Local limit theorems on some non unimodular groups

Emile Le Page and Marc Peigné

Abstract. Let $G_d$ be the semi-direct product of $\mathbb{R}^*$ and $\mathbb{R}^d$, $d \geq 1$ and let us consider the product group $G_{d,N} = G_d \times \mathbb{R}^N$, $N \geq 1$. For a large class of probability measures $\mu$ on $G_{d,N}$, one proves that there exists $\rho(\mu) \in [0,1]$ such that the sequence of finite measures

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu_n \right\}_{n \geq 1}$$

converges weakly to a non-degenerate measure.

Résumé. Soit $G_d$ le produit semi-direct de $\mathbb{R}^*$ et de $\mathbb{R}^d$ et $G_{d,N}$ le groupe produit $G_d \times \mathbb{R}^N$, $N \geq 0$. Pour une large classe de mesures de probabilité sur $G_{d,N}$ nous montrons qu’il existe $\rho(\mu) \in [0,1]$ tel que la suite de mesures finies

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu_n \right\}_{n \geq 1}$$

converge vaguement vers une mesure non nulle.

1. Introduction.

Fix two integers $d \geq 1$, $N \geq 0$ and choose a norm $\| \cdot \|$ on $\mathbb{R}^d$ and $\mathbb{R}^N$ (when $N \geq 1$). Let $G_{d,N}$ be the connected group $\mathbb{R}^* \times \mathbb{R}^d \times \mathbb{R}^N$
with the composition law

\[
\text{for all } g = (a, u, b), \text{ for all } g' = (a', u', b') \in G, \\
g \cdot g' = (aa', au' + u, b + b').
\]

We will note \( g = (a(g), u(g), b(g)) \) (or \( g = (a, u, b) \) when there is no ambiguity). The group \((G_{d,N} \cdot \cdot \cdot)\) is a non-unimodular solvable group with exponential growth and the right Haar measure \( m_D \) on \( G_{d,N} \) is

\[
m_D(da\,du\,db_1\cdots db_N) = \frac{da\,du\,db_1\cdots db_N}{a}.
\]

Note that \( G_{d,0} \) is the semi-direct product of \( \mathbb{R}^+ \) and \( \mathbb{R}^d \); in particular \( G_{1,0} \) is the affine group of the real line.

We consider a probability measure \( \mu \) on \( G \); we denote by \( \mu^{*n} \) its \( n \)th power of convolution. Under quite general assumptions on \( \mu \) we show that there exists \( \rho(\mu) \in [0,1] \) such that the sequence

\[
\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)n - \mu^{*n}} \right\}_{n \geq 0}
\]

converges weakly to a non-degenerate measure. This problem has already been tackled by Ph. Bougerol in [3] where were established local limit theorems on some solvable groups with exponential growth; in particular, for a class \( R \) of probability measures \( \mu \) on the affine group of the real line (that is \( d = 1 \) and \( N = 0 \)) he showed that the sequence

\[
\left\{ \frac{n^{3/2}}{\rho(\mu)n - \mu^{*n}} \right\}_{n \geq 0}
\]

converges weakly to a non-degenerate measure. In [7] we extend this result to a quite large class of probability measures; the new ingredient in our proof was the fact that there exists closed connections between this problem and the theory of the fluctuations of a random walk on the real line. In the present paper, we extend this result to the case \( N \geq 1 \); we first obtain uniform upperbounds in the Local limit theorem for a random walk on \( \mathbb{R}^d \), and, secondly, we use a generalisation of the Wiener-Hopf’s factorisation due to Ch. Sunyach [9].

This study is also related with the work by N. T. Varopoulos [10], [11] where upperbounds and lowerbounds for the asymptotic behaviour
of the convolution powers $\mu^n$ of a large class of probability measures are given.

From now on, we will suppose that $N \geq 1$ and we set $G = G_{d,N}$. We introduce the following conditions on $\mu$:

**Hypothesis G1.** There exists $\alpha > 0$ such that

$$\int_G (e^{\alpha |\log a|} + \|u\|^\alpha + \|b\|^2) \mu(da\,du\,db) < +\infty.$$ 

**Hypothesis G2.** $\int_G \log a \mu(da\,du\,db) = 0$ and $\int_G b \mu(da\,du\,db) = 0$.

**Hypothesis G3.** The support of $\mu$ is included in $\mathbb{R}^+ \times (\mathbb{R}^+)^d \times \mathbb{R}^N$, the image of $\mu$ by the mapping $(a,u,b) \mapsto (\log a, b)$ is aperiodic in $\mathbb{R}^{N+1}$ (see Definition 2.1) and there exists $\beta > 0$ such that

$$\int_G \|u\|^{-\beta} \mu(da\,du\,db) < +\infty.$$ 

**Hypothesis G’3.** The measure $\mu$ is absolutely continuous with respect to the Haar measure $m_D$ on $G$ and its density $\phi_\mu$ satisfies

$$\int_{[0,1] \times \mathbb{R}^N} \sqrt{s} \int_{\mathbb{R}} \phi_\mu^q(a,u,b) \, du \frac{da\,db}{a^\gamma} < +\infty.$$  

for some $\gamma$ and $q$ in $]1, +\infty[$.

We have the

**Theorem 1.1.** Let $\mu$ be a probability measure on $G$ satisfying hypotheses G1, G2 and G3 (or G’3). Then, the sequence of finite measures $\{n^{(N+3)/2} \mu^n\}_{n \geq 0}$ converges weakly to a non-degenerate Radon measure on $G$.

Note that the asymptotic behavior of the sequence $\{\mu^n\}_{n \geq 1}$ does not depend on $d$.

When $\mu$ is not centered, that is

$$\int_G \log a \mu(da\,du\,db) \neq 0$$
or
\[ \int_G b \mu(da \, du \, db) \neq 0, \]
we bring back the study to the centered case as in [7]. We introduce the following conditions on \( \mu \):

**Hypothesis G*1.** There exists \( \alpha > 0 \) such that
\[ \int_G (a^t + \|u\|^\alpha + \exp(t \|b\|)) \mu(da \, du \, db) < +\infty \]
for any \( t \in \mathbb{R} \).

**Hypothesis G*2.** One has
\[ \int_G \log a \, \mu(da \, du \, db) \neq 0 \]
with \( \mu\{g \in G : a(g) < 1\} > 0 \) and \( \mu\{g \in G : a(g) > 1\} > 0 \).

When \( \mu \) satisfies these two conditions, there exists a unique \((s_0, t_0)\) \( \in \mathbb{R} \times \mathbb{R}^N \) such that
\[ \int_G a^{s_0} e^{(t_0, b)} \mu(da \, du \, db) = \inf_{(s,t) \in \mathbb{R} \times \mathbb{R}^N} \int_G a^s e^{(t, b)} \mu(da \, du \, db). \]
Furthermore,
\[ \rho(\mu) = \int_G a^{s_0} e^{(t_0, b)} \mu(da \, du \, db) \]
belongs to \([0, 1]\). Note that the probability measure
\[ \mu_0(dg) = \frac{1}{\rho(\mu)} a(g)^{s_0} e^{(t_0, b(g))} \mu(dg) \]
satisfies hypotheses G1 and G2. The following condition is the equivalent of Hypothesis G*3 in the non centered case:

**Hypothesis G*3.** The measure \( \mu \) is absolutely continuous with respect to the Haar measure \( m_D \) on \( G \) and its density \( \phi_\mu \) satisfies
\[ \int_{[0,1] \times \mathbb{R}^N} \sqrt{q} \int_\mathbb{R} \phi_\mu^q(a, u, b) \, du \, da \, db \, a^\gamma < +\infty \]
for some $q \in ]1, +\infty[$ and $\gamma \in ]1 - s_0, +\infty[$.

**Theorem 1.2.** Let $\mu$ be a probability measure on $G$ satisfying conditions $G^*1$, $G^*2$ and $G3$ (or $G^*3$) and let

$$\rho(\mu) = \inf_{(s,t) \in \mathbb{R} \times \mathbb{R}^N} \int_G a^s e^{\langle t, b \rangle} \mu(da \, du \, db).$$

Then, the sequence of finite measures

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)n} \mu^*_{n} \right\}_{n \geq 1}
$$

weakly converges to a non-degenerate Radon measure on $G$.

The demonstration of Theorem 2.1 is closely related to the study of the fluctuations of a random walk $(X_1^n, Y_1^n)_{n \geq 0}$ on $\mathbb{R}^{N+1}$. In Section 2, we first state the classical local limit theorem on $\mathbb{R}^{N+1}$ but we add in its statement uniform upperbounds relatively to the starting point of the random walk $(X_1^n, Y_1^n)_{n \geq 0}$. This result is thus very useful to obtain a precise equivalent in Theorem 2.5 of the joint law of the random walk $(X_1^n, Y_1^n)_{n \geq 0}$ with its first entrance time $T_+$ in the half space $\mathbb{R}^+ \times \mathbb{R}^N$; a local limit theorem for the process

$$(X_1^n, \max \{0, X_1^1, \ldots, X_1^n\}, Y_1^n)_{n \geq 0}$$

is thus obtained (Theorem 2.6). In Section 3 we give the main steps of the proof of Theorem 1.1.

2. **Fluctuations of a random walk on $\mathbb{R}^{N+1}$.**

Fix an integer $N \geq 1$ and let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent $\mathbb{R} \times \mathbb{R}^N$-valued random variables with distribution $p$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_1^n, Y_1^n)_{n \geq 0}$ be the associated random walk on $\mathbb{R} \times \mathbb{R}^N$ starting from $(0,0)$ and defined by $X_0 = 0, Y_0 = 0$ and $X_1^n = X_1 + \cdots + X_n, Y_1^n = Y_1 + \cdots + Y_n$ for $n \geq 1$; the distribution of the couple $(X_1^n, Y_1^n)$ is the $n$th power of convolution $p^n$ of the measure $p$. Denote by $\mathcal{F}_n$ the $\sigma$-algebra generated by $(X_1, Y_1), \ldots, (X_n, Y_n), n \geq 1$.

Let us first recall the
Definition 2.1. Let $p$ be a probability measure on $\mathbb{R}^k$, $k \geq 1$. The measure $p$ is aperiodic on $\mathbb{R}^k$ if there is no closed and proper subgroup $H$ of $\mathbb{R}^k$ and no $\alpha \in \mathbb{R}^k$ such that $p(\alpha + H) = 1$.

Denote by $\hat{p}$ the characteristic function of $p$ defined by $\hat{p}(u,v) = \mathbb{E}[e^{iuX_1 + ivY_1}]$ for any $(u,v) \in \mathbb{R} \times \mathbb{R}^N$. Recall that the probability measure $p$ is aperiodic if and only if $|\hat{p}(u,v)| < 1$ for $(u,v) \neq (0,0)$.

For any $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^N$ let $\{T^{(k)}_{\mathcal{A}}\}_{k \geq 0}$ be the the successive times of visit of the random walk $(X^n_1, Y^n_1)_{n \geq 1}$ to the set $\mathcal{A}$; one has $T^{(0)}_{\mathcal{A}} = 0$, $T^{(1)}_{\mathcal{A}} = \inf \{n \geq 1 : (X^n_1, Y^n_1) \in \mathcal{A}\}$ and $T^{(k+1)}_{\mathcal{A}} = \inf \{n \geq T^{(k)}_{\mathcal{A}} + 1 : (X^n_1, Y^n_1) \in \mathcal{A}\}$. Note that the $T^{(k)}_{\mathcal{A}}$ are stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 1}$. We will associate to $(p, \mathcal{A})$ the transition kernel $P_{\mathcal{A}}$ defined by

$$P_{\mathcal{A}}((x,y), \mathcal{B}) = \int_{\mathbb{R} \times \mathbb{R}^N} 1_{\mathcal{A} \cap \mathcal{B}}(x' + x, y' + y) p(dx'dy'),$$

for any Borel set $\mathcal{B}$ in $\mathbb{R} \times \mathbb{R}^N$; note that for any $k \geq 1$ one has $P^{(k)}_{\mathcal{A}}((0,0), \mathcal{B}) = \mathbb{E}[\mathbb{1}_{T^{(k)}_{\mathcal{A}}} > k; (X^n_1, Y^n_1) \in \mathcal{B}]$. In order to simplify the notations we will set $T_{\mathcal{A}} = T_{\mathbb{R} \times \mathbb{R}^N}$, $P_{\mathcal{A}} = P_{\mathbb{R} \times \mathbb{R}^N}$ and $T^{(k)}_{\mathcal{A}} = T^{(k)}_{\mathbb{R} \times \mathbb{R}^N}$; similar notations will hold, with obvious modifications, when $\mathcal{A} = \mathbb{R}^+ \times \mathbb{R}^N$.

Throughout this paragraph, for any $k \geq 1$, we denote by $\lambda_k$ the Lebesgue measure on $\mathbb{R}^k$. Furthermore, for any $\delta > 0$, $\mathcal{H}_\delta(\mathbb{R}^k)$ is the space of $\mathbb{C}$-valued functions $\varphi$ on $\mathbb{R}^k$ such that

$$\sup_{x \in \mathbb{R}^k} (1 + \|x\|^\delta) |\varphi(x)| < +\infty.$$

2.1. Preliminaries.

The local limit theorem gives the asymptotic behaviour of the sequence $\{p^n(\varphi)\}_{n \geq 1}$ for any continuous function $\varphi$ with compact support on $\mathbb{R}^{N+1}$; we state it here and we precise some uniform upperbound for the sequence $\{p^n(\varphi)\}_{n \geq 1}$ when $\varphi$ belongs to $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$.

Theorem 2.2. Assume that:

1) the common distribution $p$ of the variables $(X_n, Y_n)$, $n \geq 1$, is aperiodic on $\mathbb{R}^{N+1}$,
ii) $\mathbb{E}[|X_1|^2 + |Y_1|^2] < +\infty$ and $\mathbb{E}[X_1] = 0$, $\mathbb{E}[Y_1] = 0$.

Then:

i) for any continuous function $\varphi$ with compact support on $\mathbb{R}^{N+1}$ one has

$$\lim_{n \to +\infty} n^{(N+1)/2} \mathbb{E}[\varphi(X_1^n, Y_1^n)] = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^{N+1}} \varphi(x, y) \lambda_1(dx) \lambda_N(dy),$$

where $|C|$ denotes the determinant of the positive definite quadratic form

$$C(u, v) = \mathbb{E}[(u X_1 + \langle v, Y_1 \rangle)^2].$$

ii) For any function $\varphi$ in $\mathcal{H}_d(\mathbb{R}^{N+1})$ with $\delta > 4$, the sequence

$$\{n^{(N+1)/2} \mathbb{E}[\varphi(x + X_1^n, y + Y_1^n)]\}_{n \geq 1}$$

is bounded uniformly in $(x, y) \in \mathbb{R} \times \mathbb{R}^N$.

**Proof.** The first assumption is the classical local limit theorem. To obtain the second claim, fix a non-negative function $\phi$ whose Fourier transform has a compact support $K(\hat{\phi})$. Recall that

$$\hat{\rho}(u, v) = 1 - \frac{1}{2} C(u, v) (1 + \varepsilon(u, v))$$

with $\lim_{(u, v) \to (0, 0)} \varepsilon(u, v) = 0$; so there exists $\delta > 0$ such that for $|u| + \|v\| < \delta$ one has

$$|\hat{\rho}(u, v)| \leq 1 - \frac{1}{4} C(u, v) \leq e^{-C(u,v)/4}.$$

On the other hand, by the aperiodicity of $p$ there exists $\rho = \rho(p, K(\hat{\phi}))$ such that $|\hat{\rho}(u, v)| \leq \rho$ as soon as $(u, v)$ belongs to $K(\hat{\phi})$ and $|u| + \|v\| \geq \delta$. It follows that

$$\left(2\pi n\right)^{(N+1)/2} \mathbb{E}[\phi(X_1^n, Y_1^n)]$$

$$\leq n^{(N+1)/2} \int_{|u| + \|v\| < \delta} |\phi(u, v)| |\hat{\rho}(u, v)|^n \lambda_1(du) \lambda_N(dv)$$

$$+ n^{(N+1)/2} \int_{|u| + \|v\| \geq \delta} |\phi(u, v)| |\hat{\rho}(u, v)|^n \lambda_1(du) \lambda_N(dv)$$
\[
\leq n^{(N+1)/2} \int_{|u|+|v| < \delta n^{(N+1)/2}} \left| \hat{\phi} \left( \frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}} \right) \right| e^{-n/4 |C(u/\sqrt{n}, v/\sqrt{n})|} \cdot \lambda_1(du) \lambda_N(dv) \\
+ n^{(N+1)/2} \rho^n \| \hat{\phi} \|_1 \\
\leq \| \hat{\phi} \|_{\infty} \int_{\mathbb{R} \times \mathbb{R}^N} e^{-C(u,v)/4} \lambda_1(du) \lambda_N(dv) + n^{(N+1)/2} \rho^n \| \hat{\phi} \|_1.
\]

Now set \( \hat{\phi}_{x,y}(x', y') = \hat{\phi}(x + x', y + y') \) for any \((x, y) \in \mathbb{R} \times \mathbb{R}^N\) and note that \( \hat{\phi}_{x,y}(u, v) = e^{iux + ivy} \hat{\phi}(u, v) \); the functions \( \hat{\phi}_{x,y} \) and \( \hat{\phi} \) thus have the same compact support and satisfies the equalities \( \| \hat{\phi}_{x,y} \|_1 = \| \hat{\phi} \|_1 \) and \( \| \hat{\phi}_{x,y} \|_{\infty} = \| \hat{\phi} \|_{\infty} \). For any \((x, y) \in \mathbb{R} \times \mathbb{R}^N\) one thus has

\[
\left| (2\pi n)^{(N+1)/2} \mathbb{E} [\hat{\phi}_{x,y}(X^n, Y^n)] \right| \\
\leq \| \hat{\phi} \|_{\infty} \int_{\mathbb{R} \times \mathbb{R}^N} e^{-C(u,v)/4} \lambda_1(du) \lambda_N(dv) + n^{(N+1)/2} \rho^n \| \hat{\phi} \|_1.
\]

The assertion \( \text{ii)} \) thus holds for any function \( \phi \) whose Fourier transform has a compact support. To achieve the proof of \( \text{ii)} \) it suffices to show that for any function \( \varphi \) in \( \mathcal{H}_\delta(\mathbb{R}^{N+1}) \) with \( \delta > 4 \) there exists a function \( \phi \) whose Fourier transform has a compact support and \( |\varphi| \leq \phi \). It is an immediate consequence of the following result; we thank here J. P. Conze for helpful discussions about this fact.

**Lemma 2.3.** Set

\[
h_\varepsilon(x) = \frac{1}{1 + |x|^{4+\varepsilon}},
\]

for any \( x \in \mathbb{R} \). If \( \varepsilon > 0 \) there exists a function \( \overline{h}_\varepsilon \) greater than \( h_\varepsilon \) and whose Fourier transform has a compact support in \( \mathbb{R} \).

**Proof.** Set

\[
\overline{h}_\varepsilon(x) = C \left( \frac{\sin^2 x}{x^2} + \frac{\sin^2 \alpha x}{x^2} \right)
\]

for some \( \alpha \) and \( C \) in \( \mathbb{R}^{++} \) which will depend on \( \varepsilon \). Assume \( \alpha \notin \mathbb{Q} \), the function \( \overline{h}_\varepsilon \) is strictly positive on \( \mathbb{R} \); it thus suffices to show that there exists \( \alpha \notin \mathbb{Q} \) such that

\[
\lim_{x \to +\infty} x^{2+\varepsilon}(\sin^2 x + \sin^2(\alpha x)) = +\infty.
\]
If such a real did not exist, then for any \( \alpha \notin \mathbb{Q} \) there should exist a sequence \( \{x_n\}_{n \geq 1} \) which tends to \(+\infty\) and a constant \( C_{\varepsilon} > 0 \) such that for all \( n \geq 1 \),

\[
\sin^2 x_n + \sin^2 (\alpha x_n) \leq \frac{C}{x_n^{1+\varepsilon}}.
\]

So there should exist two strictly increasing sequences of integers \( \{k_n\}_{n \geq 1} \) and \( \{l_n\}_{n \geq 1} \) such that

\[
|x_n - k_n \pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}, \quad |\alpha x_n - l_n \pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}
\]

which implies

\[
|\alpha - \frac{l_n}{k_n}| \leq \frac{C''}{k_n^{1+\varepsilon/2}}
\]

for some positive constants \( C' \) and \( C'' \). This leads to a contradiction because for almost all \( \alpha \in \mathbb{R} \) (with respect with the Lebesgue measure), this last inequality has at most a finite number of solutions in \( \mathbb{N}^2 \) [2]. The lemma is proved.

\[\text{2.2. A local limit theorem for a killed random walk on a half space.}\]

In [7], we proved the following

**Theorem 2.4.** Let the hypotheses of Theorem 2.2 hold. Then for any continuous function with compact support \( \varphi \) on \( \mathbb{R}^- \) we have

\[
\lim_{n \to +\infty} n^{3/2} \mathbb{E} \left[ [T_+ > n]; \varphi(X_D^n) \right] = \frac{1}{\sigma(X_1) \sqrt{2\pi}} \int_{-\infty}^0 \varphi(x) \lambda_1^- * U^-(dx),
\]

where \( \lambda_1^- \) denotes the restriction of the Lebesgue measure on \( \mathbb{R}^- \) and \( U^- \) is the \( \sigma \)-finite measure on \( \mathbb{R}^- \) defined by

\[
U^-(B) = \sum_{k=1}^{+\infty} \mathbb{E} \left[ 1_B(X_1^{T(k)}) \right]
\]

for any Borel set \( B \). In the same way, one has

\[
\lim_{n \to +\infty} n^{3/2} \mathbb{E} \left[ [T_+ > n]; \varphi(X_D^n) \right] = \frac{1}{\sigma(X_1) \sqrt{2\pi}} \int_{-\infty}^0 \varphi(x) \lambda_1^- * U^-(dx),
\]
where $U^-$ is the $\sigma$-finite measure on $\mathbb{R}^-$ defined by

$$U^-(B) = \sum_{k=1}^{+\infty} \mathbb{E}[1_B(X^{T(k)}_1^-)]$$

for any Borel set $B$.

Recall that the random walks $\{X^{T(k)}_1^-\}_{k \geq 1}$ and $\{X^{T(k)}_1\}_{k \geq 1}$ are transient on $\mathbb{R}^-$; it follows that the series $\sum_{k=0}^{+\infty} \mathbb{E}[T_+ > k; \varphi(x + X^k_1)]$ and $\sum_{k=0}^{+\infty} \mathbb{E}[T_{*+} > k; \varphi(x + X^k_1)]$ do converge. Furthermore one has

$$\sum_{k=0}^{+\infty} \mathbb{E}[T_+ > k; \varphi(x + X^k_1)] = \int_{-\infty}^{0} \varphi(x) U^-(dx)$$

and

$$\sum_{k=0}^{+\infty} \mathbb{E}[T_{*+} > k; \varphi(x + X^k_1)] = \int_{-\infty}^{0} \varphi(x) U^-(dx).$$

Let us now state the following

**Theorem 2.5.** Let the hypotheses of Theorem 2.2 hold. Then:

i) For any continuous function $\varphi$ with compact support on $\mathbb{R}^- \times \mathbb{R}^N$ one has

$$\lim_{n \to +\infty} n^{(N+3)/2} \mathbb{E}[T_+ > n; \varphi(X^n_1, Y^n_1)] = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^- \times \mathbb{R}^N} \varphi(x, y) \lambda^{-}_{1} * U^-(dx) \lambda_N(dy).$$

ii) For any continuous function $f$ with compact support on $\mathbb{R}$ and any $g$ in $H_\delta(\mathbb{R}^N)$ with $\delta > 4$, the sequence

$$\{n^{(N+3)/2} \mathbb{E}[T_+ > n; f(X^n_1) g(y + Y^n_1)]\}_{n \geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

In the same way, one has

$$\lim_{n \to +\infty} n^{(N+3)/2} \mathbb{E}[T_{*+} > n; \varphi(X^n_1, Y^n_1)] = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^- \times \mathbb{R}^N} \varphi(x, y) \lambda^{-}_{1} * U^-(dx) \lambda_N(dy).$$
and the sequence

\[ \left\{ n^{(N+3)/2} \mathbb{E}[[T_+ > n]; f(X^n_1) g(y + Y^n_1)] \right\}_{n \geq 1} \]

is bounded, uniformly in \( y \in \mathbb{R}^N \).

**Proof.** We prove this theorem by induction over \( N \). Theorem 2.2 deals with the case \( N = 0 \); we will suppose that this result hold for some \( N \geq 0 \) and we consider a sequence \((X_n, Y_n, Z_n)_{n \geq 1}\) of independent identically distributed random variables on \( \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \). By a classical argument in probability theory, it suffices to show the above convergence hold for \( \varphi(x, y, z) = e^{ax} 1_{\mathbb{R}^+} - (x) \phi(y) \psi(z) \) where \( a \in \mathbb{R}^+ \) and \( \phi, \psi \) are \( \mathbb{C} \)-valued functions whose Fourier transform are continuous with compact supports. By the inverse Fourier transform one has

\[
I_n = \mathbb{E}[[T_+ > n]; e^{aX^n_1} \phi(Y^n_1) \psi(Z^n_1)]
\]

\[
= \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N \times \mathbb{R}} \hat{\phi}(v) \hat{\psi}(w) \alpha_n(a, v, w) \lambda_N(dv) \lambda_1(dw)
\]

with \( \alpha_n(a, v, w) = \mathbb{E}[[T_+ > n]; e^{aX^n_1 + i(v, Y^n_1) + iwZ^n_1}] \).

The Spitzer’s factorisation for random walks on \( \mathbb{R} \) gives for all \( a > 0 \), for all \( s \in [0, 1[ \)

\[
\sum_{n=0}^{+\infty} s^n \mathbb{E}[[T_+ > n]; e^{aX^n_1}] = \exp \left( \sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E}[[X^n_1 < 0]; e^{aX^n_1}] \right).
\]

Using the fact that \( \mathbb{R}^+ \times \mathbb{R}^{N+1} \) and \( \mathbb{R}^+ \times \mathbb{R}^{N+1} \) are semi-groups in \( \mathbb{R}^{N+2} \), Ch. Sumyach extended this factorisation to the multidimensional case ([9, Corollary 3, p. 553 and Theorem 5, p. 556]); for any \( a > 0 \), \( v \in \mathbb{R}^N \), \( w \in \mathbb{R} \) and \( s \in [0, 1[ \) one thus has

\[
\sum_{n=0}^{+\infty} s^n \mathbb{E}[[T_+ > n]; e^{aX^n_1 + i(v, Y^n_1) + iwZ^n_1}] = \exp \left( \sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E}[[X^n_1 < 0]; e^{aX^n_1 + i(v, Y^n_1) + iwZ^n_1}] \right)
\]

that is

\[
(n + 1) \alpha_{n+1}(a, v, w) = \sum_{k=0}^{n} \beta_{n+1-k}(a, v, w) \alpha_k(a, v, w)
\]
with \( \beta_n(a, v, w) = \mathbb{E}[[X^n_1 < 0]; e^{aX^n_1 + i\varphi(Y^n_1) + i\omega Z^n_1}] \). Finally

\[
I_n = \frac{1}{n+1} \sum_{k=0}^n I_{n,k}
\]

with

\[
I_{n,k} = \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N \times \mathbb{R}} \beta_{n+1-k}(a, v, w) \alpha_k(a, v, w) \\
\cdot \hat{\phi}(v) \hat{\psi}(w) \lambda_N(dv) \lambda_1(dw).
\]

Set

\[
I = \frac{1}{(2\pi)^{(N+2)/2} \sqrt{|C|}} \int_{\mathbb{R}^N \times \mathbb{R}} \sum_{k=0}^{+\infty} \mathbb{E}[[T_+ > k]; e^{aX^n_k \beta}] \\
\cdot \phi(y) \psi(z) \lambda_N(dy) \lambda_1(dz),
\]

since

\[
I = \lambda^{-1}_1 * U^* - (e^a) \lambda_N(\phi) \lambda_1(\psi),
\]

it suffices to show that \( \{n^{(N+4)/2} I_n\}_{n \geq 1} \) converges to \( I \), that is

1) for all \( k > 0 \), \( \lim_{n \to +\infty} n^{(N+2)/2} I_{n,k} = I_{*,k} \),

2) \( \sum_{k=0}^{+\infty} |I_{*,k}| < +\infty \) and \( \sum_{k=0}^{+\infty} I_{*,k} = I \),

3) \( \limsup_{n \to +\infty} \limsup_{l \to +\infty} n^{(N+2)/2} \sum_{k=l}^n |I_{n,k}| = 0 \).

To prove the assertion 1), note that

\[
I_{n,k} = \mathbb{E}[[T_+ > k] \cap [X^n_{k+1} > 0]; e^{aX^n_1 \phi(Y^n_1) \psi(Z^n_1)}],
\]

by the local limit theorem on \( \mathbb{R}^{N+2} \) the assertion 1) follows with

\[
I_{*,k} = \frac{1}{2\pi^{(N+2)/2} \sqrt{|C|}} \mathbb{E}[[T_+ > k]; e^{aX^n_k \beta}] \\
\cdot \int_{\mathbb{R}^N} \phi(y) \lambda_N(dy) \int_{\mathbb{R}} \psi(z) \lambda_1(dz).
\]
The fact that the series \( \sum_{k=0}^{+\infty} |I_{*k}| \) converges is a direct consequence of Theorem 2.4. To prove the assertion 3), note that

\[
|I_{n,k}| \leq \mathbb{E} \left[ [T_+ > k] \cap [X_{n+1}^{k+1} < 0]; e^{aX_1^{n+1}} |\phi(Y_1^{n+1})| |\psi(Z_1^{n+1})| \right] \\
\leq \mathbb{E} \left[ [T_+ > k]; e^{aX_1^k} \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{ax} |\phi(y + Y_1^k)| \\
\quad \cdot |\psi(z + Z_1^k)| p^*(n+1-k) (dx \, dy \, dz) \right] \\
\leq \frac{C(a, \phi, \psi)}{(n+1-k)(N+2)/2} \mathbb{E} \left[ [T_+ > k]; e^{aX_1^k} \right] \quad \text{by Theorem 2.2.ii)} \\
\leq \frac{C_1}{(n+1-k)(N+2)/2 \, k^{3/2}} \quad \text{by Theorem 2.4.}
\]

On the other hand

\[
|I_{n,k}| \leq \|\psi\|_{\infty} \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathbb{E} \left[ [T_+ > k]; e^{aX_1^k} |\psi(y + Y_1^k)| \\
\quad \cdot e^{ax} p^*(n+1-k) (dx \, dy \, dz) \right] \\
\leq \|\psi\|_{\infty} \frac{C(a, \phi)}{k(N+3)/2} \\
\quad \cdot \mathbb{E} \left[ [X_{n+1}^{k+1} < 0]; e^{aX_{k+1}^{n+1}} \right] \quad \text{by hypothesis of induction} \\
\leq \frac{C_2}{k(N+3)/2 \sqrt{n+1-k}}.
\]

The assertion 3) follows since for any \( \varepsilon > 0 \) one has

\[
C_1 \sum_{k=l}^{n} |I_{n,k}| \leq C_1 \sum_{k=l}^{n(1-\varepsilon)/2} \frac{n(N+2)/2}{k^{3/2} (n+1-k)(N+2)/2} \\
\quad + C_2 \sum_{k=l}^{n(1-\varepsilon)/2} \frac{n(N+2)/2}{k(N+3)/2 \sqrt{n+1-k}} \\
\leq \frac{C_1}{\varepsilon(N+2)/2} \sum_{k=l}^{n(1-\varepsilon)/2} \frac{1}{k^{3/2}} \\
\quad + \frac{C_2}{\sqrt{n(1-\varepsilon)(N+3)/2} \sqrt{n+1-k}} \sum_{k=l}^{n(1-\varepsilon)/2} \frac{1}{k^{3/2}}
\]
\[ \leq C \left( \frac{1}{\varepsilon^{(N+2)/2}} + \frac{\sqrt{\varepsilon}}{(1 - \varepsilon)^{(N+3)/2}} \right). \]

Since \( \varepsilon \) is arbitrarily small, the assertion 3) follows.

The proof of ii) is also made by induction over \( N \). If \( g \in \mathcal{H}_\delta(\mathbb{R}^{N+1}) \) there exist \( \phi \in \mathcal{H}_\delta(\mathbb{R}^N) \) and \( \psi \in \mathcal{H}_\delta(\mathbb{R}^1) \) such that \( |g| \leq \phi \otimes \psi \). We set
\[ I_n(y, z) = \mathbb{E}[[T_e > n]; e^{aX^*_n} \phi(y + Y^n_1) \psi(z + Z^n_1)] \]
and we have
\[ I_n(y, z) = \frac{1}{n + 1} \sum_{k=0}^{n} I_{n,k}(y, z) \]
with
\[ I_{n,k}(y, z) = \mathbb{E}[[T_e > k] \cap [X^{n+1}_{k+1} < 0]; e^{aX^*_{n+1}} \phi(y + Y^{n+1}_1) \psi(z + Z^{n+1}_1)]. \]
As above, one has
\[ |I_{n,k}(y, z)| \leq \inf \left\{ \frac{C_1}{(n + 1 - k)(N+2)/2} \frac{C_2}{k(N+3)/2 \sqrt{n + 1 - k}} \right\} \]
which proves that the sequence
\[ \left\{ n^{(N+2)/2} \sum_{k=0}^{n} |I_{n,k}(y, z)| \right\}_{n \geq 1} \]
is uniformly bounded in \( y, z \). This achieves the proof of ii).

The convergence of the sequence
\[ \left\{ n^{(N+3)/2} \mathbb{E}[[T_{*e} > n]; \varphi(X^n_1, Y^n_1)] \right\}_{n \geq 1} \]
is obtained with similar arguments.

### 2.3. Behaviour of the process \(((X^n_1, \max \{0, X^n_1, \ldots, X^n_1\}, Y^n_1))_{n \geq 0}\)

For any \( n \geq 0 \) set \( X^n_1 = \max \{0, X^n_1, \ldots, X^n_1\} \) and let \( T_n \) be the random variable defined on \((\Omega, \mathcal{F}, \mathbb{P})\) by \( T_n = \inf \{0 \leq k \leq n : X^n_1 = \}

\(X_1^n\); for any continuous function \(\varphi\) with compact support on \(\mathbb{R}^{N+1}\) we have

\[
\mathbb{E}[\varphi(X_1^n, X_1^n - X_1^n, Y_1^n)]
\]

\[
= \sum_{k=0}^{n} \mathbb{E}[\{T_n = k\}; \varphi(X_1^k, X_1^k - X_1^n, Y_1^n)]
\]

\[
= \sum_{k=0}^{n} \mathbb{E}[\{0 < X_1^k, X_1^k < X_1^{k+1} < X_1^n \}; \varphi(X_1^k, X_1^n - X_1^{k+1}, Y_1^n)]
\]

One obtains the following factorisation

\[
\mathbb{E}[\varphi(X_1^n, X_1^n - X_1^n, Y_1^n)] = \sum_{k=0}^{n} J_{n,k}(\varphi)
\]

with

\[
J_{n,k}(\varphi)
\]

\[
= \int_{\mathbb{R}^{N+1}} \varphi(x, -x', y + y') P_k^k((0, 0), dx dy) P_{n-k}((0, 0), dx' dy').
\]

The behaviour of the process \((X_1^n, X_1^n - X_1^n, Y_1^n)\) is thus closely related to the one of the iterates of the transition kernels \(P_+\) and \(P_+\). Using this factorisation one proves the

**Theorem 2.6.** Suppose that the hypotheses of Theorem 2.2 hold.

Then, for any continuous function with compact support on \(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N\) the sequence

\[
\{n^{(N+3)/2} \mathbb{E}[\varphi(X_1^n, X_1^n - X_1^n, Y_1^n)]\}_{n \geq 1}
\]
converges to
\[
\frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N} \varphi(s, -t, y) U(s) \lambda_1^- * U(t) \lambda_N(dy) + \frac{1}{(2\pi)^{(N+1)/2}} \cdot \int_{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N} \varphi(s, -t, y) \lambda_1^- * U(s) U(t) \lambda_N(dy).
\]

Furthermore, for any continuous function \( f \) with compact support on \( \mathbb{R}^+ \times \mathbb{R}^+ \) and any \( g \) in \( \mathcal{H}_d(\mathbb{R}^N) \), the sequence
\[
\{n^{(N+3)/2} \mathbb{E}[f(\lambda_1^n, \lambda_N^n - X_1^n) g(y + Y_1^n)]\}_{n \geq 1}
\]
is bounded, uniformly in \( y \in \mathbb{R}^N \).

**Proof.** We only prove the first assertion; the second one may obtained with obvious modifications as in Theorem 2.5. Set \( \varphi(x, t, y) = \varphi_1(x) \varphi_2(t) \varphi_3(y) \) where \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) are continuous with compact support. Fix \( k \geq 0 \); by Theorem 2.5, the sequence
\[
\left\{n^{(N+3)/2} \int_{\mathbb{R}^+ \times \mathbb{R}^N} \varphi_2(x') \varphi_3(y + y') P^{n-k}((0, 0), dx' dy')\right\}_{n \geq 1}
\]
is bounded uniformly in \( y \in \mathbb{R}^N \) and converges to
\[
\frac{1}{(2\pi)^{(N+1)/2}} \int_{-\infty}^{0} \varphi_2(-t) \lambda_1^- * U(t) \lambda_N(\varphi_3).
\]
By the dominated convergence theorem, one thus obtains, for any fixed \( i \geq 1 \)
\[
\lim_{n \to +\infty} n^{(N+3)/2} \sum_{k=0}^{i} J_{n,k}(\varphi) = \frac{1}{(2\pi)^{(N+1)/2}} \sum_{k=0}^{i} \mathbb{E}[\tau^k; \varphi_1(X_1^k)] \cdot \int_{-\infty}^{0} \varphi_2(-t) \lambda_1^- * U(t) \lambda_N(\varphi_3).
\]
In the same way one has

\[
\lim_{n \to +\infty} n^{(N+3)/2} \sum_{k=n-i+1}^{n} J_{n,k}(\varphi) = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \sum_{k=0}^{i} \mathbb{E}[[T_{s}^{+} > k]; \varphi_{2}(-X_{1}^{k})] \\
\cdot \lambda_{1}^{+} \ast U_{s}^{+}(\varphi_{1}) \lambda_{N}(\varphi_{3}).
\]

Note that the sums \( \sum_{k=0}^{i} \mathbb{E}[[T_{s}^{+} > k]; \varphi_{1}(X_{1}^{k})] \) and \( \sum_{k=0}^{i} \mathbb{E}[[T_{s}^{+} > k]; \varphi_{2}(X_{1}^{k})] \) converges respectively to \( U_{s}^{+}(\varphi_{1}) \) and \( \int_{-\infty}^{0} \varphi_{2}(-t) U_{-}(dt) \).

To obtain the theorem it suffices to check that

\[
\limsup_{i \to +\infty} \limsup_{n \to +\infty} \left| n^{(N+3)/2} \sum_{k=i+1}^{n-i} J_{n,k}(\varphi) \right| = 0,
\]

one has

\[
\left| n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} J_{n,k}(\varphi) \right|
\leq n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} \mathbb{E}[[T_{s}^{+} > k]; |\varphi_{1}(X_{1}^{k})|] \\
\cdot \int_{\mathbb{R} \times \mathbb{R}^{\infty}} |\varphi_{2}(x')| |\varphi_{3}(y + y')| F_{s}^{n-k}((0, 0), dx' dy')
\leq C(\varphi_{2}, \varphi_{3}) \sum_{k=i+1}^{[n/2]} \mathbb{E}[[T_{s}^{+} > k]; |\varphi_{1}(X_{1}^{k})|] \\
\cdot \frac{n^{(N+3)/2}}{(n - k)(N+3)/2} \quad \text{by Theorem 2.5.ii)}
\leq C(\varphi) \sum_{k=i+1}^{+\infty} \frac{1}{k^{3/2}}.
\]

The same upperbound holds for the term

\[
n^{(N+3)/2} \sum_{k=[n/2]+1}^{n-i} J_{n,k}(\varphi).
\]
This achieves the proof.

3. A local limit theorem for a particular class of solvable groups.

Recall that \( G = G_{d,N} = \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^N \) with the composition law

\[
g \cdot g' = (a a', a u' + u, b + b'),
\]

for all \( g = (a, u, b) \), for all \( g' = (a', u', b') \in G_{d,N} \).

The proof of Theorem 1.1 is closed to the one of the local limit theorem for the affine group of the real line given in [7]; we just give here the main steps of the demonstration.

Let us first introduce some helpful notations. Let \( g_n = (a_n, u_n, b_n) \), \( n = 1, 2, \ldots \) be independent and identically distributed random variables with distribution \( \mu \). Denote by \( F_n \) the \( \sigma \)-algebra generated by the variables \( g_1, g_2, \ldots, g_n, n \geq 1 \). For any \( n \geq 1 \), set \( G^n_1 = g_1 \cdots g_n = (A^n_1, U^n_1, B^n_1) \); we have \( A^n_1 = a_1 \cdots a_n \), \( U^n_1 = \sum_{k=1}^n a_1 \cdots a_{k-1} u_k \) and \( B^n_1 = b_1 + \cdots + b_n \). More generally, if \( 1 \leq m \leq n \), set \( A^n_m = a_m \cdots a_n \), \( U^n_m = \sum_{k=m}^n a_m \cdots a_{k-1} u_k \), \( B^n_m = b_m + \cdots + b_n \) and set \( A^n_m = 1 \), \( U^n_m = 0 \), \( B^n_m = 0 \) otherwise.

Let \( \tilde{\mu} \) be the image of \( \mu \) by the map

\[
g = (a, u, b) \mapsto \tilde{g} = \left( \frac{1}{a}, \frac{u}{a}, b \right),
\]

if \( \tilde{g}_n = (\tilde{a}_n, \tilde{u}_n, \tilde{b}_n) \), \( n = 1, 2, \ldots \) are independent and identically distributed random variables with distribution \( \tilde{\mu} \) on \( G \), set \( \tilde{G}_m = \tilde{g}_m \cdots \tilde{g}_n = (A^n_m, U^n_m, B^n_m) \).

In order to obtain the asymptotic behaviour of the power of convolution \( \mu^{*n} \) we use the fact that the sequence \( \{U^n_1\}_{n \geq 1} \) behaves like the maximum of the variables \( A^n_1, \ldots, A^n_n \). These ideas were already used in [7]. Set \( \mathcal{A} = \{ g = (a, u, b) \in G : a > 1 \} \) and consider the transition kernel \( P_A \) associated with \( (\mu, \mathcal{A}) \) and defined by

\[
P_A(g, \mathcal{B}) = \int_G 1_{A \cap nB}(gh) \mu(dh)
\]

for any Borel set \( \mathcal{B} \subset G \) and any \( g \in G \). The probabilistic interpretation of \( P_A \) is the following one: if \( T_A = \inf \{ n \geq 1 : G^n_1 \in \mathcal{A} \} \) is the first entrance time in \( \mathcal{A} \) of the random walk \( \{G^n_1\}_{n \geq 0} \) then

\[
P^n_A(e, \mathcal{B}) = \mathbb{P} \left[ [T_A > n] \cap \left[ G^n_1 \in \mathcal{B} \right] \right], \quad \text{for all } n \geq 1.
\]
In the same way, set $A' = \{ g \in G : a(g) \geq 1 \}$, let $\tilde{P}_{A'}$ be the operator associated with $(\tilde{\mu}, A')$ and denote by $\tilde{T}_{A'}$ the first entrance time in $A'$ of the random walk $\{ \tilde{C}^n_{1} \}_{n \geq 1}$; one has

$$\tilde{P}_{A'}(e, B) = \mathbb{P}[\tilde{T}_{A'} > n] \cap [\tilde{C}^n_{1} \in B], \quad \text{for all } n \geq 1.$$

As in Section 2.3, we introduce the first time at which the random walk $\{ A^n_1 \}_{n \geq 1}$ reaches its maximum on $\mathbb{R}^+$; for any continuous function $\varphi$ with compact support on $G$, we thus obtain

$$\mathbb{E}[\varphi(G^n_1)] = \sum_{k=0}^{n} I_{n,k}(\varphi),$$

where

$$I_{n,k}(\varphi) = \int_{G \times G} \varphi\left( \frac{a'}{a}, \frac{u+u'}{a}, b+b' \right) \tilde{P}_{A'}^{k}(e, du \, db) P_{A}^{n-k}(e, da \, du \, db).$$

We now give the main steps of the proof of Theorem 1.1 under hypothesis G1, G2 and G3.

**First step. Control of the central terms of the sum** $\sum_{k=0}^{n} I_{n,k}(\varphi)$.

We show here that

$$\limsup_{i \to +\infty} \limsup_{n \to +\infty} \sum_{k=i}^{n-i} I_{n,k}(\varphi) = 0.$$

Without loss of generality, one may suppose that the support of $\varphi$ is included in $\mathbb{R}^+ \times (\mathbb{R}^+)^d \times \mathbb{R}^N$; for any $\epsilon > 0$ there exist a constant $C > 0$ and a positive function $\phi$ with compact support on $\mathbb{R}^N$ such that

$$\varphi(a, u, b) \leq C \frac{a_\epsilon}{\|u\|^{2\epsilon}} \phi(b),$$

it follows that for any $(\alpha, \beta)$ in $\mathbb{R}^+ \times \mathbb{R}^N$

$$\mathbb{E}
\left[
\left|
\tilde{T}_A > \ell
\right| \cap \varphi\left(\frac{A_l^1}{\alpha}, \frac{u+U_l^1}{\alpha}, \beta + B_l^1\right)
\right]
\leq C \alpha^\epsilon \mathbb{E}
\left[
\left|\left\{a_1 \leq 1\right\} \cap \max \{ A_2^2, \ldots, A_1^1 \} \leq \frac{1}{\alpha_1} \right\}
\right| \cdot
\frac{(A_l^1)^\epsilon}{\|u+U_l^1\|^{2\epsilon}} \phi(\beta + B_l^1)$$

$$\leq C \alpha^\epsilon \int_{G} \mathbb{E}
\left[
\max \{ A_2^2, \ldots, A_1^1 \} \phi(\beta + B_2^1)
\right] \frac{\mu(du \, dv \, db)}{a^\epsilon \|v\|^{2\epsilon}},$$
the last inequality being a consequence of the fact that \( \|u + U^i_1\| \geq \|u_1\| \)
\( \mathbb{P} \)-almost surely and
\[
1_{\{\max\{A^2_1, \ldots, A^2_i\} \leq 1/a_1\}} \leq \frac{1}{a_1^{2\varepsilon} \max\{A^2_2, \ldots, A^2_i\}^{2\varepsilon}}.
\]
By Theorem 2.6 one obtains
\[
l^{(N+3)/2} E \left[ T_A > l^i; \varphi \left( \frac{A^1_1}{\alpha}, \frac{u + U^i_1}{\alpha}, \beta + B^i_1 \right) \right] \leq C_1(\varphi) \alpha^\varepsilon.
\]
The same upperbound holds under hypotheses G1, G2 and G'3 (see [7, Lemma 3.1]).

It readily follows that
\[
n^{(N+3)/2} \sum_{k=i}^{[n/2]} I_{n,k}(\varphi) \leq 2^{(N+3)/2} \sum_{k=i}^{[n/2]} (n - k)^{(N+3)/2} I_{n,k}(\varphi)
\]
\[
\leq C_1(\varphi) \sum_{k=i}^{[n/2]} E \left[ [\tilde{T}_A > \tilde{k}] ; (\tilde{A}^k_1)_{\varepsilon} \right]
\]
\[
\leq C_2(\varphi) \sum_{k=i}^{[n/2]} \frac{1}{k^{3/2}}
\]
and so
\[
\limsup_{i \to +\infty} \limsup_{n \to +\infty} n^{(N+3)/2} \sum_{k=i}^{[n/2]} I_{n,k}(\varphi) = 0.
\]
The control of the sum \( \sum_{k=[n/2]}^{n-i} I_{n,k}(\varphi) \) goes along the same lines.

Second step. Convergence of the sequence
\[
l^{(N+3)/2} E \left[ [T_A > l^i]; \varphi \left( \frac{A^1_1}{\alpha}, \frac{u + U^i_1}{\alpha}, \beta + B^i_1 \right) \right]
\]
for any \( (\alpha, u, \beta) \in [0,1] \times (\mathbb{R}^*^+)^d \times \mathbb{R}^N \).

It is the more technical part of the proof and it uses and idea due
to Afanasev [1]. Without loss of generality, one may suppose \( \alpha = 1 \),
\( u = 0 \) and \( \beta = 0 \). For any \( n \geq 1 \), set
\[
E_n(\varphi) = n^{(N+3)/2} E \left[ [T_A > n]; \varphi(A^n_1, U^n_1, B^n_1) \right].
\]
Fix $i \in \mathbb{N}$ such that $1 \leq i \leq n/2$ and consider

$$E_n(\varphi, i) = n^{(N+3)/2} \mathbb{E}[[T_A > n]; \varphi(A_{1}^{n}, U_{1}^{i} + A_{1}^{n-i}U_{n-i+1}^{n}, B_{1}^{n})].$$

To obtain the claim, it suffices to prove that

a) $\limsup_{i \to +\infty} \limsup_{n \to +\infty} |E_n(\varphi) - E_n(\varphi, i)| = 0,$

b) for any fixed $n \in \mathbb{N}$, the sequence $\{E_n(\varphi, i)\}_{n \geq 1}$ converges to a finite limit.

**Proof of convergence a).** We use the equality

$$U_{1}^{n} = U_{1}^{i} + A_{1}^{i}U_{i+1}^{n-i} + A_{1}^{n-i}U_{n-i+1}^{n},$$

without loss of generality one may suppose that $\varphi$ is continuously differentiable, and so, for any $\varepsilon > 0$ there exists $C > 0$ and a positive function $\phi$ with compact support on $\mathbb{R}^{N}$ such that

$$|\varphi(a, u, b) - \varphi(a, v, b)| \leq C a^{\varepsilon} \|u - v\|^{\varepsilon} \phi(b),$$

consequently

$$|E_n(\varphi) - E_n(\varphi, i)|$$

$$\leq C n^{(N+3)/2} \mathbb{E}[[T_A > n]; (A_{1}^{n})^{\varepsilon} (A_{1}^{i})^{\varepsilon} \|U_{i+1}^{n-i}\|^{\varepsilon} \phi(B_{1}^{n})]$$

$$\leq C n^{(N+3)/2} \sum_{k=i+1}^{n-i} \mathbb{E}[[T_A > n]; (A_{1}^{n})^{\varepsilon} (A_{1}^{k-1})^{\varepsilon} \|u_{k}\|^{\varepsilon} \phi(B_{1}^{n})].$$

Note that for $i \leq k \leq [n/2]$ one has

$$\mathbb{E}[[T_A > n]; (A_{1}^{n})^{\varepsilon} (A_{1}^{k-1})^{\varepsilon} \|u_{k}\|^{\varepsilon} \phi(B_{1}^{n})]$$

$$\leq \mathbb{E}[[T_A > k - 1] \cap \left\{ \max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^{n}\} \leq \frac{1}{A_{k+1}^{1/\varepsilon}} \right\};$$

$$(A_{1}^{n})^{\varepsilon} (A_{1}^{k-1})^{\varepsilon} \|u_{k}\|^{\varepsilon} \phi(B_{1}^{n})]$$

$$\leq \mathbb{E}[[T_A > k - 1]; (A_{1}^{k-1})^{\varepsilon/2} a_{k}^{-\varepsilon/2} \|u_{k}\|^{\varepsilon}$$

$$\cdot \max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^{n}\}^{-3\varepsilon/2} (A_{k+1}^{n})^{\varepsilon} \phi(B_{1}^{n})].$$

By Theorem 2.6,

$$(n - k)^{(N+3)/2} \mathbb{E}[[T_A > n]; \max\{A_{k+1}^{k+1}, \ldots, A_{k+1}^{n}\}^{-3\varepsilon/2} (A_{k+1}^{n})^{\varepsilon} \phi(\beta + B_{k+1}^{n})]$$

$$\leq 0.$$

Therefore, we have

$$|E_n(\varphi) - E_n(\varphi, i)| \leq C n^{(N+3)/2} \sum_{k=i+1}^{n-i} \mathbb{E}[[T_A > n]; (A_{1}^{n})^{\varepsilon} (A_{1}^{k-1})^{\varepsilon} \|u_{k}\|^{\varepsilon} \phi(B_{1}^{n})].$$

Finally, $\limsup_{n \to +\infty} \mathbb{E}[[T_A > n]; (A_{1}^{n})^{\varepsilon} \|u_{k}\|^{\varepsilon} \phi(B_{1}^{n})] = 0$ and the proof is complete.
is bounded, uniformly in \( \beta \in \mathbb{R}^N \) and so

\[
(n - k)^{(N + 3)/2} E \left[ [T_A > n]; (A_1^n)^{\varepsilon} (A_1^{k-1})^{\varepsilon} \| u_k \|^\varepsilon \phi(B_1^n) \right] \leq \frac{C_1}{k^{3/2}}.
\]

When \([n/2] \leq k \leq n - i\) one obtains by a similar argument

\[
k^{(N + 3)/2} E \left[ [T_A > n]; (A_1^n)^{\varepsilon} (A_1^{k-1})^{\varepsilon} \| u_k \|^\varepsilon \phi(B_1^n) \right] \leq \frac{C_2}{(n - k)^{3/2}}.
\]

Finally one has

\[
\| \mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i) \| \leq C_3 \frac{1}{\sqrt{i}},
\]

convergence a) follows.

**Proof of convergence b).** Fix an integer \(i\); we have

\[
\mathbb{E}_n(\varphi, i)
\]

\[
= \int_G E_n(\varphi, g, h_1, h_2, \ldots, h_i) P_\lambda^g(e, dg) \mu(dh_1) \mu(dh_2) \cdots \mu(dh_i)
\]

with

\[
E_n(\varphi, g, h_1, h_2, \ldots, h_i)
\]

\[
= \mathbb{E} \left[ \max \left\{ A_{i+1}^{i+1}, \ldots, A_{i+1}^{n-i} \right\} \leq \frac{1}{a(g)} \right]
\]

\[
\cap \left[ \frac{1}{a(g)} \leq \min \left\{ \frac{1}{a(g) a(h_1)} \cdots \frac{1}{a(g) a(h_1) \cdots a(h_i)} \right\} \right]
\]

\[
\times \phi(a(g) A_{i+1}^{n-i} a(h_1) \cdots a(h_i), u(g) + a(g) A_{i+1}^{n-i} u(h_1 \cdots h_i),
\]

\[
B_1^{n-i} + b(h_1) + \cdots + b(h_i).
\]

Using Theorem 2.6, one may see that, for any \(g, h_1, \ldots, h_i \in G\), the sequence

\[
\{ n^{(N + 3)/2} E_n(\varphi, g, h_1, h_2, \ldots, h_i) \}_{n \geq 1}
\]

converges to a finite limit. To obtain the convergence b), we have to use Lebesgue dominated convergence theorem and therefore, we have to obtain an appropriate upperbound for \( n^{(N + 3)/2} E_n(\varphi, g, h_1, h_2, \ldots, h_i) \).
Using the fact that for any $\varepsilon > 0$ there exist $C > 0$ and a positive continuous function $\phi$ with compact support on $\mathbb{R}^N$ such that $|\varphi(a, u, b)| \leq C a^{\varepsilon} \phi(b)$, one thus obtains
\[
n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \ldots, h_i) \leq C_1 a(g)^{-3\varepsilon/2} a(h_1)^{\varepsilon} \cdots a(h_i)^{\varepsilon}
\]
which allows us to use the Lebesgue dominated convergence theorem for $\varepsilon$ small enough; convergence b) follows.

Consequently, $\{n^{(N+3)/2} I_{n,0}(\varphi)\}_{n \geq 1}$ converges to a finite limit; furthermore, for any $i \geq 1$ and any compact set $K \subset \mathbb{R}^* \times \mathbb{R}^N$, the dominated convergence theorem ensures the existence of a finite limit as $n$ goes to $+\infty$ for
\[
\left\{n^{(N+3)/2} \sum_{k=0}^{i} I_{n,k}(\varphi, K)\right\}_{n \geq 1},
\]
where
\[
I_{n,k}(\varphi, K) = \int_G 1_K(g) \cdot \left( \int_G \varphi\left(\frac{a(h)}{a(g)}, \frac{u(g) + u(h)}{a(g)}, b(g) + b(h)\right) P_{A_n}^{n-k}(e, dh) \right)
\cdot \tilde{P}_A^k(e, dg).
\]
The following step shows that the indicator function $1_K$ does not disturb too much the behaviour of these integrals.

**Third step. Control of the residual terms.**

In the first step of the present proof, we have shown that, for any $\varepsilon > 0$ there exists $C_1 > 0$ such that
\[
(n - k)^{(N+3)/2} \mathbb{E}\left[|T_A > n - k|; \varphi\left(\frac{A_{1,n-k}}{\alpha}, \frac{u + U_{1,n-k}}{\alpha}, \beta + B_{1,n-k}\right)\right] \leq C_1(\varphi) \alpha^{\varepsilon}.
\]
It follows that for any $0 < \delta < 1$
\[
\sum_{k=1}^{i} \int_{(g \in G: a(g) \leq \delta)} \left( \int_G \varphi\left(\frac{a(h)}{a(g)}, \frac{u(g) + u(h)}{a(g)}, b(g) + b(h)\right) P_{A_n}^{n-k}(e, dh) \right)
\]
\[
\cdot \tilde{P}_{A^k}^k(e, dg) \\
\leq C_1 \sum_{k=1}^i \frac{1}{(n-k)(N+3)/2} \mathbb{E} \left[ \left( \tilde{T}_{A^k} > k \right); (\tilde{A}_{\tilde{T}_{A^k}})^e \right] \\
\leq C_1 \sum_{k=1}^i \frac{1}{(n-k)(N+3)/2} k^{3/2}.
\]

On the other hand for any fixed \( U > 0 \), one has
\[
\sum_{k=1}^i \int_{\{g \in G : \|u(g)\| \geq U\}} \left( \int_G \varphi \left( \frac{a(h)}{a(g)}, \frac{u(g) + u(h)}{a(g)}, b(h) + b(g) \right) \mathcal{P}_{A^k-\lambda}^n(e, dh) \right) \\
\cdot \tilde{P}_{A^k}^k(e, dg) \\
\leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^i \frac{1}{(n-k)(N+3)/2} \mathbb{E} \left[ \left( \tilde{T}_{A^k} > k \right); (\tilde{A}_{\tilde{T}_{A^k}})^e \|\tilde{U}_k\|^{e/2} \right] \\
\leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^i \frac{1}{(n-k)(N+3)/2} k^{3/2}.
\]

The last inequality being guaranteed by standart estimations. (see [7, Lemma 3.3] for more details).

References.


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Emile Le Page  
Institut Mathématique de Rennes  
Université de Bretagne Sud  
1 Rue de la Loi  
Vannes 56000, FRANCE  
emile.lepage@univ-ubs.fr

and

Marc Peigné  
Institut Mathématique de Rennes  
Université de Rennes I  
Campus de Beaulieu  
35042 Rennes Cedex, FRANCE  
peigne@univ-rennes1.fr