Partial Differential Equations — Continuous dependence on the data for nonlinear elliptic equations via symmetrization, by Maria Francesca Betta and Anna Mercaldo.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We prove the continuous dependence on the data of weak solutions to Dirichlet problem for nonlinear elliptic equations with a first order term and datum in dual spaces of classical Sobolev spaces. We deduce uniqueness results.

Key words: Rearrangements, nonlinear elliptic equations, continuous dependence on the data, uniqueness.

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1. Introduction

In this paper we are interested in continuous dependence on the data and uniqueness of weak solutions to the Dirichlet problem

\[
\begin{cases}
-\text{div}(a(x, \nabla u)) + B(x, \nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^N \) (\( N \geq 2 \)),

\[a : (x, \zeta) \in \Omega \times \mathbb{R}^N \to a(x, \zeta) = (a_i(x, \zeta)) \in \mathbb{R}^N\]

and

\[B : (x, \zeta) \in \Omega \times \mathbb{R}^N \to B(x, \zeta) \in \mathbb{R}\]

are Carathéodory functions, \( f \) belongs to the dual space \( W^{-1,p'}(\Omega) \) of \( W_0^{1,p}(\Omega) \), for some \( p \in [1, +\infty[ \).

Standard assumptions which assure the existence of a weak solution to problem (1.1) are the ellipticity of the operator

\[
a(x, \zeta) \cdot \zeta \geq \lambda |\zeta|^p, \quad \lambda > 0,\]

\[
\text{div}(a(x, \nabla u)) \geq \lambda |\nabla u|^p, \quad \lambda > 0.
\]
the growth conditions on $a$ and $B$

\begin{equation}
|a(x, \xi)| \leq c[|\xi|^{p-1} + a_0(x)], \quad c > 0, \quad a_0 \in L^p(\Omega),
\end{equation}

\begin{equation}
|B(x, \xi)| \leq B|\xi|^{p-1}, \quad B > 0,
\end{equation}

and the monotonicity of $a$

\begin{equation}
(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') > 0, \quad \xi \neq \xi',
\end{equation}

for a.e. $x \in \Omega$, for all $\xi, \xi' \in \mathbb{R}^N$.

Under these assumptions a weak solution to problem (1.1) exists (cf. [8], [9], [12]), that is a function $u \in W^{1,p}_0(\Omega)$ exists such that

\begin{equation}
\int_{\Omega} a(x, \nabla u) \cdot \nabla \phi \, dx + \int_{\Omega} B(x, \nabla u) \phi \, dx = \langle f, \phi \rangle, \quad \forall \phi \in W^{1,p}_0(\Omega).
\end{equation}

As far as uniqueness is concerned, more restrictive assumptions on the structure of the operator are required such as a monotonicity condition on $a$ stronger then (1.5)

\begin{equation}
(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha (x + |\xi| + |\xi'|)^{p-2} |\xi - \xi'|^2, \quad \xi, \xi' \in \mathbb{R}^N,
\end{equation}

where $\alpha > 0$, $\alpha > 0$ if $p \geq 2$ or $\alpha = 0$ if $p < 2$, and a local Lipschitz continuity condition on $B$

\begin{equation}
|B(x, \xi) - B(x, \xi')| \leq b(|x| + |\xi| + |\xi'|)^{p-2} |\xi - \xi'|, \quad \xi, \xi' \in \mathbb{R}^N,
\end{equation}

where $b > 0$ and $\eta = 0$ if $p \geq 2$ or $\eta > 0$ if $p < 2$.

Uniqueness results for weak solutions to (1.1) are proved under similar assumptions in [4], [6], [7], [11] and also in [1], [16] where they are obtained as a consequence of a comparison principle.

The aim of this paper is to prove the continuous dependence on the data and to deduce the uniqueness of a weak solution to (1.1) under the structural assumptions (1.7) and (1.8). Our approach is based on the classical symmetrization methods (cf. [13], [18]) which make use of isoperimetric inequalities and properties of rearrangements (see also [2], [5], [10]).

We point out that condition (1.7) is guaranteed if $a(x, 0) = 0$ and the following ellipticity condition holds

$$
\sum_{j=1}^n \frac{\partial a_i}{\partial z_j}(x, z) \xi_i \xi_j \geq (\alpha + |\xi|)^{p-2} |\xi|^2, \quad \xi \in \mathbb{R}^N.
$$

Roughly speaking this means that the operator $a$ can be reduced to a linear degenerate elliptic operator whose degeneracy is linked to the first order terms of problem (1.1). The model we have in mind is $a(x, \xi) = (\alpha + |\nabla u|^2)^{(p-2)/2} \nabla u$, which yields the so-called $p$-Laplace operator when $p \leq 2$ by our assumptions on $\alpha$. This linearization process suggests to require that the datum $f$ belongs to
a weighted dual space $H^{-1}(\Omega, m)$ for a suitable weight $m$ linked to the degeneracy of the operator (cf. [15]). Actually, at least when $p > 2$, we assume that data of (1.1) belong to the smaller dual space $H^{-1}(\Omega)$. Such an hypothesis seems to be necessary in order to prove the continuous dependence on the data. No further restrictions are required when $p \leq 2$: under this assumption we prove the continuous dependence of weak solutions on data belonging to $W^{-1,p'}(\Omega)$.

Our main results are the following

**Theorem 1.1.** Let $u, v$ be weak solutions to problem (1.1) with data $f, g \in H^{-1}(\Omega)$ respectively. Assume (1.2), (1.3), (1.4), (1.7), (1.8) and

$$2 \leq p < \frac{2N}{N-2},$$

if $N \geq 3$ and $2 \leq p < +\infty$, if $N = 2$. Then the following inequality holds true

$$(1.9) \quad \| \nabla u - \nabla v \|_{L^p} \leq C \| f - g \|^2_{H^{-1}},$$

where $C$ is a positive constant which depends on $N$, $|\Omega|$, $p$, $\alpha$, $b$, $\varepsilon$, $\| f \|_{H^{-1}}$ and $\| g \|_{H^{-1}}$; however it is bounded when $f$ and $g$ belong to bounded subset of $H^{-1}(\Omega)$.

**Theorem 1.2.** Let $u, v$ be weak solutions to problem (1.1) with data $f, g \in W^{-1,p'}(\Omega)$ respectively. Assume (1.2), (1.3), (1.4), (1.7), (1.8) and

$$\frac{2N}{N+2} < p < 2.$$

Then the following inequality holds true

$$(1.10) \quad \| \nabla u - \nabla v \|_{L^p} \leq C \| f - g \|_{W^{-1,p'}},$$

where $C$ is a positive constant which depends on $N$, $|\Omega|$, $p$, $\alpha$, $b$, $\eta$, $\| f \|_{W^{-1,p'}}$ and $\| g \|_{W^{-1,p'}}$; however it is bounded when $f$ and $g$ belong to bounded subset of $W^{-1,p'}(\Omega)$.

Obviously Theorems 1.1 and 1.2 imply in turn uniqueness of weak solutions to (1.1). They improve, at least when $p < 2$, well-known results contained in [6], [11] and [16], since we find a larger range of the values of $p$ for which uniqueness holds.

**2. Pointwise estimates**

The proofs of Theorems 1.1 and 1.2 are based on a pointwise estimate for the decreasing rearrangement of $u - v$, difference of two weak solutions $u, v$ to (1.1) corresponding to the data $f, g$ respectively.
We recall that the decreasing rearrangement of a measurable function \( w \) defined in \( \Omega \) is the function

\[
w^*(s) = \sup\{ t \geq 0 : \mu(t) > s \}, \quad s \in [0, |\Omega|],
\]

where \( \mu \) denotes its distribution function

\[
\mu(t) = |\{ x \in \Omega : |w(x)| > t \}|, \quad t \geq 0.
\]

The estimate of the decreasing rearrangement of \( \frac{u}{v} \) is proved by adapting classical symmetrization methods introduced in [13], [18] and extended to degenerate elliptic operators in [3].

**Lemma 2.1.** Let \( u, v \) be weak solutions to problem (1.1) with data \( f, g \in H^{-1}(\Omega) \) respectively. Assume (1.2), (1.3), (1.4), (1.7), (1.8) and

\[
2 \leq p < \frac{2N}{N-2},
\]

with \( N \geq 3 \). Then we have

\[
(u - v)^*(s) \leq C\|f - g\|_{H^{-1}} s^{-(N-2)/2N}, \quad s \in (0, |\Omega|),
\]

where \( C \) is a positive constant which depends on \( N, |\Omega|, p, \alpha, b, \epsilon, \|f\|_{H^{-1}} \) and \( \|g\|_{H^{-1}} \); however it is bounded when \( f \) and \( g \) belong to bounded subset of \( H^{-1}(\Omega) \).

**Lemma 2.2.** Let \( u, v \) be weak solutions to problem (1.1) with data \( f, g \in W^{-1,p'}(\Omega) \) respectively. Assume (1.2), (1.3), (1.4), (1.7), (1.8) and

\[
\frac{2N}{N+2} < p < 2.
\]

Then we have

\[
(u - v)^*(s) \leq C\|f - g\|_{W^{-1,p'}} s^{-(N-p)/Np}, \quad s \in (0, |\Omega|),
\]

where \( C \) is a positive constant which depends on \( N, |\Omega|, p, \alpha, b, \eta, \|f\|_{W^{-1,p'}} \) and \( \|g\|_{W^{-1,p'}} \); however it is bounded when \( f \) and \( g \) belong to bounded subset of \( W^{-1,p'}(\Omega) \).

**Proof of Lemma 2.1.** Denote \( w = u - v \), \( h = f - g \) and \( H \in (L^2(\Omega))^N \) the vector field such that

\[
h = -\text{div}(H).
\]
For any fixed $t \in ]0, \text{ess sup } w[\ and \ k > 0$ we consider the function

$$\varphi = \begin{cases} 
  k \text{ sign } w & \text{if } |w| > t + k, \\
  w - t \text{ sign } w & \text{if } t < |w| \leq t + k, \\
  0 & \text{otherwise},
\end{cases}$$

as test function in (1.6) with datum $f$, $g$ respectively. Then we subtract the equations and we divide by $k$,

$$\frac{1}{k} \int_{t < |w| \leq t + k} [a(x, \nabla u) - a(x, \nabla v)] \cdot \nabla w \, dx$$

$$= \int_{|w| > t + k} [B(x, \nabla u) - B(x, \nabla v)] \text{ sign } w \, dx$$

$$\quad + \frac{1}{k} \int_{t < |w| \leq t + k} [B(x, \nabla u) - B(x, \nabla v)] (w - t \text{ sign } w) \, dx$$

$$\quad + \frac{1}{k} \int_{t < |w| \leq t + k} H \cdot \nabla w \, dx.$$  

By assumptions (1.7) and (1.8), using Hölder inequality and letting $k$ goes to zero, we obtain

$$\frac{d}{dt} \int_{|w| > t} (e + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx$$

$$\leq \frac{b}{\alpha} \int_{|w| > t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx + \frac{1}{\alpha \delta^{(p-2)/2}} \left( - \frac{d}{dt} \int_{|w| > t} |H|^2 \, dx \right)^{1/2}$$

$$\times \left( - \frac{d}{dt} \int_{|w| > t} (e + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2}.$$  

On the other hand by Schwarz and isoperimetric inequalities, it follows

$$\frac{d}{dt} \int_{|w| > t} |\nabla w| \, dx$$

$$\leq \left( - \frac{d}{dt} \int_{|w| > t} (e + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2},$$

where $\omega_N$ denotes the measure of the unit ball of $\mathbb{R}^N$. 
Therefore by (2.4) and (2.5), we get

\begin{equation}
(2.6) \quad \left( -\frac{d}{dt} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \\
\leq \frac{b(-\mu'(t))^{1/2}}{\alpha N \omega_{N}^{1/N} e^{(p-2)/2} \mu(t)^{1-1/N}} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx \\
+ \frac{1}{\alpha e^{(p-2)/2}} \left( -\frac{d}{dt} \int_{|w|>\tau} |H|^2 \, dx \right)^{1/2}.
\end{equation}

Now we evaluate the first integral in the right-hand side of (2.6). By Schwarz inequality and coarea formula, we get

\begin{equation}
(2.7) \quad \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx \\
= \int_{\tau}^{+\infty} \left( -\frac{d}{d\tau} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx \right) \, d\tau \\
\leq \int_{\tau}^{+\infty} \left( -\frac{d}{d\tau} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \\
\times \left( -\frac{d}{d\tau} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \, dx \right)^{1/2} \, d\tau.
\end{equation}

Denote by $K, H : [0, |\Omega|) \to \mathbb{R}$ the functions which satisfy the following equalities

\begin{equation}
(2.8) \quad K(\mu(t))(-\mu'(t)) = -\frac{d}{dt} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \, dx,
\end{equation}

\begin{equation}
(2.9) \quad H(\mu(t))(-\mu'(t)) = -\frac{d}{dt} \int_{|w|>\tau} |H|^2 \, dx.
\end{equation}

Properties of such functions have been studied in [3], [17] (see also [14]). Collecting (2.6), (2.7), (2.8) and (2.9), we get

\begin{equation}
\left( -\frac{d}{dt} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \\
\leq \frac{b(-\mu'(t))^{1/2}}{\alpha N \omega_{N}^{1/N} e^{(p-2)/2} \mu(t)^{1-1/N}} \int_{\tau}^{+\infty} \left( -\frac{d}{d\tau} \int_{|w|>\tau} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \\
\times \left( K(\mu(\tau))(-\mu'(\tau)) \right)^{1/2} d\tau + \frac{1}{\alpha e^{(p-2)/2}} \left( H(\mu(t))(-\mu'(t)) \right)^{1/2}.
\end{equation}
By Gronwall Lemma, we deduce

\[
(2.10) \quad \left( -\frac{d}{dt} \int_{|w|>t} (\epsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \\
\leq \frac{H(\mu(t))^{1/2}(-\mu'(t))^{1/2}}{\alpha^2 \omega_N^{1/N} \epsilon^{(p-2)/2} \mu(t)^{-1-1/N}} + \frac{b(-\mu'(t))^{1/2}}{\alpha^2 N \omega_N^{1/N} \epsilon^{p-2} \mu(t)^{1-1/N}} \\
\times \int_0^\infty (H(\mu(\tau)) K(\mu(\tau)))^{1/2}(-\mu'(\tau)) \\
\times \exp\left( \frac{b}{\alpha N \omega_N^{1/N} \epsilon^{(p-2)/2}} \int_0^\tau \frac{(K(\mu(\sigma)))^{1/2}}{\mu(\sigma)^{1-1/N}} (-\mu'(\sigma)) \, d\sigma \right) \, d\tau.
\]

Taking into account (2.5), we obtain

\[
1 \leq \frac{(H(\mu(t)))^{1/2}(-\mu'(t))}{\alpha N \omega_N^{1/N} \epsilon^{p-2} \mu(t)^{-1-1/N}} + \frac{b(-\mu'(t))}{\alpha^2 N^2 \omega_N^{2/N} \epsilon^{(3/2)(p-2)} \mu(t)^{-2-2/N}} \times \int_0^\infty (H(\mu(\tau)) K(\mu(\tau)))^{1/2}(-\mu'(\tau)) \\
\times \exp\left( \frac{b}{\alpha N \omega_N^{1/N} \epsilon^{(p-2)/2}} \int_0^\tau \frac{(K(\mu(\sigma)))^{1/2}}{\mu(\sigma)^{1-1/N}} (-\mu'(\sigma)) \, d\sigma \right) \, d\tau,
\]

from which in a standard way we get

\[
(2.11) \quad -\frac{dw^+}{dr}(r) \leq \frac{H(r)^{1/2}}{\alpha N \omega_N^{1/N} \epsilon^{p-2}} r^{-1+1/N} \\
+ \frac{b}{\alpha^2 N^2 \omega_N^{2/N} \epsilon^{(3/2)(p-2)}} r^{-2+2/N} \int_0^r \frac{(H(\sigma) K(\sigma))^{1/2}}{\mu(\sigma)^{1-1/N}} (-\sigma') \, d\sigma \\
\times \exp\left( \frac{b}{\alpha N \omega_N^{1/N} \epsilon^{(p-2)/2}} \int_0^r \frac{(K(\sigma))^{1/2}}{\mu(\sigma)^{1-1/N}} \, d\sigma \right) \, d\sigma.
\]

for \( r \in (0, |\Omega|) \).

Now we evaluate the integral in the right-hand side of (2.11). To this aim we recall that the functions \( \tilde{K}, H \) are weak limit of functions having the same rearrangement as \( (\epsilon + |\nabla u| + |\nabla v|)^{p-2} \) and \( |H|^2 \) respectively. Therefore the Lebesgue norms of \( K \) and \( H \) can be estimated from above by the same norm of \( (\epsilon + |\nabla u| + |\nabla v|)^{p-2} \) and \( |H|^2 \) respectively. This implies that \( K \) belongs to \( L^{p/(p-2)}(0, |\Omega|) \) and \( H \) to \( L^1(0, |\Omega|) \) respectively. Therefore, using Hölder inequality, since \( p < \frac{2N}{N-2} \), we have
\[ (2.12) \quad \int_0^{\vert \Omega \vert} \frac{(K(z))^{1/2}}{z^{1-N}} \, dz \leq \left( \int_{\Omega} \left( \varepsilon + |\nabla u| + |\nabla v| \right)^p \, dx \right)^{(p-2)/2p} \times \left( \int_0^{\vert \Omega \vert} \frac{1}{z^{(N-1)/2p/2}} \, dz \right)^{(p+2)/2p} < +\infty, \]

and

\[ (2.13) \quad \int_0^r (H(\sigma)K(\sigma))^{1/2} \, d\sigma \leq \left( \int_{\Omega} |H|^2 \, dx \right)^{1/2} \left( \int_{\Omega} \left( \varepsilon + |\nabla u| + |\nabla v| \right)^p \, dx \right)^{(p-2)/2p} r^{1/p}. \]

Denote by $C$ a positive constant which depends only on the data and which can vary from line to line.

A priori estimates for the gradients of weak solutions to (1.1) are well-known (cf. Lemma 3.1 in [8] or [9]), that is

\[ (2.14) \quad \| \nabla u \|_{L^p} \leq C \| f \|_{W^{-1,p^\prime} \Omega}. \]

Moreover, since $p \geq 2$,

\[ (2.15) \quad \| f \|_{W^{-1,p^\prime} \Omega} \leq C \| f \|_{H^{-1} \Omega}. \]

Combining (2.11), (2.12), (2.13), (2.14) and (2.15), we have the following differential inequality

\[ (2.16) \quad - \frac{d w^*(r)}{dr} \leq C[H(r)]^{1/2} r^{1/N-1} + C \| H \|_{L^2} r^{-2+2/N+1/p}, \quad r \in (0, |\Omega|). \]

Finally we integrate such an inequality between $s$ and $|\Omega|$ and, by Hölder inequality and the property of $H$ stated above, we get

\[ w^*(s) \leq C \| H \|_{L^2} \left[ \int_{s}^{\vert \Omega \vert} r^{-2+2/N+1/p} \, dr + \left( \int_{s}^{\vert \Omega \vert} r^{2/N-2} \, dr \right)^{1/2} \right]. \]

This yields (2.1).

Remark 2.1. The previous proof can be easily adapted to the case when $N = 2$. Indeed the integration of differential inequality (2.16) yields the following pointwise estimate of $(u - v)^*$, which replaces (2.1),

\[ (u - v)^*(s) \leq C \| f - g \|_{H^{-1}} \left[ |\Omega|^{1/p} + \left( \log \frac{|\Omega|}{s} \right)^{1/2} \right], \quad s \in (0, |\Omega|). \]

The proof of Lemma 2.2 is analogous to the proof of Lemma 2.1.
Proof of Lemma 2.2. Denote \( w = u - v \), \( h = f - g \) and \( H \in (L^p(\Omega))^N \) the vector field such that (2.3) holds. As for (2.4), we get

\[
-\frac{d}{dt} \int_{|w| \geq t} |\nabla w|^2 (|\nabla u| + |\nabla v|)^{2-p} \, dx \leq \frac{b}{a} \int_{|w| \geq t} \frac{|\nabla w|}{(\eta + |\nabla u| + |\nabla v|)^{2-p}} \, dx \\
+ \frac{1}{a} \left( -\frac{d}{dt} \int_{|w| \geq t} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \right)^{1/2} \\
\times \left( -\frac{d}{dt} \int_{|w| \geq t} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2}.
\]

On the other hand by Schwarz and isoperimetric inequalities, it follows

\[
N^\frac{1}{N} \omega^1 \mu(t)^{1/N} \leq \left( -\frac{d}{dt} \int_{|w| \geq t} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \\
\times \left( -\frac{d}{dt} \int_{|w| \geq t} (|\nabla u| + |\nabla v|)^{2-p} \, dx \right)^{1/2}.
\]

Hence we have

\[
\left( -\frac{d}{dt} \int_{|w| \geq t} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \\
\leq \frac{b}{\alpha N \omega^1 \mu(t)^{1-1/N}} \left( -\frac{d}{dt} \int_{|w| \geq t} \frac{|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \\
\times \int_{|w| \geq t} \frac{|\nabla w|}{(\eta + |\nabla u| + |\nabla v|)^{2-p}} \, dx \\
+ \frac{1}{\alpha} \left( -\frac{d}{dt} \int_{|w| \geq t} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \right)^{1/2}.
\]

By Schwarz inequality and coarea formula, since \( \eta > 0 \), we have

\[
\left( -\frac{d}{dt} \int_{|w| \geq t} \frac{|\nabla w|}{(\eta + |\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \\
\leq \frac{1}{\eta^{(2-p)/2}} \int_t^{\infty} \left( -\frac{d}{d\tau} \int_{|w| \geq \tau} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} (-\mu'(\tau))^{1/2} \, d\tau.
\]
We denote by $\mathcal{K}, \mathcal{H} : [0, |\Omega|) \to \mathbb{R}$ the functions which satisfy the following equalities

\[(2.20) \quad \mathcal{K}(\mu(t))(-\mu'(t)) = -\frac{d}{dt} \int_{|w| > t} (|u| + |v|)^{2-p} \, dx,\]

\[(2.21) \quad \mathcal{H}(\mu(t))(-\mu'(t)) = -\frac{d}{dt} \int_{|w| > t} (|u| + |v|)^{2-p} |H|^2 \, dx.\]

Collecting (2.18), (2.19), (2.20) and (2.21), we get

\[
\left(\frac{d}{dt} \int_{|w| > t} \frac{|w|^2}{(|u| + |v|)^{2-p}} \, dx\right)^{1/2} \leq \frac{b(\mathcal{K}(\mu(t)))^{1/2}(-\mu'(t))^{1/2}}{\alpha \eta^{(2-p)/2} N \omega_N^{1/N} \mu(t)^{1-1/N}} \int_t^{+\infty} \left(\frac{d}{d\tau} \int_{|w| > \tau} \frac{|w|^2}{(|u| + |v|)^{2-p}} \, dx\right)^{1/2} \\
\times (-\mu'(\tau))^{1/2} \, d\tau + \frac{1}{\alpha} (\mathcal{H}(\mu(t)))^{1/2}(-\mu'(t))^{1/2}.
\]

Now we apply Gronwall lemma and use (2.17) again. Therefore, as in the proof of Lemma 2.1, we obtain

\[(2.22) \quad -\frac{d w^*}{dr}(r) \leq \frac{b(\mathcal{K}(r))^{p-2+2/N}}{\alpha^2 \eta^{(2-p)/2} N^2 \omega_N^{2/N}} \int_0^r (\mathcal{H}(\sigma))^{1/2} \times \exp\left(\frac{b}{\alpha N \omega_N^{1/N} \eta^{(p-2)/2}} \int_\sigma^r \frac{(\mathcal{K}(z))^{1/2}}{z^{1-1/N}} \, dz\right) \, d\sigma \]

\[\quad + \frac{1}{\alpha N \omega_N^{1/N}} (\mathcal{K}(r))^{1/2} (\mathcal{H}(r))^{1/p-1+1/N},\]

for $r \in (0, |\Omega|)$. Let us evaluate the integral in the right-hand side of (2.22). By the property of $\mathcal{H}$ and $\mathcal{K}$ stated above, $\mathcal{K}$ belongs to $L^{p/(2-p)}(0, |\Omega|)$ and $\mathcal{H}$ to $L^1(0, |\Omega|)$. Therefore, using Hölder inequality, since $p > \frac{2N}{N+2}$, we have

\[(2.23) \quad \int_0^{|\Omega|} \frac{(\mathcal{K}(z))^{1/2}}{z^{1-1/N}} \, dz \leq \left(\int_\Omega (|u| + |v|)^p \, dx\right)^{(2-p)/2p} \times \left(\int_0^{|\Omega|} \frac{1}{z^{((N-1)/N)(2p/(3p-2))}} \, dz\right)^{(3p-2)/2p} < +\infty
\]

and

\[(2.24) \quad \int_0^r (\mathcal{H}(\sigma))^{1/2} \, d\sigma \leq \left(\int_\Omega (|u| + |v|)^{2-p} |H|^2 \, dx\right)^{1/2} \, r^{1/2}.
\]
Taking into account (2.22), (2.23), (2.24) and the a priori estimates (2.14), we get

\begin{equation}
\frac{dw^*}{dr}(r) \leq C \left[ \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \right)^{1/2} K(r)r^{-2+2/N+1/2} \right. \\
+ \left. (K(r))^{1/2}(H(r))^{1/2}r^{1/N-1/2} \right],
\end{equation}

for \( r \in (0, |\Omega|) \). Now we integrate such an inequality between \( s \) and \( |\Omega| \) and then we use Hölder inequality and the a priori estimates (2.14). Therefore we get

\begin{equation}
w^*(s) \leq C \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \right)^{1/2} \|K\|_{L_p(2-p)}^{1/2} \times \\
\times \left[ \|K\|_{L_p(2-p)}^{1/2} \left( \int_{s}^{\Omega} r^{(-3/2+2/N)(p/2(p-1))} \, dr \right)^{(2(p-1))/p} \right. \\
+ \left. \left( \int_{s}^{\Omega} r^{(1/N-1)(p/(p-1))} \, dr \right)^{(p-1)/p} \right].
\end{equation}

Since \( H \in L^{p'}(\Omega) \), using Hölder inequality, we get

\begin{equation}
\int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \leq \| |\nabla u| + |\nabla v| \|^2_{L^p} \|H\|^2_{L^{p'}}.
\end{equation}

Combining this inequality, (2.26) and the a priori estimates (2.14), we get (2.2). \( \square \)

### 3. Continuous dependence on the data

The pointwise estimates proved in the previous section imply estimates in Lebesgue spaces of \( u - v \) in terms of the norms in dual space of the data. Indeed under the assumptions of Lemma 2.1 (see also Remark 2.1), we have the following estimate of \( L^p \)-norm of \( u - v \)

\begin{equation}
\|u - v\|_{L^p} \leq C\|f - g\|_{H^{-1}},
\end{equation}

while under the assumptions of Lemma 2.2,

\begin{equation}
\|u - v\|_{L^2} \leq C\|f - g\|_{W^{-1,p'}}.
\end{equation}

These estimates play an important role in the proof of the continuous dependence of the weak solutions to (1.1) on the data.

**Proof of Theorem 1.1.** Denote \( h = f - g \) and \( H \in (L^2(\Omega))^N \) the vector field defined by (2.3). We consider \( w = u - v \) as test function in (1.1) with data \( f \) and \( g \) respectively. Then we subtract the two equations and, using (1.7) and (1.8), we get
\[ \alpha \int_{\Omega} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \]
\[ \leq b \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w| \, dx + \int_{\Omega} |H| |\nabla w| \, dx. \]

By Hölder inequality we have
\[ \alpha \int_{\Omega} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \]
\[ \leq b \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2} \]
\[ \times \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^p \, dx \right)^{(p-2)/2p} \left( \int_{\Omega} |w|^p \, dx \right)^{1/p} \]
\[ + \frac{1}{\varepsilon^{(p-2)/2}} \left( \int_{\Omega} |H|^2 \, dx \right)^{1/2} \left( \int_{\Omega} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx \right)^{1/2}. \]

On the other hand, since \( p \geq 2 \),
\[ \int_{\Omega} |\nabla u - \nabla v|^p \, dx \leq \int_{\Omega} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla w|^2 \, dx. \]

Therefore combining (3.1), (3.3) and (2.14), we get (1.9).

The proof of Theorem 1.2 is similar to the previous proof.

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, we consider \( w = u - v \) as test function, then we subtract the two equations and we use (1.7) and (1.8)
\[ \alpha \int_{\Omega} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \leq b \int_{\Omega} \frac{|\nabla w| |w|}{(\eta + |\nabla u| + |\nabla v|)^{2-p}} \, dx + \int_{\Omega} |H| |\nabla w| \, dx, \]

where \( H \in (L^p(\Omega))^N \) is the vector field defined by (2.3) holds.

By Schwarz inequality we have
\[ \alpha \int_{\Omega} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \]
\[ \leq \frac{b}{\eta^{(2-p)/2}} \left( \int_{\Omega} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \left( \int_{\Omega} |w|^2 \, dx \right)^{1/2} \]
\[ + \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \right)^{1/2} \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} \, dx \right)^{1/2}. \]
On the other hand by Hölder inequality

\[
\int_{\Omega} (|\nabla u| + |\nabla v|)^{2-p} |H|^2 \, dx \leq \|
abla u\|_{L^p}^2 \|
abla v\|_{L^p}^2.
\]

Finally, since \( p < 2 \),

\[
\int_{\Omega} |\nabla w|^p \, dx \leq \left( \int_{\Omega} \frac{|\nabla w|^2}{(|\nabla u| + |\nabla v|)^2-p} \, dx \right)^{p/2} \left( \int_{\Omega} (|\nabla u| + |\nabla v|)^p \, dx \right)^{(2-p)/2}.
\]

Combining (3.2), (3.4), (3.5), (3.6) and the a priori estimates (2.14), we get (1.10).

References


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