A Hardy type inequality for $W^{m,1}_0(\Omega)$ functions

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Abstract. We consider functions $u \in W^{m,1}_0(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and $m \geq 2$ is an integer. For all $j \geq 0$ and $1 \leq k \leq m-1$ such that $1 \leq j+k \leq m$, we prove that
\[
\partial_j \frac{\partial^k u(x)}{d(x)^{m-j-k}} \in W^{k,1}_0(\Omega)
\]
with
\[
\left\| \partial^k \left( \frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},
\]
where $d$ is a smooth positive function which coincides with dist$(x, \partial \Omega)$ near $\partial \Omega$, and $\partial^l$ denotes any partial derivative of order $l$.

Keywords. Hardy inequality, Sobolev spaces

1. Introduction

In [4, Theorem 1.2], the following one-dimensional Hardy type inequality was proven: Suppose that $u \in W^{2,1}(0,1)$ satisfies $u(0) = u'(0) = 0$. Then $u(x)/x \in W^{1,1}(0,1)$ with $u(x)/x|_0 = 0$ and
\[
\left\| \left( \frac{u(x)}{x} \right)' \right\|_{L^1(0,1)} \leq \|u''\|_{L^1(0,1)}.
\] (1)

As explained in [4], this inequality is somehow unexpected because one can construct a function $u \in W^{2,1}(0,1)$ such that $u(0) = u'(0) = 0$ and neither $u'(x)/x$ nor $u(x)/x^2$ belongs to $L^1(0,1)$; however, as (1) shows, for such a $u$, the difference $u'(x)/x - u(x)/x^2 = (u(x)/x)'$ is in fact an $L^1$ function, reflecting a “magical” cancelation of the non-integrable terms.
With estimate (1) already proven, it was natural to raise the following question: Assume \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \) and let \( u \) be in \( W^{2,1}_0(\Omega) \). For \( x \in \Omega \), denote by \( \delta(x) = d(x, \partial \Omega) \) the distance from \( x \) to the boundary of \( \Omega \), and let \( d : \Omega \to (0, \infty) \) be a smooth function such that \( d(x) = \delta(x) \) near \( \partial \Omega \). Is it true that \( u/d \in W^{1,1}_0(\Omega) \)? If so, can one obtain the corresponding Hardy-type estimate

\[
\int_{\Omega} \left| \nabla \left( \frac{u(x)}{d(x)} \right) \right|\, dx \leq C \| \nabla^2 u \|_{L^1(\Omega)},
\]

for some constant \( C \)?

The purpose of this work is to give a positive answer to the above question. In fact, this is a special case of the following:

**Theorem 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). Given \( x \in \Omega \), we denote by \( \delta(x) \) the distance from \( x \) to the boundary \( \partial \Omega \). Let \( d : \Omega \to (0, \infty) \) be a smooth function such that \( d(x) = \delta(x) \) near \( \partial \Omega \). Suppose \( m \geq 2 \) and let \( j, k \) be non-negative integers such that \( 1 \leq k \leq m - 1 \) and \( 1 \leq j + k \leq m \). Then for every \( u \in W^{m,1}_0(\Omega) \), we have

\[
\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W^{k,1}_0(\Omega)
\]

with

\[
\left\| \frac{\partial^j u(x)}{d(x)^{m-j-k}} \right\|_{L^1(\Omega)} \leq C \| u \|_{W^{m,1}(\Omega)},
\]

where \( \partial^l \) denotes any partial derivative of order \( l \), and \( C > 0 \) is a constant depending only on \( \Omega \) and \( m \).

The rest of this paper is organized into three sections. In Section 2 we introduce the notation and give some preliminary results. In order to present the main ideas used to prove Theorem 1, we begin in Section 3 with the proof for the special case \( m = 2 \); then in Section 4 we provide the proof for the general case \( m \geq 2 \).

### 2. Notation and preliminaries

Throughout, we denote by \( \mathbb{R}^N_+ := \{(y_1, \ldots, y_N-1, y_N) \in \mathbb{R}^N : y_N > 0\} \) the upper half-space, and \( B_{\mathbb{R}^N}(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\} \); also, when \( x_0 = 0 \), we write \( B_{\mathbb{R}^N} := B_{\mathbb{R}^N}(0) \).

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). Given \( x \in \Omega \), we denote by \( \delta(x) \) the distance from \( x \) to the boundary \( \partial \Omega \), that is,

\[
\delta(x) := \text{dist}(x, \partial \Omega) = \inf\{|x - y| : y \in \partial \Omega\}.
\]

For \( \epsilon > 0 \), the tubular neighborhood of \( \partial \Omega \) in \( \Omega \) is the set

\[
\Omega_\epsilon := \{x \in \Omega : \delta(x) < \epsilon\}.
\]

The following well known result (see e.g. Lemma 14.16 in [5]) shows that \( \delta \) is smooth in some neighborhood of \( \partial \Omega \).
Lemma 2.1. Let \( \Omega \) and \( \delta : \Omega \to (0, \infty) \) be as above. Then there exists \( \epsilon_0 > 0 \), only depending on \( \Omega \), such that \( \delta_{|\Omega_{\epsilon_0}} : \Omega_{\epsilon_0} \to (0, \infty) \) is smooth. Moreover, for every \( x \in \Omega_{\epsilon_0} \) there exists a unique \( y_x \in \partial \Omega \) so that

\[
x = y_x + \delta(x)\nu_{\partial \Omega}(y_x),
\]

where \( \nu_{\partial \Omega} \) denotes the unit inward normal vector field on \( \partial \Omega \).

Since \( \partial \Omega \) is smooth, for fixed \( \bar{x}_0 \in \partial \Omega \), there exists a neighborhood \( \mathcal{V}(\bar{x}_0) \subset \partial \Omega \), a radius \( r > 0 \) and a map

\[
\Phi : B_r^{N-1} \to \mathcal{V}(\bar{x}_0)
\]

which defines a smooth diffeomorphism. Define

\[
\mathcal{N}_+^r(\bar{x}_0) := \{ x \in \Omega_{\epsilon_0} : y_x \in \mathcal{V}(\bar{x}_0) \},
\]

where \( \epsilon_0 \) and \( y_x \) are as in Lemma 2.1. We denote by \( \Phi : B_r^{N-1} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^N \) the map defined by

\[
\Phi(\tilde{y}, t) := \tilde{\Phi}(\tilde{y}) + y_N \cdot \nu_{\partial \Omega}(\tilde{\Phi}(\tilde{y})),
\]

where \( \tilde{y} = (y_1, \ldots, y_{N-1}) \), and we write

\[
\mathcal{N}(\bar{x}_0) := \Phi(B_r^{N-1} \times (-\epsilon_0, \epsilon_0)).
\]

Lemma 2.2. The map \( \Phi|_{B_r^{N-1} \times (0, \epsilon_0)} \) is a diffeomorphism and

\[
\mathcal{N}_+^r(\bar{x}_0) = \Phi(B_r^{N-1} \times (0, \epsilon_0)).
\]

Proof. This is a direct corollary of the definition and Lemma 2.1. \( \square \)

Remark 2.1. The map \( \Phi|_{B_r^{N-1} \times (0, \epsilon_0)} \) gives a local coordinate chart which straightens the boundary near \( \bar{x}_0 \). This type of coordinates are sometimes called flow coordinates (see e.g. [3] and [6]).

From now on, \( C > 0 \) will always denote a constant only depending on \( \Omega \) and possibly the integer \( m \geq 2 \). The following is a direct, but very useful, corollary.

Corollary 2.1. Let \( f \in L^1(\mathcal{N}_+^r(\bar{x}_0)) \) and \( \Phi \) be given by (5). Then

\[
\frac{1}{C} \int_{B_r^{N-1}} \int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))| \, dy_N \, d\tilde{y} \leq \int_{\mathcal{N}_+^r(\bar{x}_0)} |f(x)| \, dx \leq C \int_{B_r^{N-1}} \int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))| \, dy_N \, d\tilde{y}.
\]
Proof. Since $\Phi|_{B^{N-1}_{L}(0,0)}$ is a diffeomorphism, we know that for all $(\tilde{y}, y_N) \in B^{N-1}_{L} \times (0,\varepsilon_0)$ we have
\[ 1/C \leq |\det D\Phi(\tilde{y}, y_N)| \leq C. \]
The result then follows from the change of variables formula. □

The following lemma provides us with a partition of unity in $\mathbb{R}^N$, constructed from the neighborhoods $N(\tilde{x}_0)$. Consider the open cover of $\partial \Omega$ given by $\{V(\tilde{x}) : \tilde{x} \in \partial \Omega\}$, where $V(\tilde{x}) \subset \partial \Omega$ is defined in (3). By the compactness of $\partial \Omega$, there exist $\{\tilde{x}_1, \ldots, \tilde{x}_M\} \subset \partial \Omega$ so that $\partial \Omega = \bigcup_{l=1}^{M} V(\tilde{x}_l)$. Notice that by the definition of $N(\tilde{x}_0)$ in (6), $\bigcup_{l=1}^{M} N(\tilde{x}_l)$ is also an open cover of $\partial \Omega$ in $\mathbb{R}^N$. The following is a classical result (see e.g. [2, Lemma 9.3] and [1, Theorem 3.15]).

Lemma 2.3 (partition of unity). There exist functions $\rho_0, \rho_1, \ldots, \rho_M \in C^\infty(\mathbb{R}^N)$ such that
(i) $0 \leq \rho_l \leq 1$ for all $l = 0, 1, \ldots, M$ and $\sum_{l=0}^{M} \rho_l(x) = 1$ for all $x \in \mathbb{R}^N$,
(ii) $\text{supp } \rho_l \subset N(\tilde{x}_l)$ for all $l = 1, \ldots, M$,
(iii) $\rho_0|_{\partial \Omega} \in C^\infty(\Omega)$.

In order to simplify notation, we will denote by $\partial^l$ any partial differential operator of order $l$ where $l$ is a positive integer.\(^1\) Also, $\partial_i$ will denote the partial derivative with respect to the $i$-th variable, and $\partial^2_{ij} = \partial_i \partial_j$.

Remark 2.2. We conclude this section by showing that, to prove Theorem 1, it is enough to prove estimate (2) for smooth functions with compact support. Suppose $u \in W^{m,1}_0(\Omega)$. Then there exists a sequence $\{u_n\} \subset C^\infty(\Omega)$ such that $\|u - u_n\|_{W^{m,1}(\Omega)} \to 0$ as $n \to \infty$. In particular, after maybe extracting a subsequence, one can assume that
\[ \partial^l u_n \to \partial^l u \quad \text{a.e. in } \Omega \text{ for all } 0 \leq l \leq m. \]

Since $d$ is smooth, the above implies that for a.e. $x \in \Omega$ and all $j \geq 0$ and $1 \leq k \leq m - 1$ with $1 \leq j + k \leq m$,
\[
\partial^k \left( \frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) = \frac{\partial^{j+k} u(x)}{d(x)^{m-j-k}} + \partial^j u(x) \partial^k \left( \frac{1}{d(x)^{m-j-k}} \right) \\
= \lim_{n \to \infty} \frac{\partial^{j+k} u_n(x)}{d(x)^{m-j-k}} + \partial^j u_n(x) \partial^k \left( \frac{1}{d(x)^{m-j-k}} \right) \\
= \lim_{n \to \infty} \partial^k \left( \frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right).
\]

Therefore, Fatou’s Lemma applies and we obtain
\[
\left\| \partial^k \left( \frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq \liminf_{n \to \infty} \left\| \partial^k \left( \frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)}. 
\]

\(^1\) In general, one would say: “For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we denote by $\partial^\alpha$ the partial derivative of order $l = |\alpha| = \alpha_1 + \cdots + \alpha_N$. Since we only care about the order of the operator, it makes sense to abuse the notation and identify $\alpha$ with its order $|\alpha| = l$.\]
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Once (2) has been proven for $u_n \in C_0^\infty(\Omega)$, we get

$$\left\| \partial^k \left( \frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u_n\|_{W^{m,1}(\Omega)},$$

and thus we can conclude that

$$\left\| \partial^k \left( \frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \liminf_{n \to \infty} \|u_n\|_{W^{m,1}(\Omega)} = C \|u\|_{W^{m,1}(\Omega)}.$$

Finally estimate (2) together with the fact that $\partial^j u_n(x)/d(x)^{m-j-k} \in C_0^\infty(\Omega)$ and $C_0^\infty(\Omega)W^{k,1}(\Omega) = W^{k,1}_0(\Omega)$ implies that $\partial^j u(x)/d(x)^{m-j-k} \in W^{k,1}_0(\Omega)$.

3. The case $m = 2$

We begin this section by proving estimate (2) in Theorem 1 for $\Omega = \mathbb{R}^N_+$, $m = 2$, $j = 0$ and $k = 1$.

**Lemma 3.1.** Suppose that $u \in C_0^\infty(\mathbb{R}^N_+)$. Then for all $i = 1, \ldots, N$,

$$\left\| \frac{\partial}{\partial y_i} \left( \frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}^N_+)} \leq 2 \|u\|_{W^{2,1}(\mathbb{R}^N_+)}.$$

**Proof.** Consider first the case $i = N$. This is similar to (1), but for completeness, we provide the proof. Notice that we can write

$$\frac{\partial}{\partial y_N} \left( \frac{u(\tilde{y}, y_N)}{y_N} \right) = \frac{1}{y_N^2} \int_0^{y_N} \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \, dt,$$

hence by integrating we obtain

$$\int_{\mathbb{R}^{N-1}} \int_0^{\infty} \left| \frac{\partial}{\partial y_N} \left( \frac{u(\tilde{y}, y_N)}{y_N} \right) \right| \, d\tilde{y} \, dy_N \leq \int_{\mathbb{R}^{N-1}} \int_0^{\infty} \frac{1}{y_N^2} \int_0^{y_N} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| \, dt \, d\tilde{y} \, dy_N \leq \int_{\mathbb{R}^{N-1}} \int_0^{\infty} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| \, dt \, d\tilde{y}.$$

hence

$$\int_{\mathbb{R}^N} \left| \frac{\partial}{\partial y_N} \left( \frac{u(y)}{y_N} \right) \right| \, dy \leq \int_{\mathbb{R}^N} \left| \frac{\partial^2}{\partial y_N^2} u(y) \right| \, dy.$$
When $1 \leq i \leq N - 1$, we need to estimate $\int_{\mathbb{R}^N} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy$. To do so, consider the change of variables $y = \Psi(x)$, where

$$\Psi(x_1, \ldots, x_i, \ldots, x_N) = (x_1, \ldots, x_i + x_N, \ldots, x_N).$$

Notice that $\det D\Psi(x) = 1$, hence

$$\int_{\mathbb{R}^N} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy = \int_{\mathbb{R}^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx.$$

Observe that if we let $v(x) = u(\Psi(x))$, we can write

$$\frac{1}{x_N} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N} \left( \frac{v(x)}{x_N} \right) - \frac{\partial}{\partial y_N} \left( \frac{u(y)}{y_N} \right) \bigg|_{y = \Psi(x)}.$$  \hspace{\stretch{1}} (9)

Applying estimate (7) to $u$ and $v$ yields

$$\int_{\mathbb{R}^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx \leq \int_{\mathbb{R}^N} \frac{\partial}{\partial x_N} \left( \frac{v(x)}{x_N} \right) \bigg| dx + \int_{\mathbb{R}^N} \frac{\partial}{\partial y_N} \left( \frac{u(y)}{y_N} \right) \bigg|_{y = \Psi(x)} \bigg| dx$$
$$= \int_{\mathbb{R}^N} \frac{\partial}{\partial x_N} \left( \frac{v(x)}{x_N} \right) \bigg| dx + \int_{\mathbb{R}^N} \frac{\partial}{\partial y_N} \left( \frac{u(y)}{y_N} \right) \bigg| dy$$
$$\leq \int_{\mathbb{R}^N} \frac{\partial^2 v(x)}{\partial x_N^2} \bigg| dx + \int_{\mathbb{R}^N} \frac{\partial^2 u(y)}{\partial y_N^2} \bigg| dy.$$  \hspace{\stretch{1}} (10)

Finally, notice that

$$\frac{\partial^2 v(x)}{\partial x_N^2} = \frac{\partial^2 u(y)}{\partial y_N^2} \bigg|_{y = \Psi(x)} + 2 \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \bigg|_{y = \Psi(x)} + \frac{\partial^2 u(y)}{\partial y_i^2} \bigg|_{y = \Psi(x)}.$$  \hspace{\stretch{1}} (10)

Thus, after reversing the change of variables when needed, we obtain

$$\int_{\mathbb{R}^N} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy = \int_{\mathbb{R}^N} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx$$
$$\leq 2 \int_{\mathbb{R}^N} \frac{\partial^2 u(y)}{\partial y_N^2} \bigg| dy + 2 \int_{\mathbb{R}^N} \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \bigg| dy + \int_{\mathbb{R}^N} \frac{\partial^2 u(y)}{\partial y_i^2} \bigg| dy$$
$$\leq 2\|u\|_{W^{2,1}(\mathbb{R}^N)}.$$

Recall (see Section 2) that for every $\tilde{x}_0 \in \partial \Omega$, we have the neighborhood $\mathcal{N}_+(\tilde{x}_0) \subset \Omega$ given by (4) and the diffeomorphism $\Phi : B^{N-1}_r(0, \epsilon_0) \to \mathcal{N}_+(\tilde{x}_0)$ given by (5). Moreover, we know that $\Phi(\lambda)$ is smooth over $\mathcal{N}_+(\tilde{x}_0)$.

**Lemma 3.2.** Let $\tilde{x}_0 \in \partial \Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by (4), and suppose $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $i = 1, \ldots, N$,

$$\left\| \frac{\partial}{\partial x_i} \frac{u(x)}{\delta(x)} \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C\|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$
Proof. We first use Corollary 2.1 to obtain
\[
\int_{\mathcal{N}_+(\epsilon_0)} \partial_i \left( \frac{u(x)}{\delta(x)} \right) \, dx \leq C \int_{B^{N-1}_\epsilon} \int_0^{\epsilon_0} \left| \partial_i \left( \frac{u(x)}{\delta(x)} \right) \right|_{x=\Phi(\tilde{y}, y_N)} \, dy_N \, d\tilde{y}.
\]
Let \( v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N)) \). We claim that
\[
\int_{B^{N-1}_\epsilon} \int_0^{\epsilon_0} \left| \partial_i \left( \frac{u(x)}{\delta(x)} \right) \right|_{x=\Phi(\tilde{y}, y_N)} \, dy_N \, d\tilde{y}
\leq C \sum_{j=1}^N \int_{B^{N-1}_\epsilon} \int_0^{\epsilon_0} \left| \partial_j \left( \frac{v(\tilde{y}, y_N)}{y_N} \right) \right| \, dy_N \, d\tilde{y}. \quad (11)
\]
We will prove (11) at the end, so that we can conclude the argument. Since \( v \in C^\infty_0(B^{N-1}_\epsilon \times (0, \epsilon_0)) \subset C^\infty_0(R_{N-1}^N) \), we can apply Lemma 3.1 to obtain
\[
\int_{B^{N-1}_\epsilon} \int_0^{\epsilon_0} \left| \partial_j \left( \frac{v(\tilde{y}, y_N)}{y_N} \right) \right| \, dy_N \, d\tilde{y} \leq C \|v\|_{W^{2,1}(B^{N-1}_\epsilon \times (0, \epsilon_0))}.
\]
Notice that the chain rule and the fact that \( \Phi \) is a diffeomorphism imply that for all \( 1 \leq i, j \leq N \),
\[
|\partial_{ij} v(\tilde{y}, y_N)| \leq C \left( \sum_{p,q=1}^N |\partial_{pq}^2 u(x)|_{x=\Phi(\tilde{y}, y_N)} + \sum_{p=1}^N |\partial_p u(x)|_{x=\Phi(\tilde{y}, y_N)} \right),
\]
so with the aid of Corollary 2.1, we can write
\[
\|v\|_{W^{2,1}(B^{N-1}_\epsilon \times (0, \epsilon_0))} \leq C \int_{B^{N-1}_\epsilon} \int_0^{\epsilon_0} \left( \sum_{p,q=1}^N |\partial_{pq}^2 u(x)|_{x=\Phi(\tilde{y}, y_N)} + \sum_{p=1}^N |\partial_p u(x)|_{x=\Phi(\tilde{y}, y_N)} \right) \, dy_N \, d\tilde{y}
\leq C \int_{\mathcal{N}_+(\epsilon_0)} \left( \sum_{p,q=1}^N |\partial_{pq}^2 u(x)| + \sum_{p=1}^N |\partial_p u(x)| \right) \, dx \leq C \|u\|_{W^{2,1}(\mathcal{N}_+(\epsilon_0))}.
\]
To conclude, we need to prove (11). To do so, notice that \( u(x) = v(\Phi^{-1}(x)) \), and \( \delta(x) = c(\Phi^{-1}(x)) \), where \( c(\tilde{y}, y_N) = y_N \). Thus, by using the chain rule we obtain
\[
\left| \partial_i \left( \frac{u(x)}{\delta(x)} \right) \right|_{x=\Phi(\tilde{y}, y_N)} = \sum_{j=1}^N \partial_j \left( \frac{v(y)}{c(y)} \right) \bigg|_{y=\Phi(\tilde{y}, y_N)} \cdot \partial_i (\Phi^{-1})_j(\Phi(\tilde{y}, y_N)),
\]
and since \( \Phi \) is a diffeomorphism, we obtain
\[
\left| \partial_i \left( \frac{u(x)}{\delta(x)} \right) \right|_{x=\Phi(\tilde{y}, y_N)} \leq C \sum_{j=1}^N \left| \partial_j \left( \frac{v(y)}{c(y)} \right) \right|_{y=\Phi(\tilde{y}, y_N)}.
\]
Estimate (11) then follows by integrating the above inequality. \( \square \)
Lemma 3.2 applies and we obtain
\[ \partial_i \left( \frac{u(x)}{d(x)} \right) \leq C \| u \|_{W^{2,1}(\Omega)} \] (12)
for \( u \in C_0^\infty(\Omega) \) and \( i = 1, \ldots, N \). To do so, we use the partition of unity given by Lemma 2.3 to write \( u(x) = \sum_{l=0}^M u_l(x) \) on \( \Omega \) where \( u_l(x) := \rho_l(x) u(x), l = 0, 1, \ldots, M \). Now, without loss of generality, we can assume that \( d(x) = \delta(x) \) for all \( x \in \Omega_{\epsilon_0} \), and that \( d(x) \geq C > 0 \) for all \( x \in \text{supp} \rho_0 \cap \Omega \). Notice that in \( \text{supp} \rho_0 \cap \Omega \), we have
\[ \frac{u_0}{d} \in C_0^\infty(\text{supp} \rho_0 \cap \Omega) \quad \text{with} \quad \| \frac{u_0}{d} \|_{W^{1,1}(\text{supp} \rho_0 \cap \Omega)} \leq C \| u_0 \|_{W^{1,1}(\text{supp} \rho_0 \cap \Omega)}. \]
To take care of the boundary part, notice that \( u_l \in C_0^\infty(\mathcal{N}_+(\tilde{\epsilon}_l)) \) for \( l = 1, \ldots, M \), so Lemma 3.2 applies and we obtain
\[ \partial_i \left( \frac{u_l(x)}{\delta(x)} \right) \leq C \| u_l \|_{W^{2,1}(\mathcal{N}_+(\tilde{\epsilon}_l))} \quad \text{for all } l = 1, \ldots, M. \]
To conclude, notice that \( \partial_i \left( \frac{u(x)}{d(x)} \right) = \sum_{l=1}^M \partial_i \left( \frac{u_l(x)}{\delta(x)} \right) + \partial_i \left( \frac{u_0(x)}{d(x)} \right) \) on \( \Omega \) and that \( |\rho_l(x)|, |\partial_i \rho_l(x)| \) and \( |\partial_i^2 \rho_l(x)| \) are uniformly bounded for all \( l = 0, 1, \ldots, M \), therefore
\[
\left\| \partial_i \left( \frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \leq \sum_{l=1}^M \left\| \partial_i \left( \frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{\epsilon}_l))} + \left\| \partial_i \left( \frac{u_0(x)}{d(x)} \right) \right\|_{L^1(\text{supp} \rho_0 \cap \Omega)} \\
\leq C \left( \sum_{l=1}^M \| u_l \|_{W^{2,1}(\mathcal{N}_+(\tilde{\epsilon}_l))} + \| u_0 \|_{W^{1,1}(\text{supp} \rho_0 \cap \Omega)} \right) \\
\leq C \left( \sum_{l=1}^M \| u \|_{W^{2,1}(\mathcal{N}_+(\tilde{\epsilon}_l))} + \| u \|_{W^{1,1}(\text{supp} \rho_0 \cap \Omega)} \right) \leq C \| u \|_{W^{2,1}(\Omega)},
\]
thus completing the proof. \( \Box \)

4. The general case \( m \geq 2 \)
To prove the general case, we need to generalize Lemma 3.1 in the following way

Lemma 4.1. Suppose \( u \in C_0^\infty(\mathbb{R}_N^m) \). Then for all \( m \geq 1 \) and \( i = 1, \ldots, N \) we have
\[
\left\| \partial_i \left( \frac{u(x)}{y_m^{m-1}} \right) \right\|_{L^1(\mathbb{R}_N^m)} \leq C \| u \|_{W^{1,m}(\mathbb{R}_N^m)},
\]
Proof. The case \( m = 1 \) is a trivial statement, whereas \( m = 2 \) is exactly what we proved in Lemma 3.1. So from now on we suppose \( m \geq 3 \). We first notice that when \( i = N \), the result follows from the proof of [4, Theorem 1.2] when \( j = 0 \) and \( k = 1 \). We refer the reader to [4] for the details.
When \(1 \leq i \leq N - 1\), we can proceed as in the proof of Lemma 3.1. Define \(v(x) = u(\Psi(x))\) where \(\Psi\) is given by (8). Notice that when \(m \geq 3\), instead of equation (9) we have
\[
\frac{1}{x_N^{m-1}} \frac{\partial u}{\partial y_i} (\Psi(x)) = \frac{\partial}{\partial x_N} \left( \frac{u(x)}{x_N^{m-1}} \right) - \frac{\partial}{\partial y_N} \left( \frac{u(x)}{y_N^{m-1}} \right) \bigg|_{y = \Psi(x)},
\]
and instead of (10) we have
\[
\frac{\partial^m v(x)}{\partial x_N^m} = \sum_{l=0}^{m} \binom{m}{l} \frac{\partial^l u(y)}{\partial y_N^{m-l}} \bigg|_{y = \Psi(x)}.
\]
Hence the estimate is reduced to the already proven result for \(i = N\). We omit the details. \(\Box\)

We also have the analog of Lemma 3.2.

**Lemma 4.2.** Let \(\tilde{x}_0 \in \partial \Omega\) and \(\mathcal{N}_+ (\tilde{x}_0)\) as in Lemma 3.2. Let \(u \in C_0^\infty (\mathcal{N}_+ (\tilde{x}_0))\). Then for all \(m \geq 1\) and \(i = 1, \ldots, N\) we have
\[
\left\| \frac{\partial^i}{\partial x_N^i} \left( \frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1 (\mathcal{N}_+ (\tilde{x}_0))} \leq C \| u \|_{W^{m,1} (\mathcal{N}_+ (\tilde{x}_0))}.
\]

**Proof.** The proof involves only minor modifications from the proof of Lemma 3.2, which we provide in the next few lines. Corollary 2.1 gives
\[
\int_{\mathcal{N}_+ (\tilde{x}_0)} \left| \frac{\partial}{\partial x_N} \left( \frac{u(x)}{\delta(x)^{m-1}} \right) \right| \, dx \leq C \int_{B_N^{N-1} \times (0, \epsilon_0)} \left| \frac{\partial}{\partial \tilde{y}_i} \left( \frac{u(\tilde{y}, \tilde{y}_N)}{(\tilde{y}_N)^{m-1}} \right) \right|_{\tilde{y} = \Phi(\tilde{x}, \tilde{y}_N)} \, d\tilde{y}_N \, d\tilde{y}.
\]
If \(v(\tilde{y}, \tilde{y}_N) = u(\Phi(\tilde{y}, \til{y}_N))\), then
\[
\int_{B_N^{N-1}} \int_{0}^{\epsilon_0} \left| \frac{\partial}{\partial \til{y}_i} \left( \frac{u(x)}{\delta(x)^{m-1}} \right) \right|_{x = \Phi(\til{y}, \til{y}_N)} \, d\til{y}_N \, d\til{y} 
\leq C \sum_{j=1}^{N} \int_{B_N^{N-1}} \int_{0}^{\epsilon_0} \left| \frac{\partial}{\partial \til{y}_j} \left( \frac{v(\til{y}, \til{y}_N)}{(\til{y}_N)^{m-1}} \right) \right| \, d\til{y}_N \, d\til{y}.
\]
Just as for (11), estimate (13) follows from the fact that \(\Phi\) is a smooth diffeomorphism. Since \(v \in C_0^\infty (B_N^{N-1} \times (0, \epsilon_0)) \subset C_0^\infty (\mathbb{R}_+^N)\), we can apply Lemma 4.1 and obtain
\[
\int_{B_N^{N-1}} \int_{0}^{\epsilon_0} \left| \frac{\partial}{\partial \til{y}_j} \left( \frac{v(\til{y}, \til{y}_N)}{(\til{y}_N)^{m-1}} \right) \right| \, d\til{y}_N \, d\til{y} \leq C \| v \|_{W^{m,1} (B_N^{N-1} \times (0, \epsilon_0))}.
\]
Notice that by the chain rule and the fact that \(\Phi\) is a smooth diffeomorphism, we get
\[
|\partial^m v(\til{y}, \til{y}_N)| \leq C \sum_{l \leq m} |\partial^l u(x)|_{l = \Phi(\til{y}, \til{y}_N)},
\]
where the left hand side is a fixed \( m \)-th order partial derivative, and on the right hand side the summation contains all partial derivatives operators of order \( l \leq m \). Again with the aid of Corollary 2.1, we can write

\[
\|v\|_{W^m,1(B^{N-1} \times (0, \epsilon_0))} \leq C \sum_{l \leq m} \int_{B^{N-1}} \int_0^{\epsilon_0} (|\partial^l u|_{x=\Phi(y, y_N)}) \, dy_N \, dy \\
\leq C \sum_{l \leq m} \int_{\mathcal{N}_x} |\partial^l u(x)| \, dx \leq C\|u\|_{W^{m,1}(\mathcal{N}_x(\tilde{\epsilon}_0))}. \tag*{\Box}
\]

And of course we have

**Lemma 4.3.** Suppose \( u \in C_0^\infty(\Omega) \). Then for all \( m \geq 1 \) and \( i = 1, \ldots, N \) we have

\[
\left\| \frac{\partial_i}{\lambda(x)^{m-1}} \right\|_{L^1(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)}.
\]

We omit the proof, because it is almost a line by line copy of the proof of the estimate (12) in Section 3 using the partition of unity. We are now ready to prove Theorem 1.

**Proof Theorem 1.** For any fixed integer \( m \geq 3 \), just as in the case \( m = 2 \), it is enough to prove the estimate (2) for \( u \in C_0^\infty(\Omega) \). Notice that since

\[
\|\partial_j^l u\|_{W^{m-l,1}(\Omega)} \leq \|u\|_{W^{m,1}(\Omega)} \quad \text{for all } 0 \leq j \leq m,
\]

it is enough to show

\[
\left\| \partial^k \left( \frac{u(x)}{\lambda(x)^{m-k-1}} \right) \right\|_{L^1(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)} \tag{14}
\]

for \( u \in C_0^\infty(\Omega) \) and \( 1 \leq k \leq m - 1 \). We proceed by induction on \( k \). The case \( k = 1 \) corresponds exactly to Lemma 4.3. If one assumes the result for \( k \), then we have to estimate, for \( i = 1, \ldots, N \),

\[
\partial_i \partial^k \left( \frac{u(x)}{\lambda(x)^{m-k-1}} \right) = \partial^k \left( \frac{\partial_i u(x)}{\lambda(x)^{m-k-1}} \right) - (m - k - 1) \partial^k \left( \frac{u(x) \partial_i \lambda(x)}{\lambda(x)^{m-k}} \right).
\]

Using the induction hypothesis for \( \hat{m} = m - 1 \) yields

\[
\left\| \partial^k \left( \frac{\partial_i u(x)}{\lambda(x)^{m-k-1}} \right) \right\|_{L^1(\Omega)} \leq C\|\partial_i u\|_{W^{m-1,1}(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)};
\]

on the other hand, by using the induction hypothesis and the fact that \( d \) is smooth in \( \Omega \), we obtain

\[
\left\| \partial^k \left( \frac{u(x) \partial_i d(x)}{\lambda(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \leq C\|u \partial_i d\|_{W^{m,1}(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)}.
\]
A Hardy type inequality for $W^{m,1}_0(\Omega)$ functions

Therefore
\[
\left\| \partial_i \partial^k \left( \frac{u(x)}{d(x)^{m-k-1}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},
\]
thus concluding the proof. □

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