Abstract. Let \( \psi_j^h \) and \( E_j^h \) denote the eigenfunctions and eigenvalues of a Schrödinger-type operator \( H_h \) with discrete spectrum. Let \( \psi(x, \xi) \) be a coherent state centered at a point \((x, \xi)\) belonging to an elliptic periodic orbit, \( \gamma \) of action \( S_\gamma \) and Maslov index \( \sigma_\gamma \). We consider “weighted Weyl estimates” of the following form: we study the asymptotics, as \( h \to 0 \) along any sequence

\[
h = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma},
\]

\( l \in \mathbb{N}, \alpha \in \mathbb{R} \) fixed, of

\[
\sum_{|E_j - E| \leq c h} |(\psi(x, \xi), \psi_j^h)|^2.
\]

We prove that the asymptotics depend strongly on \( \alpha \)-dependent arithmetical properties of \( c \) and on the angles \( \theta \) of the Poincaré mapping of \( \gamma \). In particular, under irrationality assumptions on the angles, the limit exists for a non-open set of full measure of \( c \)'s. We also study the regularity of the limit as a function of \( c \).

1. Introduction and results.

Consider a Schrödinger operator \( H = -h^2 \Delta + V(x) \) with \( V \) smooth, either on \( M = \mathbb{R}^m \) (in which case we assume \( V \) tends to infinity at infinity and therefore \( H \) has discrete spectrum) or on a compact Riemannian
manifold, $M$. In [7] we considered “trace formulae” associated to projectors on coherent states in the following sense. For $(x, \xi) \in \mathbb{R}^{2m}$ and $a \in \mathcal{S}(\mathbb{R}^m)$ define the coherent state $\psi^a_{x,\xi}$ as:

\[
(1) \quad \psi^a_{x,\xi}(y) = \rho(y - x) (2\pi\hbar)^{-m/4} 2^{-m/4} e^{-i\xi y / \hbar} e^{i\xi x / \hbar} \hat{a}\left(\frac{y - x}{\sqrt{\hbar}}\right).
\]

Here $\rho$ is a cut-off function near zero and $\hat{a}$ is the Fourier transform of $a$, (in the manifold case $(x, \xi) \in T^* M$ and the above definition is in local coordinates near $x$). Let $\psi_j$ and $E_j$ the eigenfunctions and eigenvalues of $H$. Then if $\varphi$ is a Schwartz function whose Fourier transform is compactly supported and $E = |\xi|^2 + V(x)$, we have

\[
(2) \quad \sum_j \varphi\left(\frac{E_j - E}{\hbar}\right) |(\psi_{x,\xi}, \psi_j)|^2 \sim \sum_{j=0} c_j^2(x;\xi) \hbar^{-m+1/2+j},
\]

for $\hbar \to 0$. (If $E \neq |\xi|^2 + V(x)$, the left-hand side tends to 0 rapidly in $\hbar$.) Although the form of the asymptotic expansion does not depend on $(x, \xi)$, the coefficient $c_0(x, \xi)$ is highly sensitive to the point $(x, \xi)$ being periodic or not with respect to the classical flow. In case $(x, \xi)$ is either not periodic or is on a hyperbolic trajectory, we proved in [7] (using a Tauberian theorem) that, for every $c \in \mathbb{R}$,

\[
(3) \quad \sum_{|E_j - E| \leq \hbar c} |(\psi_{x,\xi}, \psi_j)|^2 = \sum_{c \in \mathbb{Z}} \chi_{[-\infty,c]}(x,\xi) \hbar^{-m+1/2} + o(\hbar^{-m+1/2}),
\]

as $\hbar \to 0$ possibly along certain sequence. (Here $\chi_{[-\infty,c]}$ is the characteristic function of the interval $[-c, c]$.) The main goal of this paper is to study the case where $(x, \xi)$ belongs to an elliptic closed trajectory.

Our results are related to the existence of quasi-modes near an elliptic trajectory. Recall that if $H$ is as before and $\gamma$ is a closed elliptic trajectory of the Hamiltonian $|\xi|^2 + V(x)$ with energy $E$, period $T_\gamma$, action $S_\gamma$, Maslov index $\sigma_\gamma$ and Poincaré mapping of angles $\theta_j$, $j = 1, \ldots, m-1$, then one can construct (see [9], [3], [8], [7]) quasi-modes of $H$ (namely solutions of the Schrödinger equation modulo a remainder), microlocalized near $\gamma$, of quasi-energies

\[
(4) \quad E_{QM}^{k,l} = E + \frac{\hbar}{T_\gamma} \left(2\pi l - \frac{S_\gamma}{\hbar}\right) + \sum_{j=1}^{m-1} \left(k_j + \frac{1}{2}\right) \theta_j + \sigma_\gamma,
\]
for \((k, l) \in \mathbb{Z}^m, l \) large. The remainder is \(O(\hbar^2)\) uniformly as

\[
\left| 2\pi l - \frac{S_\gamma}{\hbar} \right| \quad \text{and} \quad |k| := \sum j k_j
\]

remain bounded. The existence of these quasi-modes implies that part of the spectral density of \(H\) concentrates near the quasi-energies defined by (4), but this doesn’t say anything about \(E_{QM}^{k,l}\) as \(|k| \to \infty\) and does not involve the rest of the spectrum. The results of this paper will indicate that the rescaled localized spectral density

\[
\sum_j \delta\left( \frac{E_j - \lambda}{\hbar} \right) |(\psi_j(x,\xi), \psi_j)|^2
\]

(which is the rescaled spectral density microlocalized at the point in phase space \((x, \xi)\)) has a certain semiclassical limit whose singularities are indeed precisely the quasi-energies (4), and this time with no restriction on \(|k|\).

We will now state our results, valid for more general quantum Hamiltonians: Let \(H_h = \sum_{l=0}^L \hbar^l P_l(x, D_x)\) where \(P_l\) is a differential operator of order \(l\) on \(\mathbb{R}^m\) (or \(M\)) of principal symbol \(P^0_l\), sub-principal symbol \(P^{-1}_l\) (formally \(P_l\) is regarded as acting on half-densities) and smooth coefficients. Let \(\mathcal{H}(x, \xi) = \sum_{l=0}^L P^0_l(x, \xi)\) and \(\mathcal{H}_{sub}(x, \xi) = \sum_{l=0}^L P^{-1}_l(x, \xi)\) be the principal and sub-principal symbols of \(H_h\). We assume that \(P_L\) is elliptic, \(\mathcal{H}\) is positive, and in case \(M = \mathbb{R}^m\), that \(\mathcal{H}\) tends polynomially to infinity at infinity. We will also suppose for simplicity that \(\mathcal{H}_{sub}(x, \xi) = 0\).

Let \(E_j^h\) and \(\psi_j^h\) denote the eigenvalues and eigenvectors of \(H_h\). Let us suppose that \((x, \xi)\) belongs to an elliptic trajectory of period \(T_\gamma\), action \(S_\gamma\), Maslov index \(\sigma_\gamma\) and Poincaré mapping of angles \(\theta = (\theta_1, \ldots, \theta_{m-1})\). We will use throughout the notations

\[
k = (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1},
\]

\[
k \theta := \sum_{j=1}^{m-1} k_j \theta_j \quad \text{and} \quad (k + \frac{1}{2}) \theta := \sum_{j=1}^{m-1} \left(k_j + \frac{1}{2}\right) \theta_j.
\]
Theorem 1.1. Assume that \( \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi) \) are rational. Then, for every \( \alpha \in [0, 2\pi) \), as \( h \to 0 \) along the sequence

\[
\tag{7} h = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma}, \quad l \in \mathbb{N},
\]

one has

\[
\tag{8} \sum_{|E_j - E| \leq ch} | (\psi_{(x, \xi)}, \psi_j) |^2 = h^{-m+1/2} \mathcal{L}_\alpha(c) + o(h^{-m+1/2}),
\]

for all \( c \) such that

\[
\tag{9} c \neq \pm \frac{1}{T_\gamma} \left( 2\pi j + \left( k + \frac{1}{2} \right) \theta + \alpha \right), \quad \text{for all } j \in \mathbb{Z}, \ k \in \mathbb{N}^{m-1}.
\]

Moreover, as a function of \( c \) the limit \( \mathcal{L}_\alpha(c) \) is a step function constant on the intervals defined by (9).

Next we consider the irrational case:

Theorem 1.2. Assume that \( 1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi) \) are linearly independent over the rationals. Then there exists a set \( \mathcal{M}_\alpha \) of values of \( c \), of full Lebesgue measure, such that for all \( c \in \mathcal{M}_\alpha \)

\[
\tag{10} \sum_{|E_j - E| \leq ch} | (\psi_{(x, \xi)}, \psi_j) |^2 = h^{-m-1/2} \mathcal{L}_\alpha(c) + o(h^{-m-1/2}),
\]

for \( h \) as in (7). Moreover, as a function of \( c \), \( \mathcal{L}_\alpha(c) \) is locally Lipschitz on \( \mathcal{M}_\alpha \) in the sense that for all \( c \in \mathcal{M}_\alpha \) there exists \( \beta_c > 0 \) such that

\[
\tag{11} | \mathcal{L}_\alpha(c') - \mathcal{L}_\alpha(c) | \leq \beta_c | c' - c |, \quad \text{for all } c' \in \mathcal{M}_\alpha.
\]

Finally there exists a rapidly decreasing family \( \{ g_k \}_{k \in \mathbb{N}^{m-1}} \) (related to the microlocalization of the symbol \( \psi_{(x, \xi)} \)) such that

\[
\tag{12} \{ c : \text{ for all } k \in \mathbb{N}^{m-1} \ | 1 - e^{i(cT_{\gamma}+(k+1/2)\theta+\alpha)} | \geq \varepsilon g_k \} \subset \mathcal{M}_\alpha,
\]

for all \( \varepsilon > 0 \). (For a precise definition of the set \( \mathcal{M}_\alpha \) see Lemma 3.3.)
Remark. In the rational case the discontinuities of the function $\mathcal{L}_\alpha$ are located exactly at the values of the $E_{Q,M}^{k,l}$ defined before by (4), for the values of $\hbar$ given by (7). In the irrational case in order to prove that $\mathcal{L}_\alpha(c)$ exists we need that $c$ be at some distance from the quasi-energies $E_{Q,M}^{k,l}$ (unless the symbol $a$ of the quasi-mode is chosen very judiciously, in which case we can work with $c$ in the complement of the set of all quasi-energies). In all cases this suggests that the weighted spectral measure, (5), in the semi-classical limit, is particularly singular exactly at the values of the $E_{Q,M}^{k,l}$ defined before. We hope to provide a rigorous proof of a precise statement of this elsewhere.

The paper is organized as follows: In Section 3 we prove the existence of the functions $\mathcal{L}_\alpha$ which are studied in Section 4. In Section 5 we finish the proof of the main Theorems, using a Tauberian argument that we recall in Section 2. Finally, in the appendix we review and extend slightly a result on Hölder continuity of function such as $\mathcal{L}_\alpha$ using wavelets.

2. A Tauberian lemma.

In this section we refine the Tauberian lemma of [2] and [7].

Consider an expression of the following form

$$\gamma_{E,h}^w(\varphi) = \sum_j w_j(h) \varphi \left( \frac{E_j(h) - E}{\hbar} \right),$$

defined for all $\varphi \in \mathcal{R}$ where $\mathcal{R}$ will henceforth denote the set of all Schwartz functions on the line with compactly supported Fourier transform.

Let $\mathcal{M}^\alpha$ a subset of $\mathbb{R}^+$ of full Lebesgue measure in a bounded interval.

We introduce the following notations. Fix a positive function $f \in \mathcal{R}$ satisfying $f(0) = 1$ and $\hat{f}(0) = 1$. For every $a > 0$, define

$$f_a(r) := a^{-1} f \left( \frac{r}{a} \right)$$

and for every $a > 0$ and $c > 0$

$$\varphi_{a,c} := f_a \ast \chi_{[-c,c]},$$
where \( \chi_{[-c, c]} \) is the characteristic function of the interval \([-c, c]\).

The Tauberian lemma in question is:

**Theorem 2.1** (See [2] and [7]). Let \( \mathcal{M}^\alpha \) a subset of \( \mathbb{R}^+ \) of full Lebesgue measure in a bounded interval. Suppose \( w_j(h), E_j(h), E \) and \( \Upsilon^w_h \) itself satisfy all of the following:

1) There exists a positive function \( \omega(h) \), defined on an interval \((0, h_0)\), and a functional \( F_0 \) on \( \mathcal{R} \), such that for all \( \varphi \in \mathcal{R} \)

\[
\Upsilon^w_{E,h}(\varphi) = F_0(\varphi) \omega(h) + o(\omega(h)), \quad h \to 0.
\]

2) for all \( c \in \mathcal{M}^\alpha \) the limit

\[
\mathcal{L}_\alpha(c) = \lim_{a \to 0} F_0(\varphi_{a, c})
\]

exists.

3) \( \mathcal{L}_\alpha \) is a continuous function on \( \mathcal{M}^\alpha \).

4) There exists a \( k \in \mathbb{Z} \) such that \( h^k = O(\omega(h)) \), \( h \to 0 \).

5) There exists an \( \varepsilon > 0 \) such that for every \( \varphi \) there is a constant \( C_\varphi \) such that for all \( E' \in [E - \varepsilon, E + \varepsilon] \)

\[
|\Upsilon^w_{E,h}(\varphi)| \leq C_\varphi \omega(h)
\]

(rough uniformity in \( E \)).

6) The \( w_j(h) \) are non-negative and bounded: there exists a constant \( C \geq 0 \) such that for all \( j \) and all \( h, 0 < h < h_0 \)

\[
0 \leq w_j(h) \leq C.
\]

7) The eigenvalues \( E_j(h) \) satisfy the following rough estimate: for each \( C_1 \) there exist constants \( C_2, N_0 \) such that for all \( k \)

\[
\# \{ j : E_j(h) \leq C_1 + kh \} \leq C_2 (h^{-1} k)^{N_0}.
\]

Define the weighted counting function by

\[
N^w_{E,c}(h) = \sum_{j : |x_j(h)| \leq c} w_j(h),
\]
where

\begin{equation}
  x_j(h) := \frac{E_j(h) - E}{h}.
\end{equation}

Then the conclusion is: for all \( c \in \mathcal{M}^\alpha \),

\begin{equation}
  N_{E,E}(h) = \mathcal{L}_\alpha(c) \omega(h) + o(\omega(h)), \quad h \to 0.
\end{equation}

**Proof.** Except for the fact that the set \( \mathcal{M}^\alpha \) of allowed \( c \)'s is not \( \mathbb{R}^+ \), this theorem is precisely [2, Theorem 6.3]. Proceeding exactly as in the proof of the [2, inequalities (188)], one shows that for all \( R > 0 \), for all \( N \in \mathbb{N} \) exists \( C > 0 \), \( C_N > 0 \) such that for all \( a \in (0, R) \) and for all \( \eta \), \( 0 < \eta < c \),

\begin{equation}
  \frac{1}{\omega(h)} \left( 1 - \frac{a}{\eta} \right) N_{E,E-c-\eta}(h) \leq \frac{1}{\omega(h)} \Upsilon_{E,h}(\varphi_{a,c}) \leq \frac{1}{\omega(h)} N_{E,E-c-\eta}(h) + C_N \left( \frac{a}{\eta} \right)^N.
\end{equation}

Let \( c \in \mathcal{M}^\alpha \) be given. We begin by observing that by the first of the inequalities (23)

\begin{equation}
  \frac{1}{\omega(h)} N_{E,E}(h) \leq \frac{1}{\omega(h)} \Upsilon_{E,h}(\varphi_{a,c}) + C_1 \frac{a}{\eta},
\end{equation}

where we have also used the fact that \( N_{E,E}(h)/\omega(h) \) is bounded (a trivial consequence of (16)). For every \( \eta \) such that \( 0 < \eta < c \) one can take the limit in (24) as \( h \to 0 \) to obtain that

\begin{equation}
  \limsup_{h \to 0} \frac{1}{\omega(h)} N_{E,E}(h) \leq \mathcal{F}_0(\varphi_{a,c+\eta}) + C_1 \frac{a}{\eta}.
\end{equation}

If we now assume that \( \eta + c \in \mathcal{M}^\alpha \) we can take the limit as \( a \to 0 \) to obtain

\begin{equation}
  \limsup_{h \to 0} \frac{1}{\omega(h)} N_{E,E}(h) \leq \mathcal{L}_\alpha(c + \eta).
\end{equation}

By the assumption that \( \mathcal{M}^\alpha \) has full measure, we can find a sequence \( \{\eta_j\} \) such that for all \( j \), \( c + \eta_j \in \mathcal{M}^\alpha \) and \( \eta_j \to 0 \). Taking the limit in
(26) of $\mathcal{L}_\alpha(c + \eta_j)$ as $j \rightarrow \infty$ and using the fact that $\mathcal{L}_\alpha$ is continuous at $c$ we obtain

$$\limsup_{h \to 0} \frac{1}{\omega(h)} N_{E,c}(h) \leq \mathcal{L}_\alpha(c).$$

A similar argument starting with the second inequality (23) shows that

$$\liminf_{h \to 0} \frac{1}{\omega(h)} N_{E,c}(h) \geq \mathcal{L}_\alpha(c),$$

which finishes the proof.

3. The existence of $\mathcal{L}_\alpha(c)$.

In this section we prove the existence of the coefficients $\mathcal{L}_\alpha(c)$ in the limits (8) and (10) (see (36) below).

**Lemma 3.1.** There exists a rapidly decreasing family of non-negative numbers, $\{c_k\}_{k \in \mathbb{N}^m - 1}$, such that for all $\varphi \in \mathcal{R}$ the first coefficient $c^\varphi_0(x, \xi)$ in (2) can be written as

$$c^\varphi_0(x, \xi) = \sum_{n=-\infty}^{+\infty} \sum_{k \in \mathbb{N}^m - 1} \varphi(n T_\gamma) c_k e^{in((k+1/2)\theta + \alpha)}.$$

**Proof.** In [7] we proved that the first coefficient $c^\varphi_0(x, \xi)$ in (2) can be written as

$$2^{2n} \pi^{(3n+1)/2} c^\varphi_0(x, \xi)
\sum_{n=-\infty}^{+\infty} \varphi(n T_\gamma) e^{in S_\gamma / h + \sigma} \int_{-\infty}^{+\infty} (a, Z((s \dot{x}, s \dot{\xi})) U^a) ds ,$$

where $(\dot{x}, \dot{\xi})$ is the tangent vector to the classical flow at $(x, \xi)$, $Z$ is the Weyl/Heisenberg operator defined by

$$Z(e, f)(a)(\eta) = e^{-i\epsilon f / 2} e^{i\epsilon n} a(\eta - f)$$

and $U$ is the metaplectic representation of the linearized flow at time $T_\gamma$.

(We should point out that in the manifold case $a$ defines intrinsically a
smooth vector in the metaplectic representation of $T_{(x,\xi)}(T^*M)$, and $U$ and $Z$ are operators in that representation space.) Denoting by $S$ the linearized flow at time $T_\gamma$, we also showed that one can find a symplectic mapping $R$ such that $R^{-1}SR$ is block-diagonal of the form

$$(32) \quad R^{-1}SR = \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_\theta \end{pmatrix},$$

where $\mu \in \mathbb{R}$ and $A_\theta$ is the direct sum of rotations of angles $\theta_1, \ldots, \theta_{m-1}$. Furthermore, the transformation $R$ maps the vector $(s \dot{x}, s \dot{\xi})$ to the vector $(s, 0)$.

Let us denote $a' := \text{Mp}(R)^{-1}a$ and $V := \text{Mp}(R^{-1}SR)$, where $\text{Mp}(R)$ denotes the metaplectic representation of the mapping $R$. Then, letting $Z(s) := Z((s \dot{x}, s \dot{\xi}))$ and

$$W(s) := \text{Mp}(R)^{-1} Z(s) \text{Mp}(R) = Z(s, 0, 0, 0),$$

one has

$$\left( a, Z(s) U^n a \right) = \left( a', W(s) V^n a' \right).$$

Denote the variables of $a'$ by $(\eta_1, \eta_2)$ where $\eta_1 \in \mathbb{R}$ and $\eta_2 \in \mathbb{R}^{m-1}$, and let $e^{i\Theta(D_{\eta_2}^2 + \eta_2^2)/2}$ denote the direct sum of the propagators of one-dimensional Harmonic oscillators at times $\theta_1, \ldots, \theta_{m-1}$, acting on $a'$ by acting on the $\eta_2$ variables. If $e^{i n \beta_{\eta_1}^2 / 2}$ denotes the metaplectic quantization of

$$(33) \quad \begin{pmatrix} 1 & n \mu \\ 0 & 1 \end{pmatrix},$$

we get that (30) becomes

$$2^{2n} \pi^{(3n+1)/2} e_0^2 (x, \xi)$$

$$= \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n T_\gamma) e^{i n \alpha}$$

$$\cdot \int e^{i n \beta_{\eta_1}^2 / 2} e^{i n \Theta(D_{\eta_2}^2 + \eta_2^2) / 2} \left( a' \right) \left( \eta_1 - s, \eta_2 \right) d\eta_1 ds.$$
formula plus the fact that on the Fourier transform side the operator $e^{i m \mu^2} / 2$ is multiplication by $e^{-i m \mu^2} / 2$ ($\zeta$ being the dual variable), one gets

$$2^{2n} \pi^{(3n+1)/2} C_0^\zeta (x, \xi)$$

(34)

$$= \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n T_\gamma) e^{i m \alpha} \int a^{s}(0, \eta_2) e^{i n \theta(D_{\eta_2}^{2} + \eta_2^{2})/2} a^{s}(0, \eta_2) d\eta_2 ,$$

where $a^{s}$ is the Fourier transform of $a^s$ with respect to $\eta_1$. Let $b(x) := a^s(0, x)$ and let us decompose $b$ on the Hermite basis, $h_k$, of eigenfunctions of the harmonic oscillator

(35)

$$b = \sum_{k \in \mathbb{N}^{m-1}} b_k h_k .$$

Then, letting $c_k := |b_k|^2$ we get (29) and the family $\{c_k\}$ is non-negative. It is also rapidly decreasing since the function $b$ is Schwartz.

**Remark.** For a given quantum Hamiltonian $H$, the coefficients $\{c_k\}$ depend only on the symbol $a$ of the coherent state. Observe that the proof shows that given any rapidly decreasing family $\{c_k\}$ one can find an $a$ giving rise to it.

We next prove the existence of the limit

(36)

$$\mathcal{L}_a(c) := \lim_{a \to 0} c_0^{(f_\alpha^s x_{\zeta}^{[c, c]})} (x, \xi) ,$$

for $f$ as in the Tauberian lemma and $c_0^{\zeta} (x, \xi)$ as in (29). Let $\phi_a(c) := c_0^{(f_\alpha^s x_{\zeta}^{[c, c]})} (x, \xi)$, that is

(37)

$$\phi_a(c) := c + \sum_{n \neq 0, k} \hat{f}(a n) \frac{\sin(n c T_\gamma)}{n T_\gamma} c_k e^{i m (k+1/2) \theta + \alpha} .$$

We must then prove that the limit $\mathcal{L}_a(c) = \lim_{a \to 0} \phi_a(c)$ exists.

To lighten up the notation a bit, let us define

(38)

$$d_k := (k + \frac{1}{2}) \theta + \alpha , \quad k \in \mathbb{N}^{m-1} ,$$
keeping in mind the notation (6). Let $0 < a < 1$, then
\[
\phi_1(c) - \phi_a(c) = \frac{1}{T_\gamma} \sum_{(n,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}} \sin(c n T_\gamma) c_k e^{i \pi d_k} \int_a^1 \hat{f}'(t n) \, dt
\]

(39) \[
= \frac{1}{T_\gamma} \sum_{(n,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}} \left( e^{i m c T_\gamma} - e^{-i m c T_\gamma} \right) c_k e^{i \pi d_k} \int_a^1 \hat{f}'(t n) \, dt .
\]

Applying the Poisson summation formula to the series over $n$, we get (after a calculation)

(40) \[
\phi_1(c) - \phi_a(c) = -\frac{\pi}{T_\gamma} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}} c_k \int_a^1 \left( g \left( \frac{1}{t} (2\pi j + c T_\gamma + d_k) \right) \right) \frac{dt}{t},
\]

where $g(x) := x f(x)$.

**Lemma 3.2.** Define
\[
\mathcal{M}_0^\theta = \left\{ c \in \mathbb{R} : \text{for all } (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1}, \ c \neq \pm \frac{1}{T_\gamma} (2\pi j + d_k) \right\}.
\]

If $\theta_1/(2\pi), \ldots, \theta_m/(2\pi)$ are rational and $c \in \mathcal{M}_0^\theta$, then each of the limits

(41) \[
\lim_{a \to 0} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}} c_k \int_a^1 g \left( \frac{1}{t} (2\pi j \pm c T_\gamma + d_k) \right) \frac{dt}{t},
\]

exists (and is finite). Moreover, the convergence is locally uniform in $c$.

**Proof.** By the rationality assumption the complement of $\mathcal{M}_0^\theta$ is discrete. Therefore, if $c \in \mathcal{M}_0^\theta$ there exists $\varepsilon$ such that
\[
0 < \varepsilon \leq |2\pi j \pm c T_\gamma + d_k|, \quad \text{for all } (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1}.
\]

The function $g$ is rapidly decreasing: for all $N \in \mathbb{N}$ \( \exists C_N > 0 \) such that for all $x \in \mathbb{R}$, $|g(x)| \leq C_N (1 + |x|)^{-N}$. Therefore

(42) \[
\left| g \left( \frac{2\pi j \pm c T + d_k}{t} \right) \right| \leq C_N \frac{t^N}{t^{N} + (2\pi j \pm c T + d_k)^N} \leq C_N \frac{t^N}{(2\pi j \pm c T + d_k)^N},
\]
and so for all \((j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1}\) and for all \(a \in (0, 1)\)

\[
\int_a^1 \left| g \left( \frac{1}{t} \left( 2\pi j + cT + d_k \right) \right) \right| \frac{dt}{t} \leq \frac{C_N}{N} \frac{1 - a^{N+1}}{|2\pi j + cT + d_k|^N}.
\]

This shows that each of the integrals in the series (41) extends to a continuous function of \(a \in [0, 1)\). Moreover, since the family

\[ M_{k,j} := \frac{c_k}{|2\pi j + cT + d_k|^N}, \quad (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1} \]

is absolutely convergent (for \(N\) sufficiently large) and it dominates the absolute values of the terms of (41), we are done.

We now turn to the irrational case.

**Lemma 3.3.** Assume that \(1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)\) are linearly independent over the rationals. Let

\[
\mathcal{M}_\pm^\alpha := \left\{ c \in \mathcal{M}_0^\alpha : \sum_{k \in \mathbb{N}^{m-1}} c_k \left( \pm \left( d_k + \frac{cT}{2\pi} \right) \right)^{-2} < \infty \right\},
\]

where \(\{x\}\) denotes the fractional part of \(x\), and let

\[
\mathcal{M}^\alpha := \mathcal{M}_+^\alpha \cap \mathcal{M}_-^\alpha .
\]

Then, if \(c \in \mathcal{M}^\alpha\), each of the limits

\[
\lim_{a \to 0} \sum_{j,k} c_k \int_a^b g \left( \frac{1}{t} \left( 2\pi j + cT + d_k \right) \right) \frac{dt}{t}
\]

exists and is finite. Moreover, the convergence is locally uniform in \(c\).

**Proof.** It is enough to consider one of the series above, say the one with the plus sign. Let \(c \in \mathcal{M}^\alpha\) and define

\[
O^+ := \{ (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1} : 2\pi j + cT + d_k > 0 \},
\]

and

\[
O^- := \{ (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1} : 2\pi j + cT + d_k < 0 \}.
\]
Since $c \in \mathcal{M}_0^\mathbf{a}$, $\mathbb{Z} \times \mathbb{N}^{m-1} = O^+ \cup O^-$. Recalling that $g(x) = x f(x)$ and that $f$ as well as the $c_k$ are non-negative, we see that the terms with $(j, k) \in O^\pm$ have the sign $\pm$ and therefore each of

$$
\sum_{(j, k) \in O^\pm} c_k \int_a^1 g \left( \frac{1}{t} (2\pi j + cT \gamma + d_k) \right) \frac{dt}{t}
$$

is a decreasing function of $\alpha$. It therefore suffices to show that

$$
\lim_{\alpha \to 0} \sum_{(j, k) \in O^+} c_k \int_a^1 g \left( \frac{1}{t} (2\pi j + cT \gamma + d_k) \right) \frac{dt}{t} < \infty
$$

and similarly for the series over $O^-$. Specializing (43) to $N = 2$, we see that exists $C > 0$ such that for all $\alpha \in (0, 1)$ and for all $(j, k) \in O^+$

$$
(46) \quad \int_a^1 g \left( \frac{2\pi j + cT + d_k}{t} \right) \frac{dt}{t} \leq \frac{C}{(2\pi j + cT + d_k)^2}.
$$

(The last denominator is not zero if $(j, k) \in O^+$. ) Therefore, the Lemma will be proved provided we show the convergence of the double series of scalars

$$
(47) \quad \sum_{(j, k) \in O^+} M_{k, j},
$$

where

$$
(48) \quad M_{k, j} = c_k \left( j + \frac{cT + d_k}{2\pi} \right)^{-2},
$$

that is

$$
\xi = \left( \frac{\theta_1}{2\pi}, \ldots, \frac{\theta_{m-1}}{2\pi} \right) \quad \text{and} \quad \beta = \frac{1}{2\pi} \left( cT + \alpha + \sum_{j=1}^{m-1} \frac{\theta_j}{2} \right).
$$

Since the terms in (47) are positive, we can prove its convergence by first summing over $j$ with $k$ fixed, and then summing over $k \in \mathbb{N}^{m-1}$. Observe that

$$
(50) \quad (j, k) \in O^+ \quad \text{if and only if} \quad j \geq [-k \xi - \beta] + 1,
$$
where \([x]\) denotes the greatest integer less than or equal to \(x\). For every \(k\) consider the series

\[
(51) \quad \sum_{j=-[k \xi + \beta]+1}^{+\infty} M_{k,j}.
\]

(If \(x \not\in \mathbb{Z}\), then \([-x] = -[x] - 1\), and since \(c \in \mathcal{M}^0\), for all \(k \in \mathbb{N}^{m-1}\), \(k \xi + \beta \not\in \mathbb{Z}\)). Comparing this series with the integral

\[
(52) \quad \int_{-\lfloor k \xi + \beta \rfloor}^{\infty} \frac{dx}{(x + k \xi + \beta)^2},
\]

we find that

\[
(53) \quad \sum_{j=-[k \xi + \beta]+1}^{+\infty} M_{k,j} \leq M_{k,[-k \xi + \beta]} + \frac{c_k}{-[k \xi + \beta] + k \xi + \beta},
\]

or with the notation \(\{x\} = \) fractional part of \(x = x - [x]\),

\[
(54) \quad \sum_{j=-[k \xi + \beta]+1}^{+\infty} M_{k,j} \leq \frac{c_k}{\{k \xi + \beta\}^2} + \frac{c_k}{\{k \xi + \beta\}}.
\]

Therefore convergence of (47) follows from the convergence of

\[
\sum_{k \in \mathbb{N}^{m-1}} \frac{c_k}{\{k \xi + \beta\}^2}.
\]

But since by assumption \(c \in \mathcal{M}^0\), this series converges.

In conclusion we have shown that \(\mathcal{L}_\alpha(c)\) exists for \(c\) as defined by the Lemmas.

**Remarks.** In the irrational case:

1) To find examples of numbers \(c\) in \(\mathcal{M}^2\), it suffices to find a family \(\{g_k\}_{k \in \mathbb{N}^{m-1}}\) of positive numbers such that \(\sum g_k^{-2} c_k < \infty\). Then if

\[
(55) \quad |2\pi j + c T + (k + \frac{1}{2}) \theta + \alpha| > g_k, \quad \text{for all} \ (j, k) \in \mathbb{Z} \times \mathbb{N}^{m-1},
\]
then \( c \in \mathcal{M}^\alpha \). Defining \( \hat{g}_k = \varepsilon g_k \), \( \varepsilon > 0 \), one see that still \( \sum \hat{g}_k^{-2} c_k < \infty \) and therefore associated to \( \hat{g}_k \) (by (55)) is a subset of \( \mathcal{M}_\alpha \) whose intersection with any interval \( I \) has a \( \alpha \)-measure in \( I \) arbitrary small as \( \varepsilon \to 0 \); therefore \( \mathcal{M}^\alpha \) has full measure.

2) The set \( \mathcal{M}^\alpha \) is related to the rate of decay of the \( c_k \) (that is to the properties of the symbol, \( a \), of the coherent states), as well as to irrationality properties of \( \theta/(2\pi) \). At one extreme, we can choose \( a \) such that only finitely-many of the coefficients \( c_k \) are non-zero (see the remark following Lemma 3.1). In that case \( \mathcal{M}^\alpha = \mathcal{M}_0^\alpha \) is just the complement of the set of quasi-energies of the quasi-modes associated with the trajectory.

4. Properties of the function \( \mathcal{L}_\alpha \).

Having established the existence of the function \( \mathcal{L}_\alpha(c) \), we now derive some of its properties.

**Rational case.** Let us go back to the identity \( \mathcal{L}_\alpha(c) = \lim_{a \to 0} \phi_a(c) \) where \( \phi_a \) is defined in (37). Applying in (37) the Poisson summation formula to the series over \( n \) with \( k \) fixed one obtains

\[
\mathcal{L}_\alpha(c) = \lim_{a \to 0} \frac{1}{a} \left( F_c * f \left( \frac{\cdot}{a} \right) \right)(0),
\]

where

\[
F_c(y) = \int_{-c}^{c} \sum_{j, k} c_k \delta(T \gamma(x - y) - 2\pi j - d_k) \, dx.
\]

For each \( c > 0 \) the function \( F_c \) is a step function; indeed

\[
F_c = \sum_{j, k} c_k X_{[-c - (2\pi j + d_k)/T, c - (2\pi j + d_k)/T]}.
\]

Since \( f(\cdot/a)/a \to \delta \), we obtain

\[
\mathcal{L}_\alpha(c) = \sum_{c_k : c T < -c - c T} c_k, \quad \text{for all } c \in \mathcal{M}_0^\alpha,
\]

which is clearly a step function (i.e. a locally constant function) of \( c \in \mathcal{M}_0^\alpha \).
Irrational case. To study the function $L_\alpha(c)$ on $\mathcal{M}^\alpha$ as defined by (36), we will use a wavelet decomposition.

Let $g \in L^2$ be a function satisfying $\int g(x) \, dx = 0$ and $\int x \, g(x) \, dx = 0$. If it exists, the wavelet coefficient of $L_\alpha(c) - c$ is

$$T(a, b) = \frac{1}{a} \int g\left(\frac{x - b}{a}\right) \left(L_\alpha(x) - x\right) \, dx.$$

Plugging in (60) the expression

$$L_\alpha(x) - x = \sum_{n \neq 0 \atop k} \frac{\sin (n x T_\gamma)}{n T_\gamma} c_k e^{i n d_k} c_k,$$

one finds, supposing $\hat{g}$ even

$$T(a, b) = \frac{1}{2 i} \sum_{n \neq 0 \atop k} \frac{1}{n T_\gamma} \hat{g}(a n) \sin (n b T_\gamma) c_k e^{i n (d_k)}.$$

The following result shows that such a decomposition is indeed valid.

**Proposition 4.1.** Let $g$ as before, $\hat{g}$ being compactly supported and even, and let us suppose that $\varphi$ is a compactly supported function satisfying

$$\int \overline{\varphi}(a) \hat{g}(a) \frac{da}{a} = \int \overline{\varphi}(-a) \hat{\varphi}(-a) \frac{da}{a} = 1.$$

Then, for all $c \in \mathcal{M}^\alpha$,

$$L_\alpha(c) - c = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{+\infty} \varphi\left(\frac{c - b}{a}\right) T(a, b) \, db,$$

where

$$T(a, b) = \frac{1}{2 i} \sum_{n \neq 0 \atop k} \frac{1}{n T_\gamma} \hat{g}(a n) \sin (n b T_\gamma) c_k e^{i n (d_k)}.$$
Weighted Weyl estimates near an elliptic trajectory

Proof.
\[
\int_\varepsilon^+ \frac{da}{a} \int_{-\infty}^{\infty} \frac{db}{b} \varphi \left( \frac{a - b}{a} \right) T(a, b)
= \int_\varepsilon^+ \frac{da}{a} \sum_{n \neq 0} \frac{1}{n T_\gamma} \left( \hat{\varphi}(a n) e^{i n c T_\gamma} - \hat{\varphi}(-a n) e^{-i n c T_\gamma} \right)
\cdot e^{i n d_x} \hat{g}(a n) c_k
\]
\[
= \int_\varepsilon^+ \frac{da}{a} \sum_{n \neq 0} \frac{\hat{\varphi}(a n) \hat{g}(a n) \sin (n c T_\gamma)}{n T_\gamma} e^{i n ((k+1/2) \theta + \alpha)} c_k
\]
\[
= \sum_{n \neq 0} \psi(\varepsilon n) \sin (n c T_\gamma) e^{i n d_x} c_k,
\]
where
\[
\psi(\varepsilon) := \int_\varepsilon^+ \frac{da}{a} \varphi(a) \hat{g}(a).
\]
Noting that \( \psi'(a) = \hat{\varphi}(a) \hat{g}(a)/a \) is compactly supported and \( \psi(0) = 1 \) by hypothesis one get the result, thanks to Lemma 3.3.

The next result, thanks to the result of the Appendix will enable us to prove the Lipschitz continuity on \( M^\alpha \).

Proposition 4.2.

(67) \( T(a, b) = O(a) \), near 0 almost everywhere and uniformly in b.

Proof. Since \( \int x g(x) \, dx = \int g(x) \, dx = 0, g'(0) = 0 \). So one can find a \( C^\infty \) function \( f \) such that \( \hat{g}(\xi) = \xi f(\xi) \) and \( f(0) = 0 \). Then
\[
T(a, b) = a \sum_{n \neq 0} f(a n) \sin (b n T_\gamma) e^{i n d_x} c_k,
\]
and it is easy to check, by the same argument as in Lemma 3.3, that if \( b \in M^\alpha \),
\[
\sum_{n \neq 0} f(a n) \sin (b n T_\gamma) e^{i n ((k+1/2) \theta + \alpha)} c_k
\]
is bounded.

5. End of proofs.

The convergence statements in both theorems are immediate consequences of the Tauberian lemma of Section 2, applied to the following objects

\[
\Upsilon_h(a, c) = \sum_j w_j(h) \varphi \left( \frac{E_j(h) - E}{h} \right),
\]

where

\[
w_j(h) = |\langle \psi(x, \xi), \psi_j \rangle|^2.
\]

The weighted counting function is therefore

\[
\sum_{|E_j(h) - E| \leq \varepsilon} |\langle \psi(x, \xi), \psi_j \rangle|^2.
\]

The functional of the Tauberian lemma is

\[
\mathcal{F}_0(\varphi) := c_0^\varphi(x, \xi)
\]

as defined by (29). We must check that the above objects satisfy the assumptions of the Tauberian lemma.

a) Theorem 1.1. It is easy to see that the functional \( \mathcal{F}_0 \) defined where \( c_0^\varphi(x, \xi) \) is defined by (29) satisfies the hypothesis 2 of the Tauberian Lemma of Section 2 if we take for \( \mathcal{M}^\alpha \) the set defined by (9). Moreover the other hypotheses are satisfied as in [7]. Then just apply the Tauberian Lemma.

b) Theorem 1.2. The Lipschitz continuity of \( \mathcal{F}_0 \) is an immediate consequence of Proposition 4.2 together with Theorem A.1 below. The fact that \( \mathcal{M}^\alpha \) is of full Lebesgue measure, is a classical result of Diophantine analysis (recall that the sequence \( \{g_k \} \) in the remark 1, Section 3 is rapidly decreasing).
Appendix. Wavelets and Hölder continuity.

In this appendix we will prove an easy extension of results of [6], [5] and [4]. Let $\mathcal{M}^\alpha$ a bounded subset of $\mathbb{R}$ of full Lebesgue measure.

**Theorem A.1.** Let $g$ be a be a continuously differentiable compactly supported function. Let $f$ defined and bounded on $\mathcal{M}^\alpha$. Let us suppose that $f$ admits a “scale-space coefficient $T(a,b)$” decomposition with respect to $g$, namely

$$f(x) = \int_0^\infty \int_{-\infty}^{+\infty} g\left(\frac{x - b}{a}\right)T(a, b) \frac{da}{a} \, db,$$

for all $x \in \mathcal{M}^\alpha$.

Let us suppose moreover that

$$T(a, b) = o(a^\alpha),$$

near $0$ almost everywhere and uniformly in $b$. Then $F$ is $\alpha$-Hölder continuous on $\mathcal{M}^\alpha$; by this we mean

$$|f(x_1) - f(x_2)| = O_{x_1}(\|x_2 - x_1\|^\alpha), \quad \text{for all } x_1, x_2 \in \mathcal{M}^\alpha.$$

**Proof.** The proof is absolutely equivalent to the one in [4], so we will only sketch it. Let us write first:

$$f(x) = \left(\int_0^1 \frac{da}{a} + \int_1^{\infty} \frac{da}{a}\right) \int db g\left(\frac{x - b}{a}\right)T(a, b)$$

$$= f_s(x) + f_l(x),$$

$f_l$ is obviously $C^\infty$. We concentrate on $f_s$.

Let $x_1, x_2 \in \mathcal{M}^\alpha$, $x_1 < x_2$, we cut $f_s$ in three pieces.

$$f_s(x_1) - f_s(x_2) = \int_0^{x_2 - x_1} \frac{da}{a} \int db g\left(\frac{x_2 - b}{a}\right)T(a, b)$$

$$- \int_0^{x_1 - x_1} \frac{da}{a} \int db g\left(\frac{x_2 - b}{a}\right)T(a, b)$$

$$+ \int_{x_2 - x_1}^1 \frac{da}{a} \int db \left(\frac{1}{a} g\left(\frac{x_2 - b}{a}\right) - \frac{1}{a} g\left(\frac{x_1 - b}{a}\right)\right)$$

$$\cdot T(a, b)$$

$$= T_1 - T_2 + T_3.$$
We now analyze each term:

- \( T_1 \) and \( T_2 \). Since \( T(a, b) = O(a^\alpha) \) almost everywhere, we have

\[
|T_i| = \int_0^{x_2-x_1} \frac{da}{a} \int db \left| \frac{1}{a} g \left( \frac{x_i-b}{a} \right) \right| C \ a^\alpha
= O(|x_2 - x_1|^\alpha) \|g\|_{L_1} \frac{C}{a^\alpha}.
\]

- \( T_3 \). If \( g \) is continuously differentiable let us write

\[
g \left( \frac{x_2-b}{a} \right) - g \left( \frac{x_1-b}{a} \right) = \frac{x_2-x_1}{a} g' \left( \frac{x'-b}{a} \right)
\]

with \( x_1 \leq x' \leq x_2 \). So

\[
|T_3| \leq \int_{x_2-x_1}^1 \frac{da}{a} \int db \left| \frac{1}{a^2} g' \left( \frac{x'-b}{a} \right) \right| |T(a, b)||x_2 - x_1|
= O(|x_2 - x_1|) \|g'\|_{L_1} \int_{x_2-x_1}^1 \frac{da}{a} a^{\alpha-1}
= O(|x_2 - x_1|^\alpha).
\]

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