Functions of class $C^1$ subject to a Legendre condition in an enhanced density set

Silvano Delladio

Abstract. We investigate certain second-order differential properties of functions and forms of class $C^1$ at the points around which a suitable Legendrian condition is “very densely verified”. In particular we provide a generalization of the classical identity $d^2 = 0$ on differential forms and some results about second-order osculating properties of graphs. Particular emphasis is placed on the case when the condition is verified in a locally finite perimeter set. A conjecture about the $C^2$-rectifiability of the horizontal projection of a Legendrian rectifiable set is discussed.

1. Introduction

In this paper we investigate certain second-order differential properties of functions and forms of class $C^1$ at the points around which a suitable Legendriam condition is “very densely verified”. In order to make more precise the argument behind such a rough expression, we introduce the following definition.

Definition 1.1. Let $\Omega$ be a measurable subset of $\mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is said to be a “point of enhanced density of $\Omega$” if

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B(x,r) \setminus \Omega)}{r^{n+1}} = 0.$$  

We say that “$\Omega$ is an enhanced density set” whenever a.e. $x \in \Omega$ is a point of enhanced density of $\Omega$.

The first application we give of this notion is in Theorem 2.1, which generalizes the classical identity $d^2 = 0$ on smooth differential forms: If $\lambda$ and $\mu$ are $C^1$-differential forms in $\mathbb{R}^n$ such that $d\lambda = \mu$ in a measurable set $\Omega$, then $d\mu(x) = 0$ whenever $x$ is a point of enhanced density of $\Omega$. Combining Theorem 2.1 with the theory developed in [6], we get Theorem 3.1: If $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$

Mathematics Subject Classification (2010): Primary 28Axx, 58A10; Secondary 26A45, 26A30.

Keywords: Enhanced density, parabolic blow-up, $C^2$-rectifiable sets.
and \( \Psi \in C^1(\mathbb{R}^n; M_{k,n}) \) satisfy \( Df = \Psi \) in a measurable set \( \Omega \), then there exists the approximate tangent paraboloid to the graph of \( f|\Omega \) at each point of enhanced density of \( \Omega \).

In Lemma 4.1 we state that: Every locally finite perimeter set is an enhanced density set. This fact (quite unexpected for us) makes enhanced density sets a common topic in many geometric measure theoretic contexts. As a consequence of Lemma 4.1 we obtain the “global versions” of the two results mentioned above, namely Theorem 2.1 and Theorem 3.1, under the additional assumption that \( \Omega \) has locally finite perimeter. In particular, Corollary 4.2 follows: If \( f \in C^1(\mathbb{R}^n, \mathbb{R}^k) \) and \( \Psi \in C^1(\mathbb{R}^n; M_{k,n}) \) satisfy \( Df = \Psi \) in a locally finite perimeter set \( \Omega \), then the graph of \( f|\Omega \) is a \( n \)-dimensional \( C^2 \)-rectifiable set.

A conjecture about the \( C^2 \)-rectifiability of the horizontal projection of a Legendrian rectifiable set is posed in the last section, where a sketch-proof based on Corollary 4.2 is provided too.

Acknowledgements. We wish to thank Giovanni Alberti for some useful discussions.

2. Differential forms of class \( C^1 \). A generalization of the classical identity \( d^2 = 0 \)

**Theorem 2.1.** Let \( \lambda \) and \( \mu \) be differential forms of class \( C^1 \) in \( \mathbb{R}^n \), respectively of degree \( h \) and \( h+1 \) (with \( h \geq 0 \)). If \( x_0 \) is a point of enhanced density of

\[
K := \{ x \in \mathbb{R}^n \mid d\lambda(x) = \mu(x) \}
\]

then

(i) \( x_0 \in K \);

(ii) \( \mu \) is closed at \( x_0 \), i.e., \( d\mu(x_0) = 0 \).

**Proof.** (i) It follows by observing that \( K \) is closed.

(ii) For \( h + 1 \geq n \) the assertion is trivial, hence we can assume

\[ h \leq n - 2. \]

Let \( \rho \in (0, 1) \) and consider \( \varphi \in C^2_c(B(0, 1)) \) such that

\[ 0 \leq \varphi \leq 1, \quad \varphi|B(0, \rho) \equiv 1 \]

and

\[ |D_h \varphi| \leq \frac{2}{1 - \rho} \quad (h = 1, \ldots, n). \]

For \( r > 0 \) and \( x \in \mathbb{R}^n \), define

\[ \varphi_r(x) := \varphi \left( \frac{x - x_0}{r} \right). \]
Then

$$D_h \varphi_r(x) = \frac{1}{r} D_h \varphi \left( \frac{x - x_0}{r} \right)$$

hence

$$(2.1) \quad |D_h \varphi_r| \leq \frac{2}{r(1 - \rho)}$$

Given an arbitrary \((n - 2 - h)\)-form \(\theta\) of class \(C^2\) in \(\mathbb{R}^n\), one has

$$d(\varphi_r \wedge \mu \wedge \theta) = d\varphi_r \wedge \mu \wedge \theta + \varphi_r d\mu \wedge \theta + (-1)^{h+1} \varphi_r \mu \wedge d\theta$$

by the differentiation formula for the wedge product of forms (see, e.g., Section 4.1.6 of [12]). If set for simplicity \(B_r := B(x_0, r)\) and observe that

$$\int_{B_r} d(\varphi_r \wedge \mu \wedge \theta) = \int_{\partial B_r} \varphi_r \mu \wedge \theta = 0$$

by the Stokes theorem, we get

$$\int_{B_r} \varphi_r d\mu \wedge \theta = -\int_{B_r} d\varphi_r \wedge \mu \wedge \theta + (-1)^h \int_{B_r} \varphi_r \mu \wedge d\theta$$

$$= -\int_{B_r \cap K} d\varphi_r \wedge d\mu \wedge \theta - \int_{B_r \setminus K} d\varphi_r \wedge \mu \wedge \theta +$$

$$+ (-1)^h \int_{B_r \cap K} \varphi_r d\mu \wedge d\theta + (-1)^h \int_{B_r \setminus K} \varphi_r \mu \wedge d\theta$$

$$= \int_{B_r} -d\varphi_r \wedge d\mu \wedge \theta + (-1)^h \varphi_r d\mu \wedge d\theta +$$

$$+ \int_{B_r \setminus K} d\varphi_r \wedge (d\mu - \theta) \wedge \theta + (-1)^h \varphi_r (\mu - d\lambda) \wedge d\theta.$$

But in the last member of this equality the integral over \(B_r\) is zero. Indeed, since \(\varphi_r\) and \(\theta\) are of class \(C^2\), a standard computation based on the differentiation formula for the wedge product of forms (see, e.g., Section 4.1.6 of [12]) shows that

$$-d\varphi_r \wedge d\lambda \wedge \theta + (-1)^h \varphi_r d\lambda \wedge d\theta = d(d\varphi_r \wedge \lambda \wedge \theta + (-1)^h \varphi_r \lambda \wedge d\theta)$$

hence

$$\int_{B_r} -d\varphi_r \wedge d\lambda \wedge \theta + (-1)^h \varphi_r d\lambda \wedge d\theta = 0$$

by the Stokes theorem.

Thus

$$\int_{B_r} \varphi_r d\mu \wedge \theta = \int_{B_r \setminus K} d\varphi_r \wedge (d\mu - \lambda) \wedge \theta + (-1)^h \varphi_r (\mu - d\lambda) \wedge d\theta.$$
It follows from (2.1) that there exists a number $C$, not depending on $r$ and $\rho$, such that
\[
\left| \int_{B_r} \varphi_r \, d\mu \wedge \theta \right| \leq C \, L^n(B_r \setminus K) \left( \frac{1}{r(1-\rho)} + 1 \right).
\]
On the other hand, the triangle inequality implies
\[
\left| \int_{B_r} \varphi_r \, d\mu \wedge \theta \right| \geq \left| \int_{B_r} \varphi_r \, d\mu \wedge \theta \right| - \left| \int_{B_r \setminus B_{r\rho}} \varphi_r \, d\mu \wedge \theta \right|,
\]
hence there are two numbers $C_1$ and $C_2$, which do not depend on $r$ and $\rho$, such that
\[
\rho^n \left| \int_{B_{r\rho}} d\mu \wedge \theta \right| \leq \left| \int_{B_r} \varphi_r \, d\mu \wedge \theta \right| + C_1 \, L^n(B_r \setminus K) \left( \frac{1}{r(1-\rho)} + 1 \right) + C_2 \frac{(r^n - \rho^n r^n)}{r^n}
\]
Passing to the limit for $r \downarrow 0$, we obtain
\[
\rho^n \left| \langle d\mu(x_0) \wedge \theta(x_0), dx_1 \wedge \ldots \wedge dx_n \rangle \right| \leq C_2(1 - \rho^n).
\]
Then, letting $\rho \uparrow 1$, we find
\[
\langle d\mu(x_0) \wedge \theta(x_0), dx_1 \wedge \ldots \wedge dx_n \rangle = 0.
\]
The conclusion follows at once from the arbitrariness of $\theta$.

\textbf{Remark 2.1.} The classical identity $d^2 \omega = 0$ for a differential form $\omega$ of class $C^2$ in $\mathbb{R}^n$ follows immediately by applying Theorem 2.1 with $\lambda := \omega$ and $\mu := d\omega$.

\textbf{Remark 2.2.} The closure of $\mu$ at $x_0$ is false, in general, if one simply assumes that $x_0$ is a density point (instead of an enhanced density point). For example, consider
\[
\mu(x) := -x_2 \, dx_1 + x_1 \, dx_2, \quad x = (x_1, x_2) \in \mathbb{R}^2.
\]
By Theorem 1 in [1] we can find $\lambda \in C^1(\mathbb{R}^2)$ supported in $B(0,1)$ and such that
\[
L^2\{x \in B(0,1) \mid d\lambda(x) = \mu(x)\} \geq 1.
\]
But $\{x \in B(0,1) \mid d\lambda(x) = \mu(x)\}$ is just the set $K$ defined in the statement of Theorem 2.1, hence $L^2(K) \geq 1$. So the density points of $K$ form a set of positive measure and at each of them (as long as at each point in $\mathbb{R}^2$) the form $\mu$ is not closed. We observe, incidentally, that $K$ does not contain enhanced density points.
3. Second order osculating properties of \( EC^1 \) graphs

3.1. A Whitney-type extension problem

We begin this subsection with a definition.

**Definition 3.1.** Let \( \Omega \) be any subset of \( \mathbb{R}^n \). Then \( EC^1(\Omega, \mathbb{R}^k) \) denotes the class of maps \( f \in C^1(\mathbb{R}^n, \mathbb{R}^k) \) for which there exists \( \Psi \in C^1(\mathbb{R}^n; M_{k,n}) \) such that \( \Psi|\Omega = (Df)|\Omega \).

A Whitney-type extension problem concerning \( f \in EC^1(\Omega, \mathbb{R}^k) \) arises naturally:

**Problem.** Does \( f|\Omega \) have an extension of class \( C^2 \), namely a map \( h \in C^2(U, \mathbb{R}^k) \) with \( U \) open subset of \( \mathbb{R}^n \), \( \Omega \subset U \) and \( h|\Omega = f|\Omega \)?

**Remark 3.1.** In the special case when \( \Omega \) is open, the answer to the question set in Problem above is trivially “yes” (with \( U = \Omega \) and \( h = f|\Omega \)). In general the answer is “no”. A counterexample can be obtained for \( n = k = 1 \) by considering the function introduced in the Appendix of [4] and defined as

\[
 f(x) := \int_0^x \text{dist}(t, \Omega)^{1/2} dt \quad (x \in \mathbb{R})
\]

where \( \Omega \) is a suitable Cantor-like closed subset of \([0, 1]\) of positive measure. Then \( f \) belongs to \( EC^1(\Omega, \mathbb{R}) \), because it satisfies the assumptions in Definition 3.1 with \( \Psi := 0 \). However it has been proved in [4] that the graph of \( f|\Omega \) has zero-measure intersection with every \( C^2 \) graph, namely it is not a 1-dimensional \( C^2 \)-rectifiable. In particular \( f|\Omega \) has no extension of class \( C^2 \).

3.2. Parabolic blow-up of a \( EC^1(\Omega, \mathbb{R}^k) \) graph over a point of enhanced density

Let \( \Omega \) be a subset of \( \mathbb{R}^n \), \( f \in EC^1(\Omega, \mathbb{R}^k) \) and \( \Psi \) be as in Definition 3.1. Then consider \( x_0 \in \mathbb{R}^n \) and the second degree polynomial \( \Gamma^{x_0} \) defined as

\[
 \Gamma^{x_0}(\xi) := \frac{1}{2} \sum_{i,j=1}^{n} \xi_i \xi_j D_j \Psi_{i1}(x_0) \quad (\xi \in \mathbb{R}^n)
\]

where \( \Psi_{i1} := (\Psi_{11}, \ldots, \Psi_{ki}) \).

**Remark 3.2.** If \( \Omega \) is an open subset of \( \mathbb{R}^n \) then \( f|\Omega \) is of class \( C^2 \) and \( \Psi|\Omega = D(f|\Omega) \), compare with Remark 3.1. Hence, for \( x_0 \in \Omega \), the map \( \Gamma^{x_0} \) is just the second degree term in the second order Taylor polynomial of \( f|\Omega \) at \( x_0 \).

Let \( G_{f|\Omega} \) and \( G_{\Gamma^{x_0}} \) denote the graphs of \( f|\Omega \) and \( \Gamma^{x_0} \), respectively. Also consider the family of nonhomogeneous dilatations

\[
 T^{x_0}_\rho : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k \quad (\rho > 0)
\]
defined by
\[ T^x_\rho(x, y) := \left( \frac{x - x_0}{\rho}, \frac{y - f(x_0) - Df(x_0)(x - x_0)}{\rho^2} \right). \]

The following theorem holds.

**Theorem 3.1.** Let $\Omega$ be measurable and $x_0$ be a point of enhanced density of $\Omega$. Then $\Gamma^{x_0}$ is independent from the choice of $\Psi$ and one has
\[ \mathcal{H}^n \bigcap T^x_\rho (G_{f, \Omega}) \to \mathcal{H}^n \bigcap G_{\Gamma^{x_0}} \quad (\text{as } \rho \downarrow 0) \]

in the weak* sense of measures.

**Proof.** The assertion about the independence of $\Gamma^{x_0}$ from the choice of $\Psi$ follows at once by Lemma 3.1 below. In order to prove the second claim, consider the $1$-forms of class $C^1$ in $\mathbb{R}^n$
\[ \omega_i := \sum_{j=1}^{n} \Psi_{ij} dx_j \quad (i = 1, \ldots, k) \]

and the corresponding sets
\[ K_i := \{ x \in \mathbb{R}^n \mid df_i(x) = \omega_i(x) \}. \]

Observe that
\[ \mathcal{L}^n (\Omega \setminus K_i) = 0 \quad (i = 1, \ldots, k) \]
hence $x_0$ has to be a point of enhanced density of each $K_i$. Theorem 2.1 implies that the forms $\omega_i$ are closed at $x_0$, namely the Schwartz-like equality
\[ \frac{\partial \Psi_{ij}}{\partial x_m}(x_0) = \frac{\partial \Psi_{im}}{\partial x_j}(x_0) \]
holds for all $i = 1, \ldots, k$ and $j, m = 1, \ldots, n$. The conclusion follows easily from Corollary 4.2 of [6].

**Remark 3.3.** The argument (based on Theorem 2.1) used in the proof of Theorem 3.1 shows that Corollary 4.2 of [6] holds even without assuming the Schwartz-like condition.

The following lemma is completely elementary. However, since we have no reference to cite for it, we will provide a short proof.

**Lemma 3.1.** Let a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, a subset $\Omega$ of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ be given such that:

(i) $\varphi$ is differentiable (hence continuous!) at $x$;
(ii) $x$ is a point of density of $\Omega \cap \varphi^{-1}(0)$.

Then $\nabla \varphi(x) = 0$. 

Proof. Assume \( \nabla \varphi(x) \neq 0 \) and show that a contradiction follows. For all \( j = 1, 2, \ldots \), we can find \( x_j \in \Omega \cap B(x, 1/j) \setminus \{ x \} \) such that \( \varphi(x_j) = 0 \) and

\[
u_j := \frac{x_j - x}{\| x_j - x \|} \rightarrow \frac{\nabla \varphi(x)}{\| \nabla \varphi(x) \|} \quad (\text{as} \ j \rightarrow \infty).
\]

On the other hand,

\[
0 = \varphi(x_j) = \varphi(x + \| x_j - x \| u_j) = \| x_j - x \| \nabla \varphi(x) \cdot u_j + o(\| x_j - x \|),
\]

namely

\[
\nabla \varphi(x) \cdot u_j \rightarrow 0 \quad (\text{as} \ j \rightarrow \infty).
\]

It follows that \( \| \nabla \varphi(x) \| = 0 \), which contradicts our assumption.

3.3. The case when \( \Omega \) is an enhanced density set: \( C^2 \)-rectifiability of \( EC^1(\Omega, \mathbb{R}^k) \) graphs

Recall that a subset of a Euclidean space is said to be a \( n \)-dimensional \( C^2 \)-rectifiable set if \( \mathcal{H}^n \)-almost all of it may be covered by countably many \( n \)-dimensional submanifolds of class \( C^2 \). The notion of \( C^k \)-rectifiable set has been introduced in [4] and provides a natural setting for the description of singularities of convex functions and convex surfaces [2], [3]. More generally, it can be used to study the singularities of surfaces with generalized curvatures [3]. Rectifiability of class \( C^2 \) is strictly related to the context of Legendrian rectifiable subsets of \( \mathbb{R}^N \times S^{N-1} \) (see [13], [14], [7], [8]). The level sets of a \( W^{k,p}_{loc} \) mapping between manifolds are rectifiable sets of class \( C^k \) [5]. Applications to geometric variational problems can be found in [9].

We know from Remark 3.1 that the graph of \( f|\Omega \), with \( f \in EC^1(\Omega, \mathbb{R}^k) \), does not need to be \( C^2 \)-rectifiable. The following result shows that \( C^2 \)-rectifiability occurs whenever \( \Omega \) is an enhanced density set.

**Theorem 3.2.** Let \( \Omega \) be an enhanced density subset of \( \mathbb{R}^n \) and \( f \in EC^1(\Omega, \mathbb{R}^k) \). Then the graph of \( f|\Omega \) is a \( n \)-dimensional \( C^2 \)-rectifiable set.

**Proof.** Let \( M \) denote the graph of \( f|\Omega \) and \( x_0 \) be a point of enhanced density of \( \Omega \). Then it will be enough to show that the conditions (i), (ii) and (iii) in the statement (c) of Theorem 3.5 in [4] are verified at \( (x_0, f(x_0)) \). Since (i) and (iii) are trivial, we just have to prove the existence of the approximate tangent paraboloid to \( M \) at such a point. To this aim let \( \Psi \) be as in Definition 3.1, hence \( Df(x_0) = \Psi(x_0) \).

Without affecting the generality of the argument below, we can assume that

\[
x_0 = 0, \quad f(x_0) = f(0) = 0.
\]

Let \( \Pi \) denote both the tangent space to the graph of \( f \) at \( (x_0, f(x_0)) = (0, 0) \) and the orthogonal projection from \( \mathbb{R}^n \times \mathbb{R}^k \) onto the tangent space itself. Similarly, the orthogonal complement of \( \Pi \) in \( \mathbb{R}^n \times \mathbb{R}^k \) and the corresponding orthogonal

\[
\pi_\Pi(x) = (x_1', \ldots, x_{n-k}') \quad \text{and} \quad \pi_{\Pi^\perp}(x) = (x_1, \ldots, x_{n-k}, x_{n-k+1}, \ldots, x_n).}
\]
projection operator will both be denoted by $\Pi^\perp$. Consider the map $\varphi \in C^1(\mathbb{R}^n, \Pi)$ defined as

$$\varphi(x) := \Pi(x, f(x)) \quad (x \in \mathbb{R}^n)$$

and let $r > 0$ be such that $\varphi|B(0, r) : B(0, r) \to U := \varphi(B(0, r))$ is invertible, with inverse $\mu \in C^1(U, B(0, r))$. Then define

$$\tilde{\Omega} := \varphi(\Omega \cap B(0, r))$$

and

$$\tilde{f}(u) := \Pi^\perp(\mu(u), f(\mu(u))) \quad (u \in U)$$

and

$$\tilde{\Psi}(u) := \left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\right)\right] \circ \left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\right)\right]^{-1} \quad (u \in U).$$

Observe that, for $u \in \tilde{\Omega}$, one has

$$\text{Id}_{\Pi} = D^\Pi((\varphi|B(0, r)) \circ \mu)(u) = D\varphi(\mu(u)) \cdot D^\Pi(\mu(u))$$

$$= \left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, Df(\mu(u))\right)\right] \cdot D^\Pi(\mu(u))$$

$$= \left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, Df(\mu(u))\right)\right] \cdot D^\Pi(\mu(u)).$$

In particular, it follows that $\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(0)\right)$ is invertible. Hence, by making $r$ smaller if need be, we obtain that $\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(x)\right)$ is invertible for all $x \in B(0, r)$ and also that

$$\left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\right)\right]^{-1} = D^\Pi(\mu(u))$$

whenever $u \in \tilde{\Omega}$. We get

$$D^\Pi\tilde{f}(u) = \left[\Pi^\perp \circ \left(\text{Id}_{\mathbb{R}^n}, Df(\mu(u))\right)\right] \cdot D^\Pi(\mu(u))$$

$$= \left[\Pi^\perp \circ \left(\text{Id}_{\mathbb{R}^n}, Df(\mu(u))\right)\right] \cdot \left[\Pi \circ \left(\text{Id}_{\mathbb{R}^n}, \Psi(\mu(u))\right)\right]^{-1} \cdot \tilde{\Psi}(u)$$

for all $u \in \tilde{\Omega}$.

Observe that, in our notation, the family of nonhomogeneous dilatations considered in [4] takes the following form $(\rho > 0)$:

$$\tilde{T}_\rho : \Pi \times \Pi^\perp \to \Pi \times \Pi^\perp, \quad \tilde{T}_\rho(u, v) := \left(\frac{u}{\rho}, \frac{v}{\rho^2}\right).$$

Since

$$\tilde{f}(0) = 0, \quad D^\Pi\tilde{f}(0) = 0,$$

one has

$$\tilde{T}_\rho(u, v) = \left(\frac{u}{\rho}, \frac{v - \tilde{f}(0) - D^\Pi\tilde{f}(0)u}{\rho^2}\right).$$
Moreover, 0 is a point of enhanced density of ̂Ω (relatively to Π) and the graph of ̂f| ̂Ω coincides with the graph of f|B(0,r) ∩ Ω. Then the argument used in the proof of Theorem 3.1 (based on Corollary 4.2 of [6]) shows that

\[ \mathcal{H}^n L \mathcal{T}_\rho(M) \to \mathcal{H}^n G_F \quad \text{(as } \rho \downarrow 0), \]

where G_F is the graph of the second degree polynomial

\[ \mathcal{F}(\xi) := \frac{1}{2} \sum_{i,j=1}^n \xi_i \xi_j D_j \tilde{\Psi}_{si}(0) \quad (\xi \in \Pi). \]

This definitely proves that condition (ii) in the statement (c) of Theorem 3.5 in [4] holds.

\[ \square \]

4. The case of Caccioppoli sets, corollaries

The following somehow unexpected (nevertheless very simple to prove!) result holds.

Lemma 4.1. Let Ω be a locally finite perimeter subset of \( \mathbb{R}^n \). Then

\[ \lim_{r \downarrow 0} \frac{L^n(B(x,r) \setminus \Omega)}{r^{n+1}} = 0 \]

at a.e. \( x \in \Omega \). In particular \( \Omega \) is an enhanced density set.

Proof. By applying Theorem 1 in §6.1 of [11] (with \( f = \varphi_\Omega \)), one has

\[ 0 = \lim_{r \downarrow 0} \frac{1}{r^n} \left( \frac{1}{r} \int_{B(x,r)} |\varphi_\Omega - 1| dx \right)^{\frac{n}{n+1}} = \lim_{r \downarrow 0} \left( \frac{1}{x^{n+1}} \int_{B(x,r)} \varphi_{\mathbb{R}^n \setminus \Omega} dx \right)^{\frac{n}{n+1}} \]

for a.e. \( x \in \Omega \). Hence the conclusion follows. \( \square \)

From Lemma 4.1 and Theorem 2.1 we get at once the following result.

Corollary 4.1. Let \( \lambda \) and \( \mu \) be differential forms of class C^1 in \( \mathbb{R}^n \), respectively of degree \( h \) and \( h + 1 \) (with \( h \geq 0 \)). Assume that \( d\lambda = \mu \) almost everywhere in a locally finite perimeter subset \( \Omega \) of \( \mathbb{R}^n \). Then \( \mu \) is closed at almost every point in \( \Omega \).

The combination of Lemma 4.1 and Theorem 3.2 gives the second corollary.

Corollary 4.2. Let \( \Omega \) be a locally finite perimeter subset of \( \mathbb{R}^n \) and \( f \in EC^1(\Omega, \mathbb{R}^k) \). Then the graph of \( f|\Omega \) is a n-dimensional C^2-rectifiable set.

Remark 4.1. Let \( \Omega \) be a measurable subset of \( \mathbb{R}^n \) and let \( f \in C^1(\mathbb{R}^n, \mathbb{R}^k) \). Denote by \( [G_f|\Omega] \) the multiplicity one rectifiable current naturally associated to the
graph $G_{f|\Omega}$ of $f|\Omega$, namely the one carried by $G_{f|\Omega}$ and oriented by $\eta$ such that

$$\eta(x, f(x)) = \frac{\wedge^n (\text{Id}_{\mathbb{R}^n}, Df(x))(e_1 \wedge \ldots \wedge e_n)}{\| \wedge^n (\text{Id}_{\mathbb{R}^n}, Df(x))(e_1 \wedge \ldots \wedge e_n) \|}$$  \hspace{1cm} (at a.e. $x \in \Omega$)

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In particular, for the purely-horizontal stratum $\eta_{(0)}$ of $\eta$, the following equality holds:

$$\| \wedge^n (\text{Id}_{\mathbb{R}^n}, Df(x))(e_1 \wedge \ldots \wedge e_n) \|_{\eta_{(0)}}(x, f(x)) = e_1 \wedge \ldots \wedge e_n$$  \hspace{1cm} (at a.e. $x \in \Omega$).

Observe that, if $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denotes the orthogonal projection, then $\pi_{\#}[G_{f|\Omega}]$ is just the current $[\Omega]$ corresponding to the canonically oriented integration of forms over $\Omega$. Indeed, for every smooth differential $n$-form $\omega$ with compact support in $\mathbb{R}^n$, one has (compare with Section 26 of [16])

$$\pi_{\#}[G_{f|\Omega}](\omega) = \int_{G_{f|\Omega}} \langle \omega \circ \pi, \eta_{(0)} \rangle \, dH^n$$

$$= \int_{\Omega} \langle \omega(x), \eta_{(0)}(x, f(x)) \rangle \| \wedge^n (\text{Id}_{\mathbb{R}^n}, Df(x))(e_1 \wedge \ldots \wedge e_n) \| \, dH^n(x)$$

$$= \int_{\Omega} \langle \omega, e_1 \wedge \ldots \wedge e_n \rangle \, dH^n = [\Omega](\omega).$$

Hence and by recalling Sections 26.21 of [16] or Proposition 3, Section 2.3, Chapter 2 in [15], we easily obtain that $\Omega$ is a locally finite perimeter set if and only if $\partial[\!\!\!\!\!\!G_{f|\Omega}]$ has locally finite mass.

By virtue of the previous remark, Corollary 4.2 can be restated in the following form (which is interesting in view of the next section).

**Corollary 4.3.** Let $\Omega$ be a measurable subset of $\mathbb{R}^n$, $f \in EC^1(\Omega, \mathbb{R}^k)$ and assume that $\partial[\!\!\!\!\!\!G_{f|\Omega}]$ has locally finite mass. Then $G_{f|\Omega}$ is a $n$-dimensional $C^2$-rectifiable set.

**Remark 4.2.** The example considered in Remark 3.1 shows that without the finiteness conditions assumed in Corollaries 4.2 and 4.3, the $C^2$-rectifiability of $G_{f|\Omega}$ fails to be true (in general).

5. Legendrian rectifiable subsets of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. $C^2$-rectifiability of the horizontal projection (open problem)

5.1. Legendrian rectifiable subsets of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

Set

$$X := \mathbb{R}^{n+1}, \quad Y := \mathbb{R}^{n+1}.$$ 

Throughout this section $X$ and $Y$ will also denote, respectively, the projection maps

$$X \times Y \to X, \quad (x; y) \mapsto x.$$
and
\[ X \times Y \to Y, \quad (x,y) \mapsto y. \]
Let \( x_j \) and \( y_j \) be the canonical coordinates of \( X \) and \( Y \), respectively. If \( e_1, \ldots, e_{n+1} \) is the canonical basis of \( X \), then define
\[ X_* := \text{span}\{e_1, \ldots, e_n\}. \]
Indicate by \( J \) the trivial map identifying \( X \) and \( Y \), namely
\[ J : X \to Y, \quad J(x_1, \ldots, x_{n+1}) := (x_1, \ldots, x_{n+1}). \]
Consider the usual contact form on \( X \times Y \)
\[ \alpha = \sum_{j=1}^{n+1} y_j \, dx_j. \]
Then define the notion of Legendrian rectifiable subset of \( X \times Y \), compare with [13].

**Definition 5.1.** A \( n \)-dimensional rectifiable subset \( G \) of \( X \times (Y \setminus \{0\}) \) is called "Legendrian" if
\[ \alpha(x; y)|T_{(x;y)}G = 0 \]
for \( H^n \llcorner G \) almost every \((x; y)\). Here \( T_{(x;y)}G \) denotes the approximate tangent plane to \( G \) at \((x; y)\) in \( X \times Y \).

**Remark 5.1.** Let \( R \) be the set of points \((x; y)\) in a Legendrian rectifiable set \( G \) at which \( T_{(x;y)}G \) exists and \( X|T_{(x;y)}G \) has rank \( n \). Then for almost every \((x; y) \in R \) there exists \( T_x(XG) \) (i.e., the approximate tangent plane to \( XG \) at \( x \)) and one has
\[ T_x(XG) = X(T_{(x;y)}G). \]
As a consequence, the condition (5.1) yields
\[ y \perp T_x(XG) \]
at \( H^n \llcorner R \) almost every \((x; y)\). The area formula (see, e.g., Section 3.2 of [12] and Chapter 2 of [16]) yields \( H^n(X(G \setminus R)) = 0 \). Hence \( X(R) \) and \( X(G) \) coincide, possibly except for a zero set.

We have the following characterization.

**Proposition 5.1.** Let \( \Sigma \) be a \( n \)-dimensional rectifiable subset of \( X \). Then \( \Sigma \) is the image through \( X \) of a Legendrian rectifiable subset of \( X \times Y \) if and only if the following condition is verified:
There exist countably many measurable subsets \( \Omega_j \) of \( X_* \), functions \( f_j \in EC^1(\Omega_j, \mathbb{R}) \) and linear isometries \( L_j \) in \( X \) such that
\[ \Sigma = \bigcup_j L_j \Gamma_j, \quad \Gamma_j := \{(x_*, f_j(x_*)) \in X \mid x_* \in \Omega_j\} \]
possibly except for a zero set.
Proof. Assume that $\Sigma = X(G)$ where $G$ is a Legendrian rectifiable set in $X \times Y$. Consider

$$R := \{ (x; y) \in G \mid T_{(x,y)}G \text{ exists and } X|T_{(x,y)}G \text{ has rank } n \}$$

and observe that

$$X(R) = \Sigma$$

possibly except for a zero set. By Remark 5.1, we may find countably many functions $f_j \in C^1(X_*)$, measurable subsets $\Omega_j$ of $X_*$, maps $\Phi_j \in C^1(X_*, X\setminus \{0\})$ and linear isometries $L_j$ in $X$ such that:

- One has
  $$R = \bigcup_j R_j, \quad R_j := \{ (L_j(x_*, f_j(x_*)); J L_j \Phi_j(x_*)) \mid x_* \in \Omega_j \}$$
  possibly except for a zero set, hence (5.2) holds. Observe that

  $$L_j(T_{(x_*, f_j(x_*))} \Gamma_j) = T_{(x_*, f_j(x_*))}X(R_j) = T_{(x_*, f_j(x_*))}\Sigma$$

  at a.e. $x_* \in \Omega_j$ and for all $j$;

- The vector $\Phi_j(x_*)$ is orthogonal to $T_{(x_*, f_j(x_*))} \Gamma_j$ at a.e. $x_* \in \Omega_j$ and for all $j$.

It follows that, for every $j$, there exists a measurable function $c_j : \Omega_j \rightarrow \mathbb{R}$ satisfying

$$\Phi_j = c_j (\nabla f_j - e_{n+1})$$

a.e. in $\Omega_j$. Since $\Phi_j$ does not vanish, $c_j$ does not vanish too. In particular, $\Phi_j \cdot e_{n+1}$ is nonzero a.e. in $\Omega_j$. Hence, without loss of generality, we can even assume that $\Phi_j \cdot e_{n+1}$ does not vanish in all of $X_*$. Then the map

$$\Psi_j := \frac{\Phi_j}{\Phi_j \cdot e_{n+1}} + e_{n+1}$$

belongs to $C^1(X_*)$ and one has $\nabla f_j = \Psi_j$ a.e. in $\Omega_j$. It follows that $f_j \in EC^1(\Omega_j, \mathbb{R})$.

Vice versa, assume there exist countably many measurable subsets $\Omega_j$ of $X_*$, functions $f_j \in EC^1(\Omega_j, \mathbb{R})$ and linear isometries $L_j$ in $X$ satisfying (5.2). Then, for every $j$, there exists $\Psi_j \in C^1(X_*)$ such that $\nabla f_j = \Psi_j$ a.e. in $\Omega_j$. If one defines

$$\Phi_j := \Psi_j - e_{n+1} \in C^1(X_*, X\setminus \{0\})$$

the rectifiable set

$$R := \bigcup_j \{ (L_j(x_*, f_j(x_*)); J L_j \Phi_j(x_*)) \mid x_* \in \Omega_j \} \subset X \times Y$$

is Legendrian and one has $X(R) = \Sigma$. \qed
5.2. A conjecture

The facts stated above raise interesting questions related to the density properties of rectifiable sets carrying locally integral currents (in the sense of §4.1.24 of [12]). As for the case of Legendrian rectifiable sets, in particular, we believe that the following conjecture holds (with the notation of the previous subsection).

**Conjecture 5.1.** Let \( G \) be a Legendrian \( n \)-dimensional rectifiable subset of \( X \times Y \) such that \( X(G) \) carries a locally integral current. Then \( X(G) \) is a \( n \)-dimensional \( C^2 \)-rectifiable set.

The reason for our “belief” can be summarized by the following rough argument which is, however, far from being a complete proof. First of all, by Proposition 5.1, there must exist countably many measurable subsets \( \Omega_j \) of \( X_* \), functions \( f_j \in EC^1(\Omega_j, \mathbb{R}) \) and linear isometries \( L_j \) in \( X \) such that

\[
X(G) = \bigcup_j L_j \Gamma_j, \quad \Gamma_j := \{(x_*, f_j(x_*)) \in X | x_* \in \Omega_j\}
\]

possibly except for a zero set. Now, as a consequence of the additional assumption about \( X(G) \), we expect that the \( \Omega_j \) can be chosen to be enhanced density sets. We suppose that Lemma 4.1 and the decomposition theorem (§27.6 of [16]) could be useful tools for proving such an assertion. It will be the subject of future work!

The conjecture above, in a slightly less general form, was first posed and discussed in [14]. Strictly related to this subject are also the papers [7], [8], [9], [10].

References


Received February 12, 2010.

SILVANO DELLADIO: Dipartimento di Matematica, via Sommarive 14, Povo, 38123 Trento, Italy.
E-mail: delladio@science.unitn.it