quad (1.1)$$

where $d = \text{dist}(\text{spec}(A_+), \text{spec}(-A_-))$.

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For the concept of an operator angle and further details we refer to [8] and references therein.

Estimate (1.1) can equivalently be expressed as the following inequality for the norm of the difference of the orthogonal projections \( P = E_A(\mathbb{R}_+) \) and \( Q = E_B(\mathbb{R}_+) \):

\[
\| P - Q \| \leq \sin \left( \frac{1}{2} \arctan \frac{2\| V \|}{d} \right),
\]

which, in particular, implies the estimate

\[
\| P - Q \| < \frac{\sqrt{2}}{2}.
\]

Independent of the work of Davis and Kahan, inequality (1.3) has been proven by Adamyan and Langer in [1], where the operators \( A_{\pm} \) were allowed to be semibounded. The “critical” case \( d = 0 \) has been considered in the paper [9] by Kostrykin, Makarov, and Motovilov. In particular, it was shown that for any orthogonal (not necessarily spectral) projection \( P \) satisfying

\[
E_B((0, \infty)) \leq P \leq E_B([0, \infty)),
\]

there exists a unique orthogonal projection \( Q \) such that

\[
E_B((0, \infty)) \leq Q \leq E_B([0, \infty))
\]

and

\[
\| P - Q \| \leq \frac{\sqrt{2}}{2}.
\]

It is worth mentioning that a particular case of this result has been obtained earlier by Adamyan, Langer, and Tretter, in [2]. Recently, a version of the Tan \( 2\Theta \) Theorem for off-diagonal perturbations \( V \) that are relatively bounded with respect to the diagonal operator \( A \) has been proven by Motovilov and Selin in [13], Theorem 1.

In the present work we are concerned with a sesquilinear form

\[
b = a + v,
\]

where \( a \) and \( v \) are densely defined symmetric forms, and obtain several generalizations of the aforementioned results assuming that the perturbation \( v \) is given by an off-diagonal symmetric form.

To introduce the framework of an off-diagonal form-perturbation theory, we pick up a self-adjoint involution \( J \) and assume that the form \( a \) “commutes” with the involution \( J \),

\[
a[J x, y] = a[x, J y].
\]

We also assume that the form \( a_J [x, y] \) is defined on \( \text{Dom}[a] \) and is a closed positive definite form.
Our further assumption is that the form \( v \) “anticommutes” with the involution \( J \),

\[
v[Jx, y] = -v[x, Jy],
\]

and that \( v \) satisfies the estimate

\[
|v[x, x]| \leq \beta a_J[x, x], \quad x \in \text{Dom}[a_J] = \text{Dom}[a],
\]

for some \( \beta > 0 \).

The “commutation” relations (1.5) and (1.6) suggest to interpret the form \( v \) as an off-diagonal perturbation of the diagonal form \( a \) with respect to the orthogonal decomposition \( H = H_+ \oplus H_- \) with \( H_{\pm} = \text{Ran}(I \pm J) \).

In this setting one can show that the form \( b \) admits the representation

\[
b[x, y] = \langle A_J^{1/2}x, HA_J^{1/2}y \rangle, \quad x, y \in \text{Dom}[a],
\]

where \( A_J \) is the self-adjoint operator associated with the closed positive definite form \( a_J \) and \( H \) is a bounded operator with a bounded inverse. In spite of the fact that the form \( b \) may not be semibounded, there exists a unique self-adjoint operator \( B \) in \( H \) associated with the form \( b \), i.e., \( \text{Dom}(B) \subset \text{Dom}[b] \) and

\[
b[x, y] = \langle x, By \rangle \quad x \in \text{Dom}[b], \quad y \in \text{Dom}(B).
\]

This result, proven in [4], is an extension of the First Representation Theorem for closed semi-bounded quadratic forms (see, e.g., [7]). A comprehensive exposition on representation theorems for indefinite quadratic forms can be found in [4]. In particular, we mention pioneering works [11] and [12] by McIntosh, where the relationship of indefinite forms to self-adjoint operators has been considered.

In this paper we follow a different path. Based on the observation that

\[
a[x, Jy] + iv[x, Jy]
\]

is a sectorial closed form (cf. [13] and [15]), we give an alternative proof of the First Representation Theorem for block operator matrices associated with the symmetric forms of the type (1.4) (Theorem 2.4).

We also obtain (i) a relative version of the Tan 2\( \Theta \) Theorem (Theorem 3.1) (for the pair of the operators \( A = JA_J \) and \( B \) associated with the forms \( a \) and \( b \), respectively) and (ii) its variants (Theorem 4.2) in the case where the form \( a \) is semibounded, including a generalization of the relative sin \( \Theta \) Theorem obtained in [6].

We would like to emphasize that in the off-diagonal perturbation theory setting, the First Representation Theorem does not require any assumption on the magnitude of the relative bound of the off-diagonal form \( v \) with respect to the positive definite form \( a_J \).
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2. The First Representation Theorem for off-diagonal form perturbations

To introduce the notation, it is convenient to assume the following hypothesis.

**Hypothesis 2.1.** Let $a$ be a symmetric sesquilinear form on $\text{Dom}[a]$ in a Hilbert space $\mathcal{H}$. Assume that $J$ is a self-adjoint involution such that

$$J \text{ Dom}[a] = \text{Dom}[a].$$

Suppose that

$$a[Jx, y] = a[x, Jy], \quad x, y \in \text{Dom}[a_J] = \text{Dom}[a].$$

Assume, in addition, that the form $a_J$ given by

$$a_J[x, y] = a[x, Jy], \quad x, y \in \text{Dom}[a_J] = \text{Dom}[a],$$

is a positive definite closed form and denote by $m_\pm$ the greatest lower bound of the form $a_J$ restricted to the subspace

$$\mathcal{H}_\pm = \text{Ran}(I \pm J).$$

**Definition 2.2.** Under Hypothesis 2.1, a symmetric sesquilinear form $v$ on $\text{Dom}[v] \supset \text{Dom}[a]$ is said to be off-diagonal with respect to the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

if

$$v[Jx, y] = -v[x, Jy], \quad x, y \in \text{Dom}[a].$$

If, in addition,

$$v_0 \overset{\text{def}}{=} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_J[x]} < \infty, \quad (2.1)$$

the form $v$ is said to be an $a$-bounded off-diagonal form.
Remark 2.3. If \( v \) is an off-diagonal symmetric form and \( x = x_+ + x_- \) is the unique decomposition of an element \( x \in \text{Dom}[a] \) such that \( x_\pm \in \mathcal{H}_\pm \cap \text{Dom}[a] \), then
\[
v[x] = 2\text{Re} \, v[x_+, x_-], \quad x \in \text{Dom}[a]. \tag{2.2}
\]
Moreover, if \( v_0 < \infty \), then
\[
|v[x]| \leq 2v_0 \sqrt{\alpha_J[x_+] \alpha_J[x_-]}.
\tag{2.3}
\]

**Proof.** To prove (2.2), we use the representation
\[
v[x] = v[x_+ + x_-, x_+, x_-]
= v[x_+] + v[x_-] + v[x_+, x_-] + v[x_-, x_+], \quad x \in \text{Dom}[a].
\]
Since \( v \) is an off-diagonal form, we obtain that
\[
v[x_+] = v[x_+, x_+] = v[J \, x_+, J \, x_+] = -v[x_+, x_+] = -v[x_+] = 0,
\]
and similarly \( v[x_-] = 0 \). Therefore,
\[
v[x] = v[x_+, x_-] + v[x_-, x_+] = 2\text{Re} \, v[x_+, x_-], \quad x \in \text{Dom}[a].
\]

To prove (2.3), first we observe that
\[
\alpha_J[x] = \alpha_J[x_+] + \alpha_J[x_-]
\]
and, hence, combining (2.2) and (2.1), we get the estimate
\[
|2\text{Re} \, v[x_+, x_-]| \leq v_0 \alpha_J[x] = v_0 (\alpha_J[x_+] + \alpha_J[x_-]) \quad x_\pm \in \mathcal{H}_\pm \cap \text{Dom}[a].
\]
Hence, for any \( t \geq 0 \) (and, therefore, for all \( t \in \mathbb{R} \)) we get that
\[
v_0 \alpha_J[x_+] t^2 - 2|\text{Re} \, v[x_+, x_-]| t + v_0 \alpha_J[x_-] \geq 0,
\]
which together with (2.2) implies (2.3). \( \Box \)

In this setting we present an analog of the First Representation Theorem in the off-diagonal perturbation theory.

**Theorem 2.4.** Assume Hypothesis 2.1. Suppose that \( v \) is an \( \alpha \)-bounded off-diagonal with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) symmetric form. On \( \text{Dom}[b] = \text{Dom}[a] \) introduce the symmetric form
\[
b[x, y] = a[x, y] + v[x, y], \quad x, y \in \text{Dom}[b].
\]
Then

(i) there exists a unique self-adjoint operator $B$ in $\mathcal{H}$ such that $\text{Dom}(B) \subset \text{Dom}[b]$ and

$$b[x, y] = \langle x, By \rangle \quad x \in \text{Dom}[b], \ y \in \text{Dom}(B).$$

(ii) the operator $B$ is boundedly invertible and the open interval $(-m_-, m_+) \ni 0$ belongs to its resolvent set.

Proof. (i) Given $\mu \in (-m_-, m_+)$, on $\text{Dom}[a_\mu] = \text{Dom}[a]$ we introduce the positive closed form $a_\mu$ by

$$a_\mu[x, y] = a[x, Jy] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[a_\mu],$$

and denote by $\mathcal{H}_{a_\mu}$ the Hilbert space $\text{Dom}[a_\mu]$ equipped with the inner product $\langle \cdot, \cdot \rangle_{a_\mu} = a_\mu[\cdot, \cdot]$. We remark that the norms $\| \cdot \|_{a_\mu} = \sqrt{a_\mu[\cdot, \cdot]}$ on $\mathcal{H}_{a_\mu} = \text{Dom}[a_\mu]$ are obviously equivalent. Since $v$ is $a$-bounded, we conclude then that

$$v_\mu \overset{\text{def}}{=} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_\mu[x]} < \infty, \quad \mu \in (-m_-, m_+).$$

Along with the off-diagonal form $v$, introduce a dual form $v'$ by

$$v'[x, y] = iv[x, Jy], \quad x, y \in \text{Dom}[a].$$

We claim that $v'$ is an $a$-bounded off-diagonal symmetric form. It suffices to show that

$$v_\mu = v'_\mu < \infty, \quad \mu \in (-m_-, m_+),$$

where

$$v'_\mu \overset{\text{def}}{=} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v'[x]|}{a_\mu[x]}, \quad \mu \in (-m_-, m_+). \quad (2.4)$$

Indeed, let $x = x_+ + x_-$ be the unique decomposition of an element $x \in \text{Dom}[a]$ such that $x_{\pm} \in \mathcal{H}_{a_{\pm}} \cap \text{Dom}[a]$. By Remark 2.3,

$$v[x] = v[x_+, x_-] + v[x_-, x_+] = 2 \text{Re} \ v[x_+, x_-], \quad x \in \text{Dom}[a].$$

In a similar way (since the form $v'$ is obviously off-diagonal) we get that

$$v'[x] = iv[x_+ + x_-, J(x_+ + x_-)]$$

$$= iv'[x_+] - iv'[x_-] - iv[x_+, x_-] + iv[x_-, x_+]$$

$$= -iv[x_+, x_-] + \overline{iv[x_+, x_-]} = 2 \text{Im} \ v[x_+, x_-], \quad x \in \text{Dom}[a].$$
Clearly, from (2.4) follows that
\[ v'_\mu = 2 \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\text{Im} v[x_+, x_-]|}{a_\mu[x]} = 2 \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\text{Re} v[x_+, x_-]|}{a_\mu[x]} = v_\mu, \]
for all \( \mu \in (-m_-, m_+) \), which completes the proof of the claim.

Next, on \( \text{Dom}[t_\mu] = \text{Dom}[a] \) introduce the sesquilinear form
\[ t_\mu \overset{\text{def}}{=} a_\mu + i v', \quad \mu \in (-m_-, m_+). \]

The real part of \( t_\mu \),
\[ (\text{Re} t_\mu)[x, y] \overset{\text{def}}{=} \frac{1}{2} (t_\mu[x, y] + \overline{t_\mu[y, x]}), \]
equals \( a_\mu \). Hence, \( t_\mu \) is closed. Since the form \( a_\mu \) is positive definite and the form \( v' \) is an \( a_\mu \)-bounded symmetric form, the form \( t_\mu \) is a closed sectorial form with the vertex 0 and semi-angle
\[ \theta_\mu = \arctan(v'_\mu) = \arctan(v_\mu). \quad (2.5) \]

Let \( T_\mu \) be a unique \( m \)-sectorial operator associated with the form \( t_\mu \) (cf., e.g., Theorem VI.2.1 in [7]). Introduce the operator
\[ B_\mu = JT_\mu \] on \( \text{Dom}(B_\mu) = \text{Dom}(T_\mu), \mu \in (-m_-, m_+). \]

We obtain that
\[
\langle x, B_\mu y \rangle = \langle x, JT_\mu y \rangle = \langle Jx, T_\mu y \rangle = a_\mu[Jx, y] + iv'[Jx, y]
\]
\[ = a[x, y] - \mu(Jx, Jy) + i^2v[Jx, Jy] \quad (2.6) \]
\[ = a[x, y] - \mu(x, y) + v[x, y], \]
for all \( x \in \text{Dom}[a], y \in \text{Dom}(B_\mu) = \text{Dom}(T_\mu). \) In particular, \( B_\mu \) is a symmetric operator on \( \text{Dom}(B_\mu) \), since the forms \( a \) and \( v \) are symmetric, and \( \text{Dom}(B_\mu) = \text{Dom}(T_\mu) \subset \text{Dom}[a] \).

Since the real part of the form \( t_\mu \) is closed and positive definite with a positive lower bound, the operator \( T_\mu \) has a bounded inverse. This implies that the operator \( B_\mu = JT_\mu \) has a bounded inverse and, therefore, the symmetric operator \( B_\mu \) is self-adjoint on \( \text{Dom}(B_\mu) \).

As an immediate consequence, we conclude (put \( \mu = 0 \)) that the self-adjoint operator \( B \overset{\text{def}}{=} B_0 \) is associated with the symmetric form \( b \) and that \( \text{Dom}(B) \subset \text{Dom}[a] \).
To prove uniqueness, assume that $B'$ is another self-adjoint operator associated with the form $b$. Then for all $x \in \text{Dom}(B)$ and all $y \in \text{Dom}(B')$ we get that
\[
\langle x, B'y \rangle = b[x, y] = b[y, x] = \langle y, Bx \rangle = \langle Bx, y \rangle,
\]
which means that $B = (B')^* = B'$.

(ii) From (2.6) we conclude that the self-adjoint operator $B_\mu + \mu I$ is associated with the form $b$ and, hence, by the uniqueness
\[
B_\mu = B - \mu I \quad \text{on } \text{Dom}(B_\mu) = \text{Dom}(B).
\]
Since $B_\mu$ has a bounded inverse for all $\mu \in (m_-, m_+)$, so does $B - \mu I$ which means that the interval $(-m_-, m_+)$ belongs to the resolvent set of the operator $B_0$.

\[\Box\]

**Remark 2.5.** In the particular case $v = 0$, from Theorem 2.4 follows that there exists a unique self-adjoint operator $A$ associated with the form $a$.

For a different, more constructive proof of Theorem 2.4 as well as for the history of the subject we refer to our work [4].

**Remark 2.6.** For the part (i) of Theorem 2.4 to hold it is not necessary to require that the form $a_J$ in Hypothesis 2.1 is positive definite. It is sufficient to assume that $a_J$ is a semi-bounded from below closed form (see, e.g., [14]).

**Remark 2.7.** We conjecture that in the case of off-diagonal form perturbation theory in question the following domain stability property
\[
\text{Dom}[b] = \text{Dom}(|B|^{1/2})
\]
holds. In this case (see, e.g., [4]), the form $b$ is represented by the operator $B$, i.e.,
\[
b[x, y] = (|B|^{1/2} x, \text{sign}(B) |B|^{1/2} y), \quad x, y \in \text{Dom}[b],
\]
which is the content of the Second Representation Theorem. We refer however to [4] for a simple counterexample of a not off-diagonal relative bounded perturbation for which the domain stability property fails to hold. We also refer to [17], p. 53, where the domain stability problem in a more general context of the perturbation theory is discussed.

3. The Tan $2\Theta$ Theorem

The main result of this work provides a sharp upper bound for the angle between the positive spectral subspaces $\text{Ran} E_A(\mathbb{R}_+)$ and $\text{Ran} E_B(\mathbb{R}_+)$ of the operators $A$ and $B$ respectively. This result is an extension of Theorem 1 in [13].
Theorem 3.1. Assume Hypothesis 2.1 and suppose that $v$ is off-diagonal with respect to the decomposition $\Sigma = \Sigma_+ \oplus \Sigma_-$. Let $A$ be a unique self-adjoint operator associated with the form $a$ and $B$ the self-adjoint operator associated with the form $b = a + v$ referred to in Theorem 2.4.

Then the norm of the difference of the spectral projections $P = E_A(\mathbb{R}_+)$ and $Q = E_B(\mathbb{R}_+)$ satisfies the estimate

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan v \right) < \frac{\sqrt{2}}{2},$$

where

$$v = \inf_{\mu \in (-m_-,m_+)} v_\mu = \inf_{\mu \in (-m_-,m_+)} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_\mu[x]},$$

with

$$a_\mu[x,y] = a[x,Jy] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[a_\mu] = \text{Dom}[a].$$

The proof of Theorem 3.1 uses the following result borrowed from [16].

Proposition 3.2. Let $T$ be an $m$-sectorial operator of semi-angle $\theta < \pi/2$. Let $T = U |T|$ be its polar decomposition. If $U$ is unitary, then the unitary operator $U$ is sectorial with semi-angle $\theta$.

Remark 3.3. We note that for a bounded sectorial operator $T$ with a bounded inverse the statement is quite simple. Due to the equality

$$\langle x, Tx \rangle = \langle |T|^{-1/2} y, U |T|^{1/2} y \rangle = \langle y, |T|^{-1/2} U |T|^{1/2} y \rangle, \quad y = |T|^{1/2} x,$$

the operators $T$ and $|T|^{-1/2} U |T|^{1/2}$ are sectorial with the semi-angle $\theta$. The resolvent sets of the operators $|T|^{-1/2} U |T|^{1/2}$ and $U$ coincide. Therefore, since $U$ is unitary, it follows that $U$ is sectorial with semi-angle $\theta$.

Proof of Theorem 3.1. Given $\mu \in (-m_-,m_+)$, let $T_\mu = U_\mu |T_\mu|$ be the polar decomposition of the sectorial operator $T_\mu$ with vertex 0 and semi-angle $\theta_\mu$, with

$$\theta_\mu = \arctan (v_\mu)$$

(as in the proof of Theorem 2.4 (cf. (2.5))). Since $B_\mu = J T_\mu$, we conclude that

$$|T_\mu| = |B_\mu| \quad \text{and} \quad U_\mu = J^{-1} \text{sign}(B_\mu).$$

Since $T_\mu$ is a sectorial operator with semi-angle $\theta_\mu$, by a result in [16] (see Proposition 3.2), the unitary operator $U_\mu$ is sectorial with vertex 0 and semi-angle $\theta_\mu$ as well. Therefore, applying the spectral theorem for the unitary operator $U_\mu$ from (3.1) we obtain the estimate

$$\|J - \text{sign}(B_\mu)\| = \|I - J^{-1} \text{sign}(B_\mu)\| = \|I - U_\mu\| \leq 2 \sin \left( \frac{1}{2} \arctan v_\mu \right).$$
Since the open interval \((-m_-, m_+)\) belongs to the resolvent set of the operator \(B = B_0\), the involution \(\text{sign}(B_\mu)\) does not depend on \(\mu \in (-m_-, m_+)\) and, hence, we conclude that
\[
\text{sign}(B_\mu) = \text{sign}(B_0) = \text{sign}(B), \quad \mu \in (-m_-, m_+).
\]
Therefore,
\[
\|P - Q\| = \frac{1}{2} \| J - \text{sign}(B) \| = \frac{1}{2} \| J - \text{sign}(B_\mu) \| \leq \sin \left( \frac{1}{2} \text{arctan} v_\mu \right) \quad (3.2)
\]
and, hence, since \(\mu \in (-m_-, m_+)\) has been chosen arbitrarily, from (3.2) follows that
\[
\|P - Q\| \leq \inf_{\mu \in (-m_-, m_+)} \sin \left( \frac{1}{2} \text{arctan} v_\mu \right) \leq \sin \left( \frac{1}{2} \text{arctan} v \right).
\]
The proof is complete. \(\square\)

As a consequence, we have the following result that can be considered a geometric variant of the Birman–Schwinger principle for the off-diagonal form-perturbations.

**Corollary 3.4.** Assume Hypothesis 2.1 and suppose that \(v\) is off-diagonal. Then the form \(a_J + v\) is positive definite if and only if the \(a_J\)-relative bound (2.1) of \(v\) does not exceed one. In this case
\[
\|P - Q\| \leq \sin \left( \frac{\pi}{8} \right),
\]
where \(P\) and \(Q\) are the spectral projections referred to in Theorem 3.1.

**Proof.** Since \(v\) is an \(a\)-bounded form, we conclude that there exists a self-adjoint bounded operator \(V\) in the Hilbert space \(\text{Dom}[a]\) such that
\[
v[x, y] = a_J[x, Vy], \quad x, y \in \text{Dom}[a].
\]
Since \(v\) is off-diagonal, the numerical range of \(V\) coincides with the symmetric about the origin interval \([-\|V\|, \|V\|]\). Therefore, we can find a sequence \(\{x_n\}_{n=1}^{\infty}\) in \(\text{Dom}[a]\) such that
\[
\lim_{n \to \infty} \frac{v[x_n]}{a_J[x_n]} = -\|V\|,
\]
which proves that \(\|V\| \leq 1\) if and only if the form \(a_J + v\) is positive definite. If it is the case, applying Theorem 3.1, we obtain the inequality
\[
\|P - Q\| \leq \sin \left( \frac{1}{2} \text{arctan}(\|V\|) \right) \leq \sin \left( \frac{\pi}{8} \right)
\]
which completes the proof. \(\square\)

**Remark 3.5.** We remark that in accordance with the Birman–Schwinger principle, for the form \(a_J + v\) to have the negative spectrum it is necessary that the \(a_J\)-relative bound \(\|V\|\) of the perturbation \(v\) is greater than one. As Corollary 3.4 shows, in the off-diagonal perturbation theory this condition is also sufficient.
4. Two sharp estimates in the semibounded case

In this section we will be dealing with the case of off-diagonal form-perturbations of a semi-bounded operator.

**Hypothesis 4.1.** Assume that $A$ is a self-adjoint semi-bounded from below operator. Suppose that $A$ has a bounded inverse. Assume, in addition, that the following conditions hold.

(i) **The spectral condition.** An open finite interval $(\alpha, \beta)$ belongs to the resolvent set of the operator $A$. We set

$$
\Sigma_- = \spec(A) \cap (-\infty, \alpha] \quad \text{and} \quad \Sigma_+ = \spec(A) \cap [\beta, \infty].
$$

(ii) **Boundedness.** The sesquilinear form $\mathbf{v}$ is symmetric on $\Dom[\mathbf{v}] \supset \Dom(|A|^{1/2})$ and

$$
\mathbf{v} \overset{\text{def}}{=} \sup_{0 \neq x \in \Dom[\mathbf{a}]} \frac{|\mathbf{v}[x]|}{\| |A|^{1/2} x \|^2} < \infty. \tag{4.1}
$$

(iii) **Off-diagonality.** The sesquilinear form $\mathbf{v}$ is off-diagonal with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with

$$
\mathcal{H}_+ = \Ran E_A((\beta, \infty)) \quad \text{and} \quad \mathcal{H}_- = \Ran E_A((-\infty, \alpha)).
$$

That is,

$$
\mathbf{v}[J x, y] = -\mathbf{v}[x, J y], \quad x, y \in \Dom[\mathbf{a}],
$$

where the self-adjoint involution $J$ is given by

$$
J = E_A((\beta, \infty)) - E_A((-\infty, \alpha)). \tag{4.2}
$$

Let $\mathbf{a}$ be the closed form represented by the operator $A$. A direct application of Theorem 2.4 shows that under Hypothesis 4.1 there is a unique self-adjoint boundedly invertible operator $\mathbf{B}$ associated with the form

$$
\mathbf{b} = \mathbf{a} + \mathbf{v}.
$$

Under Hypothesis 4.1 we distinguish two cases (see Fig. 1 and 2).

**Case I.** Assume that $\alpha < 0$ and $\beta > 0$. Set

$$
d_+ = \dist(\min(\Sigma_+), 0) \quad \text{and} \quad d_- = \dist(\min(\Sigma_-), 0)
$$

and suppose that $d_+ > d_-$. 

**Case II.** Assume that $\alpha, \beta > 0$. Set

$$
d_+ = \dist(\min(\Sigma_+), 0) \quad \text{and} \quad d_- = \dist(\max(\Sigma_-), 0).
$$
As it follows from the definition of the quantities \( d_\pm \), the sum \( d_- + d_+ \) coincides with the distance between the lower edges of the spectral components \( \Sigma_+ \) and \( \Sigma_- \) in Case I, while in Case II the difference \( d_+ - d_- \) is the distance from the lower edge of \( \Sigma_+ \) to the upper edge of the spectral component \( \Sigma_- \). Therefore, \( d_+ - d_- \) coincides with the length of the spectral gap \((\alpha, \beta]\) of the operator \( A \) in the latter case.

![Figure 1. The spectrum of the unperturbed sign-indefinite semibounded invertible operator \( A \) in Case I.](image1)

![Figure 2. The spectrum of the unperturbed strictly positive operator \( A \) with a gap in its spectrum in Case II.](image2)

We remark that the condition \( d_+ > d_- \) required in Case I holds only if the length of the convex hull of the negative spectrum \( \Sigma_- \) of \( A \) does not exceed the one of the spectral gap \((\alpha, \beta) = (\max(\Sigma_-), \min(\Sigma_+))\).

Now we are prepared to state a relative version of the Tan 2\( \Theta \) Theorem in the case where the unperturbed operator is semi-bounded or positive.

**Theorem 4.2.** In either of Cases I or II, introduce the spectral projections

\[
P = \mathbb{E}_A((-\infty, \alpha]) \quad \text{and} \quad Q = \mathbb{E}_B((-\infty, \alpha])
\]

of the operators \( A \) and \( B \) respectively.

Then the norm of the difference of \( P \) and \( Q \) satisfies the estimate

\[
\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \left( \frac{2v}{\delta} \right) \right) < \frac{\sqrt{2}}{2},
\]

where

\[
\delta = \frac{1}{\sqrt{d_+ d_-}} \begin{cases} 
d_+ + d_- & \text{in Case I,} 
d_+ - d_- & \text{in Case II,}
\end{cases}
\]
and $v$ stands for the relative bound of the off-diagonal form $v$ (with respect to $a$) given by (4.1).

**Proof.** We start with the remark that the form $a - \mu$, where $a$ is the form of $A$, satisfies Hypothesis 2.1 with $J$ given by (4.2). Set

$$a_\mu = (a - \mu) x, \quad \mu \in (\alpha, \beta),$$

that is,

$$a_\mu[x, y] = a[x, Jy] - \mu[x, Jy], \quad x, y \in \text{Dom}[a].$$

Notice that $a_\mu$ is a strictly positive closed form represented by the operators $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively.

Since $v$ is off-diagonal, from Theorem 3.1 follows that

$$\| E_{A - \mu I}(\mathbb{R}^+) - E_{B - \mu I}(\mathbb{R}^+) \| \leq \sin \left( \frac{1}{2} \arctan v_\mu \right), \quad \mu \in (\alpha, \beta), \quad (4.6)$$

with

$$v_\mu \overset{\text{def}}{=} \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v(x)|}{a_\mu[x]}, \quad (4.7)$$

Since $v$ is off-diagonal, by Remark 2.3 we get the estimate

$$|v(x)| \leq 2v_0 \sqrt{a_0[x_+]a_0[x_-]}, \quad x \in \text{Dom}[a],$$

where $x = x_+ + x_-$ is a unique decomposition of the element $x \in \text{Dom}[a]$ with $x_\pm \in \mathcal{S}_\pm \cap \text{Dom}[a]$.

Thus, in these notations, taking into account that

$$v_0 = v,$$

where $v$ is given by (4.1), we get the bound

$$v_\mu \leq 2v \sup_{0 \neq x \in \text{Dom}[a]} \frac{\sqrt{a_0[x_+]a_0[x_-]}}{a_\mu[x]}, \quad (4.8)$$

Since $a_\mu$ is represented by $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively, we observe that

$$a_\mu[x] = \begin{cases} a_0[x_+] - \mu \|x_+\|^2 + a_0[x_-] + \mu \|x_-\|^2 & \text{in Case I}, \\ a_0[x_+] - \mu \|x_+\|^2 - a_0[x_-] + \mu \|x_-\|^2 & \text{in Case II}. \end{cases} \quad (4.9)$$

Introducing the elements $y_\pm \in \mathcal{S}_\pm$,

$$y_\pm \overset{\text{def}}{=} \begin{cases} (|A| \mp \mu I)^{1/2} x_\pm & \text{in Case I}, \\ \pm (A - \mu I)^{1/2} x_\pm & \text{in Case II}. \end{cases}$$
and taking into account (4.9), we obtain the representation
\[
\frac{\sqrt{\alpha_0(x_+)\alpha_0(x_-)}}{a_{\mu}[x]} = \frac{||A||^{1/2}(|A| - \mu I)^{-1/2}y_+|| \cdot ||A||^{1/2}(-A + \mu I)^{-1/2}y_-||}{\|y_+\|^2 + \|y_-\|^2},
\]
valid in both Cases I and II. Using the elementary inequality
\[
\|y_+\| \cdot \|y_-\| \leq \frac{1}{2}(\|y_+\|^2 + \|y_-\|^2),
\]
we arrive at the following bound
\[
\frac{\sqrt{\alpha_0(x_+)\alpha_0(x_-)}}{a_{\mu}[x]} \leq \frac{1}{2}||A||^{1/2}(|A| - \mu I)^{-1/2}|\sigma_+| \cdot ||A||^{1/2}(-A + \mu I)^{-1/2}|\sigma_-|.
\]
(4.10)

It is easy to see that
\[
||A||^{1/2}(|A| - \mu I)^{-1/2}|\sigma_+| \leq \frac{\sqrt{d_+}}{\sqrt{d_+ - \mu}}, \quad \mu \in (\alpha, \beta), \quad \text{in Cases I and II},
\]
(4.11)

while
\[
||A||^{1/2}(-A + \mu I)^{-1/2}|\sigma_-| \leq \begin{cases} 
\frac{\sqrt{d_-}}{\sqrt{d_- + \mu}}, & \mu \in (0, \beta), \quad \text{in Case I}, \\
\frac{\sqrt{d_-}}{\sqrt{\mu - d_-}}, & \mu \in (\alpha, \beta), \quad \text{in Case II}.
\end{cases}
\]
(4.12)

Choosing \(\mu = \frac{d_+ - d_-}{2} > 0\) in Case I (recall that \(d_+ > d_-\) by the hypothesis) and \(\mu = \frac{d_+ + d_-}{2}\) in Case II, and combining (4.10), (4.11), and (4.12), we get the estimates
\[
\frac{\sqrt{\alpha_0(x_+)\alpha_0(x_-)}}{a_{\frac{d_+ - d_-}{2}}[x]} \leq \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}
\]
and
\[
\frac{\sqrt{\alpha_0(x_+)\alpha_0(x_-)}}{a_{\frac{d_+ + d_-}{2}}[x]} \leq \frac{\sqrt{d_+ d_-}}{d_+ - d_-} \quad \text{in Case II}.
\]

Hence, from (4.8) it follows that
\[
v_{\frac{d_+ - d_-}{2}} \leq 2v \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}
\]
and
\[
v_{\frac{d_+ + d_-}{2}} \leq 2v \frac{\sqrt{d_+ d_-}}{d_+ - d_-} \quad \text{in Case II}.
\]
Applying (4.6), we get the norm estimates
\[
\|E_{A - \frac{d_+ - d_-}{2} I} (\mathbb{R}^+) - E_{B - \frac{d_+ + d_-}{2} I} (\mathbb{R}^+)| \leq \sin \left( \frac{1}{2} \arctan \left( \frac{2 \sqrt{d_+ d_-}}{d_+ + d_-} \right) \right) \tag{4.13}
\]
in Case I and
\[
\|E_{A - \frac{d_+ + d_-}{2} I} (\mathbb{R}^+) - E_{B - \frac{d_+ - d_-}{2} I} (\mathbb{R}^+)| \leq \sin \left( \frac{1}{2} \arctan \left( \frac{2 \sqrt{d_+ d_-}}{d_+ - d_-} \right) \right) \tag{4.14}
\]
in Case II. It remains to observe that \(\|P - Q\|\), where the spectral projections \(P\) and \(Q\) are given by (4.3), coincides with the left hand side of (4.13) and (4.14) in Case I and Case II, respectively.

The proof is complete. \(\square\)

**Remark 4.3.** We remark that the quantity \(\delta\) given by (4.5) coincides with the relative distance (with respect to the origin) between the lower edges of the spectral components \(\Sigma_+\) and \(\Sigma_-\) in Case I and it has the meaning of the relative length (with respect to the origin) of the spectral gap \((d_-, d_+)\) in Case II.

For the further properties of the relative distance and various relative perturbation bounds we refer to the paper [10] and references quoted therein.

We also remark that in Case II, i.e., in the case of a positive operator \(A\), the bound (4.4) directly improves a result obtained in [6], the relative Sin\(\Theta\) Theorem, that in the present notations is of the form
\[
\|P - Q\| \leq \frac{v}{\delta}.
\]

We conclude our exposition with considering an example of a \(2 \times 2\) numerical matrix that shows that the main results obtained above are sharp.

**Example 4.4.** Let \(\mathcal{H}\) be the two-dimensional Hilbert space \(\mathcal{H} = \mathbb{C}^2\), \(\alpha < \beta\) and \(w \in \mathbb{C}\).

We set
\[
A = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad V = \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \(v\) be the symmetric form represented by (the operator) \(V\).

Clearly, the form \(v\) satisfy Hypothesis 4.1 with the relative bound \(v\) given by
\[
v = \frac{|w|}{\sqrt{\alpha \beta}},
\]
provided that \(\alpha, \beta \neq 0\). Since \(VJ = -J V\), the form \(v\) is off-diagonal with respect to the orthogonal decomposition \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\).
In order to illustrate our results, denote by $B$ the self-adjoint matrix associated with the form $a + v$, that is,

$$B = A + V = \begin{pmatrix} \beta & w \\ w^* & \alpha \end{pmatrix}. $$

Denote by $P$ the orthogonal projection associated with the eigenvalue $\alpha$ of the matrix $A$, and by $Q$ the one associated with the lower eigenvalue of the matrix $B$.

It is well known (and easy to see) that the classical Davis–Kahan Tan $2\Theta$ Theorem (cf. (1.2)) is exact in the case of $2 \times 2$ numerical matrices. In particular, the norm of the difference of $P$ and $Q$ can be computed explicitly

$$\|P - Q\| = \sin \left( \frac{1}{2} \arctan \frac{2|w|}{\beta - \alpha} \right). \quad (4.15)$$

Since, in the case in question,

$$v_{\mu} = \sup_{0 \neq x \in \text{Dom}[a]} \frac{|v[x]|}{a_{\mu}[x]} = \frac{|w|}{\sqrt{\beta - \mu}(\mu - \alpha)}, \quad \mu \in (\alpha, \beta), \quad (4.16)$$

from (4.16) follows that

$$\inf_{\mu \in (\alpha, \beta)} v_{\mu} = \frac{2|w|}{\beta - \alpha}$$

(with the infimum attained at the point $\mu = \frac{\alpha + \beta}{2}$).

Therefore, the result of the relative tan $2\Theta$ Theorem 3.1 is sharp.

It is easy to see that if $\alpha < 0 < \beta$ (Case I), then the equality (4.15) can also be rewritten in the form

$$\|P - Q\| = \sin \left( \frac{1}{2} \arctan \frac{2\sqrt{d_+d_-}v}{d_+ + d_-} \right), \quad (4.17)$$

where $d_+ = \beta$, $d_- = -\alpha$ and $v = \frac{|w|}{\sqrt{|\omega|^2}}$.

If $0 < \alpha < \beta$ (Case II), the equality (4.15) states that

$$\|P - Q\| = \sin \left( \frac{1}{2} \arctan \frac{2\sqrt{d_+d_-}v}{d_+ - d_-} \right), \quad (4.18)$$

with $d_+ = \beta$, $d_- = \alpha$, and $v = \frac{|w|}{\sqrt{\alpha \beta}}$.

The representations (4.17) and (4.18) show that the estimate (4.4) becomes equality in the case of $2 \times 2$ numerical matrices and, therefore, the results of Theorem 4.2 are sharp.
The Tan $2\Theta$ Theorem for indefinite quadratic forms

References


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