Mathematical Analysis — Optimal regularity results in spaces of Hölder continuous functions for some infinite dimensional Ornstein-Uhlenbeck semigroup, by Giuseppe Da Prato.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We consider the elliptic equation \( \lambda \phi - L\phi = f \) where \( \lambda > 0 \), \( f \) is \( \theta \)-Hölder continuous and \( L \) is an Ornstein-Uhlenbeck operator in a Hilbert space \( H \). We show that the mapping \( D^2\phi \) (with values in the space of Hilbert–Schmidt operators on \( H \)) is \( \theta \)-Hölder continuous.

Key words: PDEs with infinitely many variables, Schauder estimates, Ornstein-Uhlenbeck semigroup.

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1. Introduction and setting of the problem

Let \( H \) be a separable real Hilbert space (norm \( | \cdot | \), inner product \( \langle \cdot, \cdot \rangle \)). We are given a linear operator \( A: D(A) \subset H \to H \) such that

Hypothesis 1.1.

(i) \( A \) is self-adjoint and there exists \( \omega > 0 \) such that

\[
\langle Ax, x \rangle \leq -\omega |x|^2, \quad x \in D(A).
\]

(ii) \( A^{-1} \) is of trace class.

As well known, Hypothesis 1.1 implies that there exists a complete orthonormal system \( (e_k) \) in \( H \) and a sequence of real numbers \( (a_k) \) greater than \( \omega \) such that

\[
Ae_k = -a_k e_k, \quad \forall k \in \mathbb{N}
\]

and

\[
\text{Tr}[A^{-1}] = \sum_{k=1}^{\infty} \frac{1}{a_k} < +\infty.
\]
Under Hypothesis 1.1 we can consider the Ornstein–Uhlenbeck semigroup defined by (see [6])

\[ R_t \varphi(x) = \int_H \varphi(e^{tA}x + y)N_Q(dy), \quad \forall t > 0, x \in H. \]  

Here \( \varphi : H \to \mathbb{R} \) is any continuous function with e.g. polynomial growth (that is such that \( |\varphi(x)| \leq M(1 + |x|^n) \) for all \( x \in H \) and some \( M > 0, n \in \mathbb{N} \)) and \( N_Q_t \) is the Gaussian measure in \( H \) with mean 0 and covariance operator \( Q_t \) given by

\[ Q_t = -\frac{1}{2} A^{-1}(1 - e^{2tA}), \quad \forall t \geq 0. \]

Note that the Gaussian measure \( N_Q_t \) is well defined since \( A^{-1} \), and consequently \( Q_t \), is of trace class.

Let us define the infinitesimal generator \( L \) of \( R_t \) through its Laplace transform (as in [2]) setting for any \( \lambda > 0 \) and for any continuous function \( f : H \to \mathbb{R} \) with polynomial growth

\[ (\lambda - L)^{-1} f(x) = \int_0^{\infty} e^{-\lambda t} R_t f(x) \, dt, \quad \forall x \in H. \]  

The operator \( L \) acts as a concrete differential operator on the space \( \delta_A(H) \) of all exponential functions defined as the linear span of all real parts of functions \( \varphi_h \) of the form

\[ \varphi_h(x) = e^{i\langle x, h \rangle}, \quad \forall x \in H, \]

where \( h \) varies in \( D(A) \). It is not difficult in fact to check that

\[ L\varphi = \frac{1}{2} \text{Tr}[D^2\varphi] + \langle x, AD\varphi \rangle, \quad \forall \varphi \in \delta_A(H). \]

This paper is devoted to the study of the elliptic equation

\[ \lambda \varphi - L \varphi = f, \]

where \( \lambda > 0 \) is a given number and \( f \) is a given function in a suitable functional space. As we shall see there is a dramatic difference between the case when \( H \) is finite or infinite dimensional. In order to better illustrate this difference it is convenient to recall what happens when \( f \) belongs to \( L^2(H, \mu) \) where \( \mu \) is the unique invariant measure of \( R_t, t \geq 0 \). The short Section 2 is devoted to recall the main results in this case. Finally, Section 3 is devoted to study (1.7) in spaces of Hölder continuous functions. We first recall previous optimal regularity result proved in [1] and [3] and then we present a new optimal regularity result. This last result will allow us to take into account a new kind of perturbations of the Ornstein–Uhlenbeck diffusion process for which it is will possible to prove existence and
uniqueness of an associated martingale problems, arguing as in [13]. These facts will be the object of a future paper.

Remark 1.2. \( R_t \) is the transition semigroup of the diffusion process \( X(t), t \geq 0 \), the solution to the differential stochastic equation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{dX(t)}{dt} = AX(t) + dW(t), \\
X(0) = x \in H,
\end{array}
\right. \\
\end{aligned}
\]

where \( W(t) \) is a cylindrical Wiener process in some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) taking values in \( H \). We can take \( W(t) \) as

\[
\langle W(t), z \rangle = \sum_{k=0}^{\infty} \beta_k \langle z, e_k \rangle, \quad \forall z \in H,
\]

where \( (\beta_k) \) is a family of mutually independent standard Brownian motions on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Then we have

\[
R_t \phi(x) = \mathbb{E}[\phi(X(t, x))], \quad t \geq 0, \ x \in H, \ \phi \in C_b(H),
\]

where \( \mathbb{E} \) denotes the expectation.

2. Optimal regularity results for \( f \in L^2(H, \mu) \)

By (1.3) it follows easily that \( \mu = N_{\frac{1}{2}A^{-1}} \), where

\[
Q_{\infty} = -\frac{1}{2}A^{-1},
\]

is the unique invariant measure for \( R_t, t \geq 0 \), that is

\[
\int_H R_t \phi(x) \mu(dx) = \int_H \phi(x) \mu(dx),
\]

for all \( \phi : H \rightarrow \mathbb{R} \) continuous and bounded. So, \( R_t \) can be uniquely extended to \( L^2(H, \mu) \) (even to \( L^p(H, \mu) \) for any \( p \geq 1 \)) which we shall denote by \( R_t^2 \). The infinitesimal generator of \( R_t^2 \) will be denoted by \( L_2 \).

The following result can be found in [7], see also [4, (10.55)].

Proposition 2.1. Let \( \lambda > 0 \) and \( f \in L^2(H, \mu) \). Then equation (1.8) has a unique solution \( \phi \in D(L_2) \) with the following properties

\[
\begin{aligned}
\phi &\in W^{2,2}(H, \mu), \\
(-A)^{1/2}D\phi &\in L^2(H, \mu; H).
\end{aligned}
\]
Moreover the following identity holds.

\[ \int_H (L_2 \varphi)^2 d\mu = \frac{1}{2} \int_H \text{Tr}[(D^2 \varphi)^2] d\mu + \int_H \|(-A)^{1/2} D\varphi\|^2 d\mu. \]

Notice that if the dimension of \( H \) is finite, equation (1.8) reduces to

\[ \lambda \varphi - \frac{1}{2} \Delta \varphi - \langle x, AD\varphi \rangle = f, \]

so that, by (2.1) it follows that both terms

\[ \Delta \varphi, \quad \langle x, AD\varphi \rangle \]

belong to \( L^2(H, \mu) \). Nothing similar happens if the dimension of \( H \) is infinite. In this case we have no information on the terms

\[ \frac{1}{2} \text{Tr}[D^2 \varphi], \quad \langle x, AD\varphi \rangle, \]

we know only that the sum of these two terms is meaningful. However, the weaker informations (2.1) and (2.2) are available. When \( H \) is infinite dimensional \( A \) is unbounded and so, identity (2.3) shows that they are in a sense optimal.

3. Optimal regularity results in space of Hölder continuous functions

3.1. Introduction

Here we consider equation (1.8) when \( f \) belongs to the space of all \( \theta \)-Hölder continuous and bounded real functions on \( H \), which we denote by \( C^\theta_b(H) \).

We start by recalling some known results.

**Theorem 3.1.** Assume that Hypothesis 1.1 holds. Let \( \theta \in (0, 1) \), \( f \in C^\theta_b(H) \), \( \lambda > 0 \) and let \( \varphi = (\lambda - L)^{-1}f \) be the solution to (1.8). Then the following statements hold.

(i) \( \varphi \) belongs to \( C^{2+\theta}_b(H) \) and there exists \( M > 0 \) (independent on \( \lambda \) and on \( f \)) such that

\[ \|\varphi\|_{C^{2+\theta}_b(H)} \leq M\|f\|_{C^\theta_b(H)}. \]

(ii) For all \( x \in H \) we have \( D\varphi(x) \in D((-A)^{1/2}) \) and \( (-A)^{1/2}D\varphi \in C^\theta_b(H) \). Moreover, there exists \( M_1 > 0 \) (independent on \( \lambda \) and on \( f \)) such that

\[ \|(-A)^{1/2}D\varphi\|_{C^\theta_b(H)} \leq M_1\|f\|_{C^\theta_b(H)}. \]
For a precise definition of $C^\theta_b(H)$ and $C^{2+\theta}_b(H)$ see the end of this subsection.

The Schauder estimate (i) was proved in [1] whereas (ii) was proved in [3]. Clearly

(ii) is a counterpart of (2.2) in the Hölder setting. The main result of this paper is the proof of a counterpart of (2.1), namely that if $f \in C^\theta_b(H)$ then

(iii) $D^2 \varphi \in C^\theta_b(H, L^2(H))$ and there exists $M_3 > 0$ (independent on $\lambda$ and on $f$) such that

$$\|D^2 \varphi\|_{C^\theta_b(H, L^2(H))} \leq M_3 \|f\|_{C^\theta_b(H)}.$$  

(3.3)

**Remark 3.2.** When $H$ is finite-dimensional, the Schauder estimates (3.1) were proved in [5]. Even in this case they are not consequence of the general results in [8] because the Ornstein–Uhlenbeck operator has unbounded coefficients.

**Remark 3.3.** A result similar to (iii) was proved for the Gross Laplacian by [11].

Let us finish this section by giving some notation and by recalling the definition of interpolation spaces needed in what follows.

3.1.1 Notations. In all the paper $H$ is a separable Hilbert space, $A : D(A) \subset H \to H$ is a linear operator fulfilling Hypothesis 1.1 and $(e_h)$ is an orthonormal basis defined by (1.2). For each $x \in H$ and any $h \in \mathbb{N}$ we set $x_h = \langle x, e_h \rangle$.

By $L^2(H)$ we denote the Hilbert space of all Hilbert–Schmidt operators from $H$ into $H$ endowed with the inner product

$$\langle T, S \rangle = \text{Tr}[TS^*], \quad \forall T \in L^2(H)$$

and the norm

$$\|T\|^2_{L^2(H)} = \text{Tr}[TT^*] = \sum_{h,k=1}^{\infty} |\langle Te_h, e_k \rangle|^2, \quad \forall T \in L^2(H).$$

Let $E$ be a Banach space. We shall denote by $C_b(H; E)$ the Banach space of all uniformly continuous and bounded functions from $H$ into $E$ endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|_E$. For any $k \in \mathbb{N}$ we denote by $C^k(H; E)$ the space of all mappings $\varphi : H \to E$ which are uniformly continuous and bounded together with their derivatives up to the $k$-th order. $C^k(H; E)$ is a Banach space with the norm

$$\|\varphi\|_k = \sum_{h=1}^{k} \sup_{x \in H} \|D^h \varphi(x)\|.$$  

Here $D^h \varphi(x)$ is the derivative of $\varphi$ at $x$ of order $h$ and $\|D^h \varphi(x)\|$ is the usual norm of the $h$-linear form $D^h \varphi(x)$. 
Finally, if \( \theta \in (0, 1) \), we shall denote by \( C^0_b(H; E) \) (resp. \( C^{k+\theta}_b(H; E) \), \( k \in \mathbb{N} \)) the subspace of \( C_b(H; E) \) (resp. \( C^k(H; E) \)) consisting of all functions \( \varphi : H \to E \) such that

\[
[\varphi]_\theta := \sup_{x, y \in H \atop x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{\theta}} < +\infty,
\]

(respectively,

\[
[\varphi]_{k+\theta} := \sup_{x, y \in H \atop x \neq y} \frac{\|D^k \varphi(x) - D^k \varphi(y)\|}{|x - y|^{\theta}} < +\infty.
\]

\( C^0_b(H; E) \) is a Banach space with the norm

\[
\|\varphi\|_\theta := \|\varphi\|_0 + [\varphi]_\theta, \quad \varphi \in C^0_b(H; E).
\]

When \( E = \mathbb{R} \) we shall write \( C^k_b(H; \mathbb{R}) = C^k_b(H) \) and \( C^{k+\theta}_b(H; \mathbb{R}) = C^{k+\theta}_b(H) \).

3.1.2 Interpolation spaces. We shall use the \( K \) method for real interpolation spaces, see e.g. [12]. Let \( X \) and \( Y \) be Banach spaces such that \( Y \subset X \) with continuous embedding. For any \( t > 0 \) and any \( x \in H \) define

\[
K(t, x) = \inf \{ \|a\|_X + t \|b\|_Y : x = a + b, a \in X, b \in Y \}.
\]

Then, for arbitrary \( \theta \in (0, 1) \), set

\[
\|x\|_{(X, Y)_{\theta, \infty}} = \sup_{t > 0} t^{-\theta} K(t, x),
\]

\( (X, Y)_{\theta, \infty} = \{ x \in X : \|x\|_{(X, Y)_{\theta, \infty}} < +\infty \} \).

As is easily seen \( (X, Y)_{\theta, \infty} \), endowed with the norm

\[
\|x\|_{(X, Y)_{\theta, \infty}},
\]

is a Banach space.

Remark 3.4. It is not difficult to check that the following statement (i):

(i) For all \( t > 0 \) there exist \( a_t \in X \) and \( b_t \in Y \) such that \( x = a_t + b_t \) and

\[
\|a_t\|_X + t \|b_t\|_Y \leq L t^\theta,
\]

implies that

(ii) \( x \in (X, Y)_{\theta, \infty} \) and \( \|x\|_{(X, Y)_{\theta, \infty}} \leq L \).

Conversely, statement (ii) implies that \( \forall \varepsilon > 0, \forall t > 0 \) there exist \( a_t \in X \) and \( b_t \in Y \) such that \( x = a_t + b_t \) and

\[
\|a_t\|_X + t \|b_t\|_Y \leq (L + \varepsilon) t^\theta.
\]
Let us recall the basic interpolation theorem, see e.g. [12].

**Theorem 3.5.** Let $X, X_1, Y, Y_1$ be Banach spaces such that $Y \subset X, Y_1 \subset X_1$ with continuous embeddings. Let moreover $T$ be a linear mapping $T : X \to X_1, T : Y \to Y_1$, such that for some $M, N > 0$

$$
\|Tx\|_{X_1} \leq M\|x\|_X, \quad \|Ty\|_{Y_1} \leq N\|y\|_Y.
$$

Then $T$ maps $(X, Y)_{\theta, \infty}$ into $(X_1, Y_1)_{\theta, \infty}$, and

$$
\|Tx\|_{(X_1, Y_1)_{\theta, \infty}} \leq M^{1-\theta}N^\theta\|x\|_{(X, Y)_{\theta, \infty}}, \quad x \in (X, Y)_{\theta, \infty}.
$$

We shall need also the following result, see [1].

**Theorem 3.6.** Let $K$ be a separable Hilbert space. Then we have

$$
(C_b(K), C^1_b(K))_{\theta, \infty} = C^0_b(K), \quad \forall \theta \in (0, 1).
$$

Moreover there exists a positive constant $\kappa_\theta$ such that

$$
\frac{1}{\kappa_\theta} \|\varphi\|_{C^0_b(K)} \leq \|\varphi\|_{(C_b(K), C^1_b(K))_{\theta, \infty}} \leq \kappa_\theta \|\varphi\|_{C^0_b(K)}.
$$

**Remark 3.7.** Let $\varphi \in C^0_b(K)$ and let $\theta \in (0, 1)$. By Remark 3.4 to prove that $\varphi \in C^0_b(K)$ it is enough to prove that for any $t \in (0, 1]$ there exist $a_t \in C^0_b(K)$ and $b_t \in C^1_b(K)$ such that $\varphi = a_t + b_t$ and

$$
\|a_t\|_0 \leq \kappa t^\theta, \quad \|b_t\|_1 \leq \kappa t^{\theta-1}
$$

for a suitable positive constant $\kappa$.

### 3.2. Estimates

We assume here that Hypothesis 1.1 holds. Under this assumption for any $t > 0$ and any $\varphi \in C^0_b(H)$ we have that $R_t\varphi \in C^\infty_b(H)$, see [7]. Moreover, the following expressions hold for the three first derivatives of $R_t\varphi$.

$$
\langle DR_t\varphi(x), \alpha \rangle = \int_H \langle \Lambda_t\alpha, Q_t^{1/2}y \rangle \varphi(e^{tA}x + y)N_Q(dy), \quad \forall x, \alpha \in H,
$$

$$
\langle D^2 R_t\varphi(x) \cdot \alpha, \beta \rangle = \int_H \langle \Lambda_t\alpha, Q_t^{-1/2}y \rangle \langle \Lambda_t\beta, Q_t^{-1/2}y \rangle \varphi(e^{tA}x + y)N_Q(dy)
\quad - \langle \Lambda_t\alpha, \Lambda_t\beta \rangle R_t\varphi(x), \quad \forall x, \alpha, \beta \in H.
$$

and, for any $x, \alpha, \beta, \gamma \in H$, ...
\( D^3 R_t \varphi(x)(x, \beta, \gamma) \)

\[
= \int_H \langle \Lambda_t x, Q_t^{-1/2} y \rangle \langle \Lambda_t \beta, Q_t^{-1/2} y \rangle \langle \Lambda_t \gamma, Q_t^{-1/2} y \rangle \varphi(e^{tA}x + y)N_Q(dy) \\
- (\langle \Lambda_t x, \Lambda_t \beta \rangle D_r R_t \varphi(x) + \langle \Lambda_t x, \Lambda_t \gamma \rangle D_r R_t \varphi(x) \\
+ \langle \Lambda_t \beta, \Lambda_t \gamma \rangle D_x R_t \varphi(x)).
\]

Here we have set

\[
\Lambda_t = Q_t^{-1/2} e^{tA} = \sqrt{2}(-A)^{1/2} e^{tA}(1 - e^{2tA})^{-1/2}.
\]

**Lemma 3.8.** There exist \( c_1 > 0 \) such that

\[
\|\Lambda_t\| \leq c_1 t^{-1/2}, \quad \forall t > 0,
\]

**Proof.** It is enough to notice that

\[
\|\Lambda_t\| = \sup_{k \in \mathbb{N}} \sqrt{2a_k e^{-\lambda_k} (1 - e^{-2\lambda_k})^{-1/2}} \\
\leq t^{-1/2} \sup_{\xi > 0} \sqrt{2\xi e^{-\xi} (1 - e^{-2\xi})^{-1/2}}, \quad t > 0.
\]

**Lemma 3.9.** Let \( \varphi \in C_b(H) \) and \( t > 0 \). Then \( D^2 R_t \varphi \in C_b(H; L^2(H)) \) and there exists \( d_1 > 0 \) such that

\[
|D^2 R_t \varphi(x)|_{L^2(H)} \leq d_1 t^{-1} \|\varphi\|_0, \quad \forall t > 0, \ x \in H.
\]

**Proof.** By (3.8) we have for all \( h, k \in \mathbb{N} \)

\[
\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle = \int_H \langle \Lambda_t e_h, Q_t^{-1/2} y \rangle \langle \Lambda_t e_k, Q_t^{-1/2} y \rangle \varphi(e^{tA}x + y)N_Q(dy) \\
- \langle \Lambda_t e_h, \Lambda_t e_k \rangle R_t \varphi(x),
\]

which can be written as

\[
\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle = \Lambda_{t,h} \Lambda_{t,k} \lambda_h(t)^{-1/2} \lambda_k(t)^{-1/2} \int_H y_h y_k \varphi(e^{tA}x + y)N_Q(dy) \\
- \Lambda_{t,h}^2 \delta_{h,k} \int_H \varphi(e^{tA}x + y)N_Q(dy),
\]

where \( y_k = \langle y, e_k \rangle \) for all \( k \in \mathbb{N} \) and for \( t > 0 \), \( \Lambda_{t,k}, \ k \in \mathbb{N}, \) is the sequence of eigenvalues of \( \Lambda_t \) defined by,

\[
\Lambda_t e_k = \Lambda_{t,k} e_k, \quad \forall t > 0, \ k \in \mathbb{N},
\]

\[3.10\]
whereas \( \lambda_k(t), h \in \mathbb{N} \), are the sequence of eigenvalues of \( Q_t \),

\[
Q_t e_k = \lambda_k(t)e_k, \quad h \in \mathbb{N}.
\]

In order to estimate \( \|D^2 R_t \varphi(x)\|_{L^2(H)} \) we proceed as in [7, Lemma 6.2.7], introducing a suitable orthonormal system in \( L^2(H, N_{Q_t}) \). More precisely, for any \( t > 0 \) we define

\[
\Phi_{h,k}(t) = \begin{cases} 2^{-1/2}(\lambda_h^{-1}(t) y_h^2 - 1), & \text{if } h = k, \\ \lambda_h^{-1}(t) \lambda_k^{-1/2}(t) y_h y_k, & \text{if } h \neq k. \end{cases}
\]

(3.15)

(3.14)

(It is not difficult to check that \( (\Phi_{h,k}(t)) \) is indeed orthonormal in \( L^2(H, N_{Q_t}) \) for any \( t > 0 \).)

Now let \( h = k \in \mathbb{N} \) and write

\[
\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle = \sqrt{2} \Lambda_{t,k}^2 \int_H \Phi_{k,k}(t) \varphi(e^{tA} x + y) N_{Q_t}(dy) = \sqrt{2} \Lambda_{t,k}^2 \langle \Phi_{k,k}(t), \varphi(e^{tA} x + \cdot) \rangle_{L^2(H, N_{Q_t})}.
\]

Recalling that \( |\Lambda_{t,k}^2| = |\Lambda_t|^2 \) for all \( k \in \mathbb{N} \) and all \( t > 0 \) we have

\[
|\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2|\Lambda_t|^4 \langle \Phi_{k,k}(t), \varphi(e^{tA} x + \cdot) \rangle_{L^2(H, N_{Q_t})}^2.
\]

Summing up on \( k \) we deduce by the Parseval inequality that

\[
\sum_{k=1}^{\infty} |\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2|\Lambda_t|^4 \int_H |\varphi(e^{tA} x + y)|^2 N_{Q_t}(dy) = 2|\Lambda_t|^4 |\varphi|^2_0.
\]

Now from (3.11) we have

\[
(3.16) \quad \sum_{k=1}^{\infty} |\langle D^2 R_t \varphi(x) \cdot e_k, e_k \rangle|^2 \leq 2c_1^4 t^{-2} |\varphi|^2_0.
\]

Let now \( h \neq k \in \mathbb{N} \) and write

\[
\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle = \Lambda_{t,h} \Lambda_{t,k} \int_H \Phi_{h,k}(t) \varphi(e^{tA} x + y) N_{Q_t}(dy) = \Lambda_{t,h} \Lambda_{t,k} \langle \Phi_{h,k}(t), \varphi(e^{tA} x + \cdot) \rangle_{L^2(H, N_{Q_t})}.
\]

Proceeding as before we see that

\[
|\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle|^2 \leq |\Lambda_t|^4 |\langle \Phi_{h,k}(t), \varphi(e^{tA} x + \cdot) \rangle_{L^2(H, N_{Q_t})}|^2.
\]

By the Parseval inequality we deduce that
Now from (3.11) we have

$$\sum_{h,k=1, h \neq k}^{\infty} |\langle D^2 R_t \varphi(x) \cdot e_h, e_k \rangle|^2 \leq |\Lambda_i|^4 \|\varphi\|_0^2.$$  

By (3.16) and (3.17) it follows that

$$\text{Tr}[(D^2 R_t \varphi(x))^2] \leq 2 c_1^4 t^{-2} \|\varphi\|_0^2,$$

which proves the result with $d_1 = \sqrt{2} c_1^2$. However, it remains to show that $D^2 R_t \varphi \in C_b(H, L_2(H))$. To this purpose let us introduce a one-to-one mapping $\psi : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$. For any $x, y \in H$ and any $N \in \mathbb{N}$ we have

$$\|D^2 R_t \varphi(x) - D^2 R_t \varphi(y)\|_{L_2(H)}^2 = \sum_{(h,k): \psi(h,k) = 1}^N (D_h D_k R_t \varphi(x) - D_h D_k R_t \varphi(y))^2 + \sum_{(h,k): \psi(h,k) = N+1}^{\infty} (D_h D_k R_t \varphi(x) - D_h D_k R_t \varphi(y))^2 := I_1 + I_2.$$  

Now $I_2$ can be made arbitrarily small by (3.18) choosing $N$ sufficiently large, then $I_1$ goes to zero when $y$ is close to $x$ because all partial derivatives of $R_t \varphi$ are Lipschitz continuous. The proof is complete. \hfill $\Box$

Now we prove

**Lemma 3.10.** Let $\varphi \in C^1_b(H)$ and $t > 0$. Then $D^2 R_t \varphi \in C_b(H; L_2(H))$ and there exists $d_2 > 0$ such that

$$\|D^2 R_t \varphi(x)\|_{L_2(H)} \leq d_2 t^{-1/2} \|\varphi\|_1, \quad \forall t > 0, x \in H.$$  

**Proof.** Let $\varphi \in C^1_b(H)$, $t > 0$. Then, differentiating (1.3) with respect to $x$ yields

$$\langle DR_t \varphi(x), \alpha \rangle = \int_H \langle D\varphi(e^{tA}x + y), e^{tA} \alpha \rangle N_Q(dy), \quad \forall t > 0, x, \alpha \in H.$$  

Now, using (3.7) with $\langle D\varphi(e^{tA}x + \cdot), e^{tA} \alpha \rangle$ replacing $\varphi$, yields

$$\langle D^2 R_t \varphi(x) \alpha, \beta \rangle = \int_H \langle \Lambda_i \beta, Q_t^{-1/2} y \rangle \langle D\varphi(e^{tA}x + y), e^{tA} \alpha \rangle N_Q(dy), \quad \forall t > 0, x, \alpha, \beta \in H.$$
Consequently for any $h, k \in \mathbb{N}$

$$
\langle D^2 R_t \varphi(x) e_h, e_k \rangle = \Lambda_{t,k} e^{-\alpha t h} \int_H \hat{\lambda}_k(t)^{-1/2} y_k D_h \varphi(e^{tA} x + y) N_{Q_h}(dy),
$$

$$
\forall t > 0, x \in H,
$$

where $\Lambda_{t,k}$ were defined in (3.13). Setting

$$
\Psi_k(t) = \hat{\lambda}_k(t)^{-1/2} y_k, \quad t > 0, k \in \mathbb{N},
$$

we can write the above identity as

$$
\langle D^2 R_t \varphi(x) e_h, e_k \rangle = \Lambda_{t,k} e^{-\alpha t h} \langle \Psi_k(t), D_h \varphi(e^{tA} x + \cdot) \rangle_{L^2(H,N_{Q_h})}.
$$

It follows that

$$
|\langle D^2 R_t \varphi(x) e_h, e_k \rangle|^2 \leq \|\Lambda_t\|^2 |\langle \Psi_h(t), D_h \varphi(e^{tA} x + \cdot) \rangle|_{L^2(H,N_{Q_h})}^2.
$$

Now, summing up on $k$ and taking into account that the system $(\Psi_h(t))$ is orthonormal on $L^2(H,N_{Q_h})$, we see by the Parseval inequality and (3.11) that

$$
\sum_{k=1}^{\infty} |\langle D^2 R_t \varphi(x) e_h, e_k \rangle|^2 \leq c_1^2 t^{-1} \int_H |D_h \varphi(e^{tA} x + y)|^2 N_{Q_h}(dy) \leq c_1^2 t^{-1} \|\varphi\|_1.
$$

Equation (3.19) follows summing up on $h$ and taking $d_2 = c_1$.

**Corollary 3.11.** Let $\varphi \in C_b^0(H), \; \theta \in (0,1)$ and $t > 0$. Then $D^2 R_t \varphi \in C_b(H;L^2(H))$ and we have

$$
\|D^2 R_t \varphi\|_{C_b(H;L^2(H))} \leq c_\theta t^{\theta/2-1} \|\varphi\|_{\theta}, \quad t > 0,
$$

where $c_\theta = d_1^{1-\theta} d_2^\theta \kappa_\theta$ and $\kappa_\theta$ is defined in (3.5).

**Proof.** Let $t > 0$ be fixed and denote by $\gamma$ the mapping

$$
\gamma : C_b(H) \rightarrow C_b(H;L^2(H)), \quad \varphi \mapsto D^2 R_t \varphi.
$$

From Lemmas 3.9 and 3.10 it follows that

(i) $\gamma$ maps $C_b(H)$ into $C_b(H;L^2(H))$ with norm less than $d_1 t^{-1}$,

(ii) $\gamma$ maps $C_b^1(H)$ into $C_b(H;L^2(H))$ with norm less than $d_2 t^{-1/2}$.

Consequently, by Theorem 3.6, we have that $\gamma$ maps $(C_b(H),C_b^1(H))_{\theta,\infty}$ into $C_b(H;L^2(H))$ with norm less than $(d_1 t^{-1})^{1-\theta} (d_2 t^{-1/2})^\theta$. Therefore

$$
\|\gamma(\varphi)\|_{C_b(H;L^2(H))} \leq (d_1 t^{-1})^{1-\theta} (d_2 t^{-1/2})^\theta \|\varphi\|_{(C_b(H),C_b^1(H))_{\theta,\infty}}.
$$
On the other hand by Theorem 3.5 we have
\[(C_b(H), C^1_b(H))_{	heta, \infty} = C^0_b(H),\]
and so the conclusion follows from (3.5).

\[\square\]

**Lemma 3.12.** Let \(\varphi \in C_b(H)\) and \(t > 0\). Then \(D^2 R_t \varphi \in C^1_b(H; L_2(H))\) and there exists \(d_3 > 0\) such that
\[
\|D^2 R_t \varphi(x)\|_{L_2(H)} \leq d_3 t^{-3/2}\|\varphi\|_{0}, \quad t > 0.
\]

**Proof.** Let \(\varphi \in C_b(H)\). Then we have
\[
\|D^2 R_t \varphi(x)\|_{L_2(H)} = \sum_{h, k, l=1}^{\infty} |D^3 R_t \varphi(x)(e_h, e_k, e_l)|^2.
\]

On the other hand, by (3.9) we have
\[
D^3 R_t \varphi(x)(e_h, e_k, e_l)
= \Lambda_{t, h} \Lambda_{t, k} \Lambda_{t, l} \int_H \lambda_h(t)^{-1/2} y_h \lambda_k(t)^{-1/2} y_k \lambda_l(t)^{-1/2} y_l \varphi(e^{tA}x + y) N_{Q_l}(dy)
- \Lambda_{t, h}^2 \Lambda_{t, l} \delta_{h, k} \int_H \lambda_l(t)^{-1/2} y_l \varphi(e^{tA}x + y) N_{Q_l}(dy)
- \Lambda_{t, k}^2 \Lambda_{t, l} \delta_{h, l} \int_H \lambda_k(t)^{-1/2} y_k \varphi(e^{tA}x + y) N_{Q_l}(dy)
- \Lambda_{t, k}^2 \Lambda_{t, l} \delta_{l, k} \int_H \lambda_h(t)^{-1/2} y_h \varphi(e^{tA}x + y) N_{Q_l}(dy).
\]

Now we define an orthonormal system on \(L^2(H, N_{Q_l})\) setting
\[
\zeta_{h, k, l} = \begin{cases}
(\lambda_h(t) \lambda_k(t) \lambda_l(t))^{-1/2} y_h y_k y_l, & \text{if } h \neq k \neq l, \\
3^{-1/2}(\lambda_h^2(t) \lambda_l(t))^{-1/2} y_h^2 y_l - \lambda_l(t)^{-1/2} y_l, & \text{if } h = k \neq l, \\
3^{-1/2}(\lambda_l^2(t) \lambda_k(t))^{-1/2} y_k^2 y_l - \lambda_k(t)^{-1/2} y_l, & \text{if } h \neq k = l, \\
3^{-1/2}(\lambda_k^2(t) \lambda_h(t))^{-1/2} y_h y_k^2 - \lambda_h(t)^{-1/2} y_l, & \text{if } k = l \neq h.
\end{cases}
\]

Assume first that \(h \neq k \neq l\) and write (3.23) as
\[
D^3 R_t \varphi(x)(e_h, e_k, e_l) = \Lambda_{t, h} \Lambda_{t, k} \Lambda_{t, l} \langle \zeta_{h, k, l}, \varphi(e^{tA}x + \cdot) \rangle_{L_2(H, N_{Q_l})},
\]
which implies
\[
|D^3 R_t \varphi(x)(e_h, e_k, e_l)|^2 \leq c_1^6 t^{-3} |\langle \zeta_{h, k, l}, \varphi(e^{tA}x + \cdot) \rangle_{L_2(H, N_{Q_l})}|^2.
\]
So, by the Parseval inequality

\[(3.24) \quad \sum_{h,k,l,h \neq k \neq l} |D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq c_1^6 t^{-3} \| \phi \|_0^2.\]

Let now \( h = k \neq l \) and write (3.23) as

\[D^3 R_{i} \phi(x)(e_h, e_k, e_l) = 3^{1/2} \Lambda_{i,h}^2 \Lambda_{i,l} \langle \zeta_h, k, l, \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{0i})},\]

which implies

\[|D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3} |\langle \zeta_h, k, l, \varphi(e^{tA}x + \cdot) \rangle_{L^2(H, N_{0i})}|^2.\]

So, by the Parseval inequality

\[(3.25) \quad \sum_{h,k,l,h \neq k \neq l} |D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3} \| \phi \|_0^2.\]

In a similar way we see that if \( h = l \neq k \) we have

\[(3.26) \quad \sum_{h,k,l,h \neq k \neq l} |D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3} \| \phi \|_0^2.\]

and if \( k = l \neq h \) we have

\[(3.27) \quad \sum_{h,k,l,h \neq k} |D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^6 t^{-3} \| \phi \|_0^2.\]

Taking into account (3.25), (3.26) and (3.27) we end up with

\[\sum_{h,k,l=1}^{\infty} |D^3 R_{i} \phi(x)(e_h, e_k, e_l)|^2 \leq 3c_1^3 t^{-3} \| \phi \|_0^2\]

and so, the conclusion follows since the fact that \( D^2 R_{i} \phi \in C_b(H, L_2(H)) \) can be proved as before.

\[\square\]

**Lemma 3.13.** Let \( \phi \in C_b^1(H) \) and \( t > 0 \). Then \( D^2 R_{i} \phi \in C_b^1(H; L_2(H)) \) and there exists \( d_4 > 0 \) such that

\[(3.28) \quad \| DD^2 R_{i} \phi(x) \|_{L_2(H)} \leq d_4 t^{-1} \| \phi \|_0, \quad t > 0.\]

**Proof.** Let \( \phi \in C_b^1(H) \) and \( h, k \in \mathbb{N} \). Then, differentiating (1.3) with respect to \( x \) in the direction \( e_h \) yields

\[\langle DR_{i} \phi(x), e_h \rangle = e^{-t x_h} \int_H D_h \phi(e^{tA}x + y)N_{0i}(dy), \quad \forall t > 0, x.\]
Now, using (3.8) with $D_h \varphi(e^{tA}x + \cdot)$ replacing $\varphi$, yields

$$D^3 R_i \varphi(x)(e_h, e_k, e_l) = \int_H \langle \Lambda_i e_k, \mathcal{Q}_l^{-1/2} y \rangle \langle \Lambda_i e_l, \mathcal{Q}_l^{-1/2} y \rangle D_h \varphi(e^{tA}x + \cdot) N_Q(dy)$$

$$- \langle \Lambda_i e_k, \Lambda_i e_l \rangle R_i D_h \varphi(x),$$

which can be written as

$$D^3 R_i \varphi(x)(e_h, e_k, e_l) = e^{-ta_k} \Lambda_{t,k} \Lambda_{t,l} \int_H \lambda_k(t)^{-1/2} \lambda_l(t)^{-1/2} y_k y_l D_h \varphi(e^{tA}x + \cdot) N_Q(dy)$$

$$- e^{-ta_k} \Lambda_{t,k}^2 \delta_{k,l} \int_H D_h \varphi(e^{tA}x + \cdot) N_Q(dy).$$

Let now $k = l$. Then recalling (3.15) we have

$$D^3 R_i \varphi(x)(e_h, e_k, e_l) = 2^{1/2} e^{-ta_h} \Lambda_{t,k}^2 \langle \Phi_{k,k}, D_h \varphi(e^{tA}x + \cdot) \rangle_{L^2(H,N_Q)},$$

from which

$$|D^3 R_i \varphi(x)(h, k, k)|^2 \leq 2c_1^4 t^2 |\langle \Phi_{k,k}, D_h \varphi(e^{tA}x + \cdot) \rangle_{L^2(H,N_Q)}|^2$$

and, summing up on $k$ and $h$

$$\sum_{h,k=1}^\infty |D^3 R_i \varphi(x)(e_h, e_k, e_l)|^2 \leq 2c_1^4 t^2 \|\varphi\|_0^2. \quad (3.29)$$

Finally, if $k \neq l$, then using again by (3.15) we have

$$D^3 R_i \varphi(x)(e_h, e_k, e_l) = e^{-ta_h} \Lambda_{t,k}^2 \langle \Phi_{k,l}, D_h \varphi(e^{tA}x + \cdot) \rangle_{L^2(H,N_Q)},$$

from which

$$|D^3 R_i \varphi(x)(e_h, e_k, e_l)|^2 \leq c_1^2 t^2 |\langle \Phi_{k,k}, D_h \varphi(e^{tA}x + \cdot) \rangle_{L^2(H,N_Q)}|^2$$

and, summing up on $k$, $l$ and $h$

$$\sum_{h,k,l=1,k \neq l}^\infty |D^3 R_i \varphi(x)(e_h, e_k, e_l)|^2 \leq c_1^2 t^2 \|\varphi\|_0^2. \quad (3.30)$$

Now the conclusion follows from (3.29) and (3.30).
Finally we prove.

**Corollary 3.14.** Let $\varphi \in C^0_b(H)$, $\theta \in (0,1)$ and $t > 0$. Then $D^2 R_t \varphi \in C^0_b(H; L^2(H))$ and we have

$$\|D^2 R_t \varphi(x)\|_{C^1_b(H; L^2(H))} \leq c_{1,0} t^{(0-3)/2} \|\varphi\|_{\theta}, \quad t > 0,$$

where $c_{0,1} = d_3 t^{-3/2} d_4^3 \kappa_0$.

**Proof.** Let $t > 0$ be fixed and denote by $\delta$ the mapping

$$\delta : C_b(H) \to C_b(H, L^2(H)), \quad \varphi \mapsto D^2 R_t \varphi.$$

From Lemmas 3.12 and 3.13 it follows that

(i) $\delta$ maps $C_b(H)$ into $C^1_b(H; L^2(H))$ with norm less than $d_3 t^{-3/2}$,

(ii) $\delta$ maps $C^1_b(H)$ into $C^1_b(H; L^2(H))$ with norm less than $d_4 t^{-1}$.

Consequently, by Theorem 3.5, we have that $\delta$ maps $(C_b(H), C^1_b(H))_{\theta, \infty}$ into $C^1_b(H; L^2(H))$ with norm $\leq (d_3 t^{-3/2})^{1-\theta} (d_4 t^{-1})^\theta$. Therefore

$$\|\delta(\varphi)\|_{C^1_b(H; L^2(H))} \leq (c_4 t^{-3/2})^{1-\theta} (c_5 t^{-1})^\theta \|\varphi\|_{(C_b(H), C^1_b(H))_{\theta, \infty}}.$$

Now the conclusion follows from Theorem 3.5.

\[\square\]

3.3. Proof of the Main Result

We are now ready to prove the main result of the paper. The proof is similar to the finite-dimensional case, see [9].

**Theorem 3.15.** Assume that Hypothesis 1.1 holds. Let $\theta \in (0,1)$, $f \in C^0_b(H)$, $\lambda > 0$ and let $\varphi = (\lambda - L)^{-1} f$ be the solution to (1.8). Then we have $D^2 \varphi \in C^0_b(H; L^2(H))$ and there exists $M_1 > 0$ (independent on $\lambda$ and on $f$) such that

$$\|D^2 \varphi\|_{C^0_b(H; L^2(H))} \leq M_1 \|f\|_{C^0_b(H)}.$$

**Proof.** Let $f \in C^0_b(H)$, $\lambda > 0$ and $\varphi = (\lambda - L)^{-1} f$. Then for any $s \geq 0$,

$$D^2 R_s \varphi(x) = \int_0^{+\infty} e^{-\lambda s} D^2 R_s f(x) ds, \quad x \in H.$$

Proceeding as in [2] it follows that the integral is well defined for each $x \in H$. Following Remark 3.7 we shall look, given $t > 0$, for $a_t \in C_b(H, L^2(H))$ and $b_t \in C^1_b(H; L^2(H))$ such that (3.6) holds. We shall set

$$a_t(x) = \int_0^{t^2} e^{-\lambda s} D^2 R_s f(x) ds, \quad x \in H,$$
and
\[ b_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} D^2 R_s f(x) \, ds, \quad x \in H. \]

By arguing as in Lemma 3.9 we see that \( a_t \) and \( b_t \) are uniformly continuous. Moreover, it is easy to check that
\[
\|a_t(x)\|_{L^2(H)} \leq \int_0^{t^2} e^{-\lambda s} \|D^2 R_s f(x)\|_{L^2(H)} \, ds, \quad x \in H,
\]
so, by (3.20) we deduce
\[
\|a_t(x)\|_{L^2(H)} \leq \int_0^{t^2} e^{-\lambda s} \|D^2 R_s f\|_{C_b(H; L^2(H))} \, ds, \quad x \in H.
\]

Finally, taking the supremum in \( x \) yields
\[
(a_t)_{C_b(H; L^2(H))} \leq c_0 \|f\|_0 \int_0^{t^2} s^{\theta/2-1} \, ds = \frac{2}{\theta} c_0 \|f\|_0 t^{\theta}. \tag{3.33}
\]

In the same way since
\[ Db_t(x) = \int_{t^2}^{+\infty} e^{-\lambda s} DD^2 R_s f(x) \, ds, \quad x \in H, \]
we deduce by (3.31) that
\[
\|Db_t\|_{C_b(H; L^2(H))} \leq c_{1, \theta} \|f\|_0 \int_{t^2}^{+\infty} s^{(\theta-3)/2} \, ds = \frac{2c_{1, \theta}}{1 - \theta} \|f\|_0 t^{\theta-1}. \tag{3.34}
\]

Therefore \( D^2 R_t \varphi \) belongs to \( (C_b(L^2(H)), C^1_b(L^2(H))))_{0, \infty} \) and so to \( C^0_b(L^2(H)) \) by Theorem 3.6.

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