An extension of the Krein-Šmulian Theorem

Antonio S. Granero

Abstract

Let $X$ be a Banach space, $u \in X^{**}$ and $K, Z$ two subsets of $X^{**}$. Denote by $d(u, Z)$ and $d(K, Z)$ the distances to $Z$ from the point $u$ and from the subset $K$ respectively. The Krein-Šmulian Theorem asserts that the closed convex hull of a weakly compact subset of a Banach space is weakly compact; in other words, every $w^*$-compact subset $K \subset X^{**}$ such that $d(K, X) = 0$ satisfies $d(\overline{co}^{w^*}(K), X) = 0$.

We extend this result in the following way: if $Z \subset X$ is a closed subspace of $X$ and $K \subset X^{**}$ is a $w^*$-compact subset of $X^{**}$, then

$$d(\overline{co}^{w^*}(K), Z) \leq 5d(K, Z).$$

Moreover, if $Z \cap K$ is $w^*$-dense in $K$, then $d(\overline{co}^{w^*}(K), Z) \leq 2d(K, Z)$. However, the equality $d(K, X) = d(\overline{co}^{w^*}(K), X)$ holds in many cases, for instance, if $\ell_1 \not\subseteq X^*$, if $X$ has $w^*$-angelic dual unit ball (for example, if $X$ is WCG or WLD), if $X = \ell_1(I)$, if $K$ is fragmented by the norm of $X^{**}$, etc. We also construct under CH a $w^*$-compact subset $K \subset B(X^{**})$ such that $K \cap X$ is $w^*$-dense in $K$, $d(K, X) = \frac{1}{2}$ and $d(\overline{co}^{w^*}(K), X) = 1$.

1. Introduction

If $X$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of $X$, respectively, and $X^*$ its topological dual. If $u \in X^{**}$ and $K, Z$ are two subsets of $X^{**}$, let $d(u, Z) = \inf\{\|u - z\| : z \in Z\}$ be the distance to $Z$ from $u$, $d(K, Z) = \sup\{d(k, Z) : k \in K\}$ the distance to $Z$ from $K$, $co(K)$ the convex hull of $K$, $\overline{co}(K)$ the norm-closure of $co(K)$ and $\overline{co}^{w^*}(K)$ the $w^*$-closure of $co(K)$.

2000 Mathematics Subject Classification: 46B20, 46B26.

Keywords: Krein-Šmulian Theorem, Banach spaces, compact sets.
This paper is devoted to investigate the connection between the distances $d(\overline{w^o}(K), Z)$ and $d(K, Z)$, when $Z \subset X^{**}$ is a subspace of $X$ (in particular, when $Z = X$) and $K$ is a $w^*$-compact subset of $X^{**}$. There exist some facts that suggest that the distance $d(\overline{w^o}(K), Z)$ is controlled by the distance $d(K, Z)$. Indeed, on the one hand, we have the classical Theorem of Krein-Šmulian (see [5, p. 51]). Using the terminology of distances, this Theorem asserts the following: if $X$ is a Banach space, every $w^*$-compact subset $K \subset X^{**}$ with $d(K, X) = 0$ (that is, $K \subset X$ is a weakly compact subset of $X$) satisfies $d(\overline{w^o}(K), X) = 0$ (that is, the closed convex hull $\overline{w^o}(K)$ of $K$ in $X$ is weakly compact).

On the other hand, if the dual $X^*$ of the Banach space $X$ does not contain a copy of $\ell_1$, it is very easy to prove that $d(K, Z) = d(\overline{w^o}(K), Z)$ for every $w^*$-compact subset $K \subset X^{**}$ of $X^{**}$ and every subspace $Z \subset X^{**}$. Indeed, in this case $\overline{w^o}(K) = \overline{w^o}(K)$ (see [9]). So, as $d(\overline{w^o}(K), Z) = d(K, Z)$ (this follows from the fact that the function $\varphi(u) := d(u, Z)$, $\forall u \in X^{**}$, is convex when $Z \subset X^{**}$ is a convex subset of $X^{**}$), we easily obtain that $d(K, Z) = d(\overline{w^o}(K), Z)$.

In view of these facts, one is inclined to conjecture that $d(K, X) = d(\overline{w^o}(K), X)$ for every $w^*$-compact subset $K \subset X^{**}$ and every Banach space $X$. Unfortunately, assuming the Continuum Hypothesis (for short, $CH$), this is not true because of the following result we will prove here.

**Theorem 1** Under $CH$, if $X = \ell_\infty(\omega^+)$ (= subspace of the elements $f \in \ell_\infty(\omega^+)$ with countable support), there exists a $w^*$-compact subset $H \subset B(X^{**})$ such that $d(H, X) = 1/2$, $H \cap X$ is $w^*$-dense in $H$ and $d(\overline{w^o}(H), X) = 1$.

However, there exist many Banach spaces $X$ for which the equality $d(K, X) = d(\overline{w^o}(K), X)$ holds, for every $w^*$-compact subset $K \subset X^{**}$, for example, the class of Banach spaces with property $J$.

**Definition 2** A Banach space $X$ has property $J$ (for short, $X \in J$) if for every $z \in B(X^{**}) \setminus X$ and for every number $b \in \mathbb{R}$ with $0 < b < d(z, X)$, there exists a sequence $\{x_n\}_{n \geq 1} \subset \mathcal{S}(B(X^*), z, b) := \{u \in B(X^*) : z(u) \geq b\}$ such that $x_n \overset{w^*}{\to} 0$.

For this class of Banach spaces with property $J$ we prove the following result.

**Theorem 3** Let $X$ be a Banach space such that $X \in J$. Then for every $w^*$-compact subset $K \subset X^{**}$ we have $d(K, X) = d(\overline{w^o}(K), X)$.

In the following corollary we state that many Banach spaces have property $J$ and, so, satisfy Theorem 3. Recall that, for a Banach $X$, the dual unit ball $(B(X^*), w^*)$ is angelic in the $w^*$-topology if, for every subset $A \subset B(X^*)$ and every $z \in A^w$, there exists a sequence $\{a_n\}_{n \geq 1} \subset A$ such that $a_n \overset{w^*}{\to} z$. 


Corollary 4 If $X$ is a Banach space such that $(B(X^*), w^*)$ is angelic (for example, if $X$ is weakly compactly generated (for short, WCG) or weakly Lindelöf determined (for short, WLD)), then $X \in J$ and, so, for every $w^*$-compact subset $K \subset X^{**}$ we have $d(K, X) = d(\text{co}w^*(K), X)$.

Although the equality $d(K, X) = d(\text{co}w^*(K), X)$ does not hold in general, we can ask whether there exists a universal constant $1 \leq M < \infty$ such that $d(\text{co}w^*(K), X) \leq M d(K, X)$ for every Banach space $X$ and every $w^*$-compact subset $K \subset X^{**}$.

The answer to this question is affirmative. We prove the following result, which extends the Krein-Šmulian Theorem.

Theorem 5 If $X$ is a Banach space, $Z \subset X$ a closed subspace of $X$ and $K \subset X^{**}$ a $w^*$-compact subset, then $d(\text{co}w^*(K), Z) \leq 5 d(K, Z)$.

When $K \cap Z$ is $w^*$-dense in $K$, the argument used in Theorem 5 gives the following result.

Theorem 6 Let $X$ be a Banach space, $Z \subset X$ a closed subspace and $K \subset X^{**}$ a $w^*$-compact subset. If $Z \cap K$ is $w^*$-dense in $K$, then $d(\text{co}w^*(K), Z) \leq 2 d(K, Z)$.

Finally, we also obtain the following result.

Theorem 7 Let $I$ be an infinite set and $X = \ell_1(I)$. Then for every $w^*$-compact subset $K \subset X^{**}$ we have $d(\text{co}w^*(K), X) = d(K, X)$.

A version of the problem we study here was considered (independently) by M. Fabian, P. Hájek, V. Montesinos and V. Zizler in [7]. They study the class of $w^*$-compact subsets $K \subset X^{**}$ such that $K \cap X$ is $w^*$-dense in $K$. Instead of distances, they deal with the notion of $\epsilon$-weakly relatively compact subsets of $X$ (for short, $\epsilon$-WRK) introduced in [8]. A bounded subset $H$ of $X$ is said to be $\epsilon$-WRK, for some $\epsilon > 0$, if $H + \epsilon B(X^*) \subset X$, that is, if $d(H, X) \leq \epsilon$. Using arguments based on the techniques of double limit due to Grothendieck and Ptáček, they prove that the constant $M = 2$ holds for this category of $w^*$-compact subsets $K \subset X^{**}$ such that $K \cap X$ is $w^*$-dense in $K$. More precisely, they prove the following beautiful result.

Theorem ([7]) Let $X$ be a Banach space and $H \subset X$ a bounded subset of $X$. Assume that $H$ is $\epsilon$-WRK for some $\epsilon > 0$. Then the convex hull $\text{co}(H)$ is $2\epsilon$-WRK. Moreover, if $(B(X^*), w^*)$ is angelic, or $X^*$ does not contain a copy of $\ell_1$, then $\text{co}(H)$ is $\epsilon$-WRK.

Observe that the Theorem of Krein-Šmulian follows from this result when $\epsilon = 0$. 

2. Proofs of the results

Let us introduce some notation and terminology (see [1], [4], [6], [11]). \(|A|\) denotes the cardinality of a set \(A\), \(\omega^+\) the first uncountable ordinal, \(\mathfrak{N}_1\) the first uncountable cardinal and \(CH\) the continuum hypothesis. A Hausdorff compact space \(K\) is said to have property \((M)\) if every Radon Borel measure \(\mu\) on \(K\) has separable support \(\text{supp}(\mu)\). If \(K\) is a convex compact subset of some locally convex linear space \(X\) and \(\mu\) is a Radon Borel probability measure on \(K\), \(r(\mu)\) denotes the barycentre of \(\mu\). Recall (see [3]) that \(r(\mu) \in K\) and that \(r(\mu)\) satisfies \(x^*(r(\mu)) = \int_K x^*(k) d\mu\) for every \(x^* \in X^*\).

If \(X\) is a Banach space, let \(X^\perp = \{z \in X^{***} : \langle z, x \rangle = 0, \ \forall x \in X\}\) denote the subspace of \(X^{***}\) orthogonal to \(X\). If \(Y \subset X\) is a subspace of \(X\), let \(Y^\perp(X^*) = \{z \in X^* : \langle z, y \rangle = 0, \ \forall y \in Y\}\) be the subspace of \(X^*\) orthogonal to \(Y\), \(Y^\perp(X^{***}) = \{z \in X^{***} : \langle z, y \rangle = 0, \ \forall y \in Y\}\), etc. So, \(X^\perp = X^\perp(X^{***})\). Recall that, if \(u \in X\) (resp., \(u \in X^{**}\)), then \(d(u, Y) = \sup\{\langle z, u \rangle : z \in B(Y^\perp(X^*))\}\) (resp., \(d(u, Y) = \sup\{\langle z, u \rangle : z \in B(Y^\perp(X^{***}))\}\). If \(A \subset X\) is a subset of \(X\), \([A]\) denotes the linear span of \(A\).

Let \(I\) be an infinite set with the discrete topology. Then:

(0) We use the symbol \(\ell_\infty(I)\) to denote the Banach space of all \(f = (f(i))_{i \in I} \in \mathbb{R}^I\) with supremum norm finite \(\|f\| := \sup\{|f(i)| : i \in I\} < \infty\). The symbol \(c_0(I)\) means its subspace consisting from \(f = (f(i))_{i \in I} \in \ell_\infty(I)\) such that the set \(\{i \in I : |f(i)| > \epsilon\}\) is finite for all \(\epsilon > 0\).

(1) If \(f \in \ell_\infty(I)\), \(\text{supp}(f) = \{i \in I : f(i) \neq 0\}\) will be the support of \(f\) and \(\tilde{f}\) the Stone-Čech extension of \(f\) to \(\beta I\), where \(\beta I\) is the Stone-Čech compactification of \(I\).

(2) Let \(cI = \bigcup\{\overline{A}^I : A \subset I, \ A\ \text{countable}\}\) and \(\ell_\infty^c(I) = \{f \in \ell_\infty(I) : \text{supp}(f)\ \text{countable}\}\). Observe that \(cI\) is an open subset of \(\beta I\) and that, if \(f \in \ell_\infty^c(I)\), then \(f \in \ell_\infty(I)\) if and only if \(\hat{f}_{|\beta I \setminus I} = 0\).

(3) Let \(\Sigma([0, 1]^I) = \{x \in [0, 1]^I : \text{supp}(x)\ \text{countable}\}\) and \(\Sigma([-1, 1]^I) = \{x \in [-1, 1]^I : \text{supp}(x)\ \text{countable}\}\).

(4) Recall that a compact space is said to be a Corson space if it is homeomorphic to some compact subset of \(\Sigma([-1, 1]^I)\).

Proof of Theorem 1. We use a modification of the Argyros-Mercurakis-Nguyen-Pennis Corson compact space without property \((M)\) [1, p. 219]. In the following we adopt the notation and terminology of [1, p. 219]. Let \(\Omega\) be the space of Erdős, that is, the Stone space of the quotient algebra \(M_\lambda/N_\lambda\), where \(\lambda\) is the Lebesgue measure on \([0, 1]\), \(M_\lambda\) is the algebra of \(\lambda\)-measurable subsets of \([0, 1]\) and \(N_\lambda\) is the ideal of \(\lambda\)-null subsets of \([0, 1]\). \(\Omega\) is
a compact extremely disconnected space (because $M_\lambda/N_\lambda$ is complete) and
there exists a strictly positive regular Borel normal probability measure $\hat{\lambda}$
on $\Omega$, determined by the condition $\hat{\lambda}(V) = \lambda(U)$, $V$ being any clopen subset
of $\Omega$ and $U$ a $\lambda$-measurable subset of $[0, 1]$ such that $V = U + N_\lambda$.

Now we proceed as in [1, 3.11 Lemma] with small changes. Write $[0, 1] = \{x_\xi : \xi < \omega^+\}$ and let $\{K_\xi : \xi < \omega^+\}$ be the well-ordered class of compact
subsets of $[0, 1]$ with strictly positive Lebesgue measure. For each $\xi < \omega^+$
we choose a compact subset $U_\xi \subset [0, 1]$ such that:

(a) $U_\xi \subset \{x_\rho : \xi < \rho < \omega^+\} \cap K_\xi$.

(b) If $\lambda(K_\xi) = 1$, then $U_\xi$ satisfies the condition $\lambda(U_\xi) > 0$. If $\lambda(K_\xi) < 1$,
$U_\xi$ satisfies the condition $\lambda(K_\xi) - (1 - \lambda(K_\xi)) < \lambda(U_\xi) \leq \lambda(K_\xi)$.

Let $V_\xi$ be the clopen subset of $\Omega$ corresponding to $U_\xi$. Then $\{V_\xi : \xi < \omega^+\}$ is
a pseudobase of $\Omega$ that witnesses the failure of the property caliber $\omega^+$, that is, if $A \subset \omega^+$ and $|A| = \aleph_1$, then $\bigcap_\xi \in A V_\xi = \emptyset$. Moreover, (b) automatically implies that $|\{\xi < \omega^+ : \lambda(U_\xi) > t\}| = \aleph_1$ for every $0 < t < 1$, whence
$|\{\xi < \omega^+ : \lambda(V_\xi) > t\}| = \aleph_1$ for every $0 < t < 1$.

Consider $A = \{A \subset \omega^+ : \cap_\xi \in A V_\xi \neq \emptyset\}$. Clearly, $A$ is an adequate family (see [11, p. 1116]) such that every element of $A$ is a countable subset
of $\omega^+$. Moreover, there are elements $A \in A$ with $|A| = \aleph_0$. Indeed, apply
a well-known result from measure theory (see Lemma 8) and the fact that
$\{\xi < \omega^+ : \lambda(V_\xi) > \delta\}$ is infinite for some (in fact, every) $0 < \delta < 1$.

So, if $K = \{1_A : A \in A\} \subset \Sigma(\{0, 1\}^{\omega^+}) \subset \ell_\infty(\omega^+)$, then $K$ is a Corson compact space with respect to the $w^*$-topology $\sigma(\ell_\infty(\omega^+), \ell_1(\omega^+))$. Define
the continuous map $T : \Omega \to K$ so that, for every $x \in \Omega$, $T(x) = 1_{A_x}$, where
$A_x = \{\xi \in \omega^+ : x \in V_\xi\}$. Observe that $A_x \in A$ and, so, $T(x) \in K$, $\forall x \in \Omega$.

Let $L = T(\Omega) \subset K$. Then $L$ is a Corson compact space without property (M),
because $L$ is nonseparable but $L$ is the support of $\mu$, where $\mu = T(\hat{\lambda})$ is the probability on $K$ image of $\hat{\lambda}$ under $T$. So, as $L \subset K$,
$K$ is also a Corson compact space without property (M).

Let $I$ be the space $\omega^+$, with the discrete topology, and $X = \ell_\infty(I)$. Then,
the dual space $X^*$ is

$$X^* = \ell_1(I) \oplus_1 M_R(cI \setminus I),$$

where $M_R(cI \setminus I)$ is the space of Radon Borel measures $\nu$ on $\beta I$ such that
$\text{supp}(\nu) \subset cI \setminus I$ and $\oplus_1$ means $\ell_1$-sum (that is, if a Banach space $Y$ has the
decomposition $Y = Y_1 \oplus_1 Y_2$ and $y \in Y$, with $y = y_1 + y_2$ and $y_1 \in Y_1, y_2 \in Y_2$,
then $\|y\| = \|y_1\| + \|y_2\|$). Observe that $\ell_1(I) \oplus_1 M_R(cI \setminus I)$ can be considered
as a 1-complemented closed subspace of $(\ell_\infty(I))^* = \ell_1(I) \oplus_1 M_R(\beta I \setminus I)$. 
The bidual of X is
\[ X^{**} = \ell_\infty(I) \oplus_\infty M_R(cI \setminus I)^*, \]
where \( \oplus_\infty \) means \( \ell_\infty \)-sum (that is, if a Banach space Y has the decomposition \( Y = Y_1 \oplus_\infty Y_2 \) and \( y \in Y \), with \( y = y_1 + y_2 \) and \( y_1 \in Y_1, y_2 \in Y_2 \), then \( \| y \| = \sup\{\| y_1 \|, \| y_2 \|\} \)). Let \( \pi_1, \pi_2 : X^{**} \to X^{**} \) be the canonical projections onto \( \ell_\infty(I) \) and \( M_R(cI \setminus I)^* \), respectively. The subspaces \( \pi_1(X^{**}) = \ell_\infty(I) \) and \( \pi_2(X^{**}) = M_R(cI \setminus I)^* \) are \( w^* \)-closed subspaces of \( X^{**} \). Moreover, the \( w^* \)-topology \( \sigma(X^{**}, X^*) \) coincides on \( \pi_1(X^{**}) = \ell_\infty(I) \) with the \( \sigma(\ell_\infty(I), \ell_1(I)) \)-topology. For \( x \in X^{**} \) we write \( x = (x_1, x_2) \), with \( \pi_1(x) = x_1 \in \ell_\infty(I) \) and \( \pi_2(x) = x_2 \in M_R(cI \setminus I)^* \). So, if \( J : X \to X^{**} \) is the canonical embedding and \( f \in X \), then \( J(f) = (f_1, f_2) \), where \( f_1 = \pi_1(f) = f \) and \( \pi_2(f) = f_2 \) is such that \( f_2(\nu) = \nu(f) = \int_{cI\setminus I} f \, dv \), for every \( \nu \in M_R(cI \setminus I) \).

The map \( \phi : \ell_\infty(I) \to X^{**} \) such that \( \phi(f) = (f, 0) \), \( \forall f \in \ell_\infty(I) \), is an isomorphism between \( \ell_\infty(I) \) and \( \pi_1(X^{**}) \), for the norm-topologies and also for the \( \sigma(\ell_\infty(I), \ell_1(I)) \)-topology of \( \ell_\infty(I) \) and the \( w^* \)-topology of \( \pi_1(X^{**}) \). So, \( H := \phi(K) = \{(k, 0) : k \in K\} \subset B(X^{**}) \) is a Corson compact space without property \( (M) \), which is homeomorphic to \( K \). Since the family \( \mathcal{A} \) is adequate (in particular, \( B \in \mathcal{A} \) if \( B \subset A \) and \( A \in \mathcal{A} \)), the subset \( \{1_A : A \in \mathcal{A}, A \text{ finite}\} \) of \( K \) is dense in \( K \). So, as \( J(1_A) = (1_A, 0) \) when \( A \subset \omega^* \) is finite, we get that \( H \cap J(X) \) is \( w^* \)-dense in \( H \), because
\[
\phi(\{1_A : A \in \mathcal{A}, A \text{ finite}\}) = \{(1_A, 0) : A \in \mathcal{A}, A \text{ finite}\} = J(\{1_A : A \in \mathcal{A}, A \text{ finite}\}) \subset H \cap J(X).
\]

**Claim 1.** \( d(H, J(X)) = \frac{1}{2} \).

Indeed, pick \( f \in K \) and assume that \( f = 1_A \), for some \( A \in \mathcal{A} \). If \( |A| < \aleph_0 \), clearly \( \phi(f) = (f, 0) = J(f) \), that is, \( \phi(f) \in J(X) \). Suppose that \( |A| = \aleph_0 \). Then \( d(\phi(f), J(X)) = \frac{1}{2} \) because:
(a) Clearly, \( \| \phi(f) - \frac{1}{2} J(f) \| = \frac{1}{2} \), whence \( d(\phi(f), J(X)) \leq \frac{1}{2} \).
(b) On the other hand, \( \| \phi(f) - J(g) \| \geq \frac{1}{2} \) for every \( g \in X \). Indeed, let \( g \in X \) and assume that \( \| \phi(f) - J(g) \| \leq \frac{1}{2} \). Then \( \| f - g \| \leq \frac{1}{2} \) in \( \ell_\infty(I) \), which implies that \( \frac{1}{2} \leq g \) on \( A \) (because \( f = 1_A \)) and so \( \tilde{g} \geq \frac{1}{2} \) on \( \overline{A}^H \).

Since \( |A| = \aleph_0 \), we can pick \( p \in \overline{A}^I \setminus I \subset cI \setminus I \). Let \( \delta_p \in M_R(cI \setminus I) \) be such that \( \delta_p(h) = \tilde{h}(p) \) for every \( h \in \ell_\infty(I) \). Notice that \( \| \delta_p \| = 1 \). Then, if \( J(g) = (g, g_2) \), we have
\[
|\phi(f) - J(g)(\delta_p)| = |g_2(\delta_p)| = | - \int_{cI \setminus I} \tilde{g} \cdot d(\delta_p) | = | - \tilde{g}(p) | \geq \frac{1}{2}.
\]
Finally, recall that there are elements \( A \in \mathcal{A} \) with \( |A| = \aleph_0 \).
Claim 2. \(d(\overline{co^{w^*}}(H), J(X)) = 1\).

Indeed, first \(d(\overline{co^{w^*}}(H), J(X)) \leq 1\) because \(\overline{co^{w^*}}(H) \subset B(X^{**})\). On the other hand, let \(\nu := \phi(\mu)\) be the probability on \(\phi(L)\) image of \(\mu\) under \(\phi\). Since \(\phi(L) \subset B(\pi_1(X^{**}))\) and \(\pi_1(X^{**})\) is a convex \(w^*\)-closed subset of \(X^{**}\), we conclude that \(\overline{co^{w^*}}(\phi(L)) \subset B(\pi_1(X^{**}))\). So, as \(r(\nu) \in \overline{co^{w^*}}(\phi(L))\), we get that \(r(\nu) = (z_0, 0)\) for some \(z_0 \in B(\ell_\infty(I))\). If \(\xi \in I\), define \(\pi_\xi : X^{**} \rightarrow \mathbb{R}\) by \(\pi_\xi(f_1, f_2) = f_1(\xi)\), for all \((f_1, f_2) \in X^{**} = \ell_\infty(I) \oplus_\infty M_R(\mathcal{C}I \setminus I)^*\). Observe that \(\pi_\xi\) is a \(w^*\)-continuous linear map on \(X^{**}\). So

\[
\pi_\xi(z_0, 0) = \pi_\xi(r(\nu)) = \int_{\phi(L)} \pi_\xi(k) d\nu = \int_L k(\xi) d\mu = \hat{\lambda}(V_\xi).
\]

Thus, for every \(0 < t < 1\) we have, by construction, \(|\{\xi \in I : z_0(\xi) > t\}| = |\{\xi \in I : \hat{\lambda}(V_\xi) > t\}| = n_1\), and this implies that \(\|z_0 - g\| \geq 1\) in \(\ell_\infty(I)\), for every \(g \in X = \ell_\infty(I)\), whence \(\|z_0 - J(g)\| \geq 1\) for every \(g \in X\), that is, \(d((z_0, 0), J(X)) \geq 1\). Finally, we obtain \(d(\overline{co^{w^*}}(H), J(X)) \geq 1\) because \((z_0, 0) \in \overline{co^{w^*}}(\phi(L)) \subset \overline{co^{w^*}}(H)\).

And this completes the proof. \(\Box\)

Remark. Theorem 1 gives, under \(CH\), a negative answer to the following question posed in Problem 3 of [7]: if \(X\) is a Banach space and \(H \subset X\) a \(\epsilon\)-WRK, is \(co(H)\) a \(\epsilon\)-WRK?

We need the following well known result from measure theory.

Lemma 8 Let \((\Omega, \Sigma, \mu)\) be a measure space with \(\mu\) positive and finite and \(\{A_n\}_{n<\omega} \subset \Sigma\) be a sequence of measurable sets with \(\mu(A_n) > \delta > 0\) for all \(n < \omega\) and some \(\delta > 0\). Then there exists an infinite subset \(I \subset \omega\) such that \(\cap_{n \in I} A_n \neq \emptyset\).

Proof. Consider the sequence \(B_n = \cup_{k \geq n} A_k\), \(n \geq 1\). The sequence \(\{B_n\}_{n \geq 1}\) is decreasing and \(\mu(B_n) > \delta\) for every \(n \geq 1\). Hence \(\mu(\cap_{n<\omega} B_n) \geq \delta\) and therefore \(\cap_{n<\omega} B_n \neq \emptyset\). Choose \(w \in \cap_{n<\omega} B_n\) and inductively a sequence \(\{A_{n_k}\}_{k<\omega}\), \(n_k < n_{k+1}\), such that \(w \in A_{n_k}\) for all \(k < \omega\). Then \(I = \{n_k : k < \omega\}\) is the desired infinite subset. \(\Box\)

Proposition 9 Let \(I\) be an infinite set and \(X = (c_0(I), \|\cdot\|_\infty)\). Then every \(w^*\)-compact subset \(K \subset X^{**}\) satisfies \(d(K, X) = d(\overline{co^{w^*}}(K), X)\).

Proof. First, recall that if \(f \in X^{**} = \ell_\infty(I)\), then

\[
d(f, X) = \sup\{|f(p)| : p \in \beta I \setminus I\}.
\]
Suppose that there exists a w*-compact subset $K \subset B(X^{**})$ such that $d(K, X) < d(\overline{co}^w(K), X)$. Then we can find two real numbers $a, b$ such that
\[ d(K, X) < a < b < d(\overline{co}^w(K), X) \leq 1. \]

Pick $z_0 \in \overline{co}^w(K)$ such that $d(z_0, X) > b$. So, there exist $\epsilon > 0$ and $p_0 \in \beta I \setminus I$ such that $|\hat{z}_0(p_0)| > b + \epsilon$, for example, $\hat{z}_0(p_0) > b + \epsilon$. Let $U \subset I$ be such that $p_0 \in \overline{U}^\beta I$ and $z_0(j) > b + \epsilon$, $\forall j \in U$. Let $\mu$ be a Radon Borel probability on $K$ such that $z_0 = r(\mu)$ and denote $A_j := \{ k \in K : k(j) \geq b \}$, $j \in U$, which is a closed subset of $K$.

**Claim.** $\mu(A_j) > \frac{\epsilon}{1 - b}, \forall j \in U$.

Indeed, let $\pi_j : \ell_\infty(I) \to \mathbb{R}$, $j \in I$, be such that $\pi_j(f) = f(j)$ for every $f \in \ell_\infty(I)$. Observe that $\pi_j$ is a w*-continuous linear map on $\ell_\infty(I)$, for every $j \in I$. Thus, for every $j \in U$ we have
\[
\begin{align*}
  z_0(j) &= \pi_j(z_0) = \pi_j(r(\mu)) = \int_K \pi_j(k) d\mu = \int_K k(j) d\mu = \\
  &= \int_{A_j} k(j) d\mu + \int_{K \setminus A_j} k(j) d\mu \leq \mu(A_j) + (1 - \mu(A_j)) b,
\end{align*}
\]
and this implies
\[
\mu(A_j) \geq \frac{z_0(j) - b}{1 - b} > \frac{\epsilon}{1 - b}.
\]

Let $V_0 \subset U$ be an arbitrary infinite subset. By Lemma 8 there exists an infinite countable subset $\mathcal{N}_0 \subset V_0$ such that $\emptyset \neq \bigcap_{j \in \mathcal{N}_0} A_j \subset K$. Pick $x_0 \in \bigcap_{j \in \mathcal{N}_0} A_j$. Then for every $q \in \overline{N}_0^\beta I \setminus I$ we have $\hat{x}_0(q) \geq b$, which implies $d(x_0, X) \geq b$, a contradiction, because $x_0$ belongs to $K$. \[\square\]

If $(X, \tau)$ is a topological space, a subset $K \subset X$ is said to be **regular in X** if and only if the interior set $\text{int}(K)$ is dense in $K$.

**Corollary 10** Let $I$ be an infinite set, $H \subset \beta I \setminus I$ a compact subset which is regular in $\beta I \setminus I$, and $Y_H = \{ f \in \ell_\infty(I) : f|_H = 0 \}$. Then for every w*-compact subset $K \subset \ell_\infty(I)$ we have $d(K, Y_H) = d(\overline{co}^w(K), Y_H)$.

**Proof.** First, observe that $d(z, Y_H) = \sup\{ |\hat{z}(x)| : x \in H \}$ for every $z \in \ell_\infty(I)$. Suppose that there exist a w*-compact subset $K \subset B(\ell_\infty(I))$ and real numbers $a, b$ such that:
\[
d(K, Y_H) < a < b < d(\overline{co}^w(K), Y_H) \leq 1.
\]
Let $z_0 \in \overline{co}^w(K)$ be such that $d(z_0, Y_H) > b$. Since $\text{int}(H)$ is dense in $H$, there exists $p_0 \in \text{int}(H)$ such that, for example, $\hat{z}_0(p_0) > b + \epsilon$, for some
$\epsilon > 0$. Let $U \subset I$ be an infinite subset such that $p_0 \in \overline{U}^I \setminus I \subset H$ and $z_0(j) > b + \epsilon$, $\forall j \in U$. By an argument similar to that of Proposition 9, we find an infinite countable subset $N_0 \subset U$ and a vector $x_0 \in K$ such that $\bar{x}_0(q) \geq b$, for every $q \in N_0 \setminus I \subset H$, which implies $d(x_0,Y_H) \geq b$, a contradiction, because $x_0 \in K$ and $d(K,Y_H) \leq a < b$. \hfill \blacksquare

We now prove Theorem 3 and Corollary 4.

**Proof of Theorem 3.** Suppose that there exist a $w^*$-compact subset $K \subset B(X^*)$ and real numbers $a, b$ such that:

$$d(K,X) < a < b < d(\overline{co}^w(K),X).$$

Pick $z_0 \in \overline{co}^w(K)$ with $d(z_0,X) > b$. Since $X \in J$ we can choose a sequence $\{x^*_n\}_{n \geq 1} \subset S(B(X^*),z_0,b)$ such that $x^*_n \overset{w^*}{\to} 0$. Let $T : X \to c_0 := c_0(\mathbb{N})$ be such that $T(x) = (x^*_n(x))_{n \geq 1}$, $\forall x \in X$. Clearly, $T$ is a linear continuous map with $\|T\| \leq 1$. Let $L = T^*(K)$, which is a $w^*$-compact subset of $B(\ell_\infty)$.

**Claim 1.** $d(L,c_0) \leq d(K,X)$.

Indeed, let $c_0^\perp = \{f \in \ell_1^*: \langle f,u \rangle = 0, \forall u \in c_0 \}$ and pick $v \in B(c_0^\perp)$. Then $\|T^*(v)\| \leq 1$ and for every $x \in X$ we have:

$$\langle T^*(v),x \rangle = \langle v,T^*x \rangle = \langle v,Tx \rangle = 0.$$

So, $T^*(B(c_0^\perp)) \subset B(X^\perp)$. Hence, if $k \in K$ and $T^*(k) = : h \in L$ we have:

$$d(h,c_0) = \sup\{\langle v,h \rangle : v \in B(c_0^\perp)\} = \sup\{\langle v,T^*(k) \rangle : v \in B(c_0^\perp)\} = \sup\{\langle T^*(v),k \rangle : v \in B(c_0^\perp)\} \leq \sup\{\langle w,k \rangle : w \in B(X^\perp)\} = d(k,X).$$

**Claim 2.** If $w_0 := T^*(z_0) \in \overline{co}^w(L)$, then $d(w_0,c_0) \geq b$.

Indeed, let $\{e_n\}_{n \geq 1}$ be the canonical basis of $\ell_1$, which satisfies $T^*(e_n) = x^*_n$, $\forall n \geq 1$. Since $x^*_n \in S(B(X^*),z_0,b)$, then

$$\langle w_0,e_n \rangle = \langle T^*(z_0),e_n \rangle = \langle z_0,T^*(e_n) \rangle = \langle z_0,x_n^* \rangle \geq b.$$

Let $\psi$ be a $w^*$-limit point of $\{e_n\}_{n \geq 1}$ in $(\ell_1^*,w^*)$. Clearly, $\psi \in B(c_0^\perp)$ and also $\psi(w_0) \geq b$ by (2.1). So, $d(w_0,c_0) \geq b$.

Therefore, $L \subset B(\ell_\infty)$ is a $w^*$-compact subset such that

$$d(L,c_0) \leq d(K,X) < a < b \leq d(w_0,c_0) \leq d(\overline{co}^w(L),c_0),$$

a contradiction to Proposition 9. \hfill \blacksquare
Of course, not every Banach space has property $J$. Indeed, if $X$ is a non-reflexive Grothendieck Banach space (for example, if $X = \ell_\infty(I)$ with $I$ infinite), then clearly $X$ does not have property $J$. Moreover, $X$ cannot be isomorphically embedded into a Banach space with property $J$.

However, the family of Banach spaces fulfilling property $J$ is very large. For example, this family includes the class of Banach spaces $X$ whose dual unit ball $(B(X^*), w^*)$ is angelic in the $w^*$-topology. Recall that every WCG (even every WLD) Banach space belongs to this class (see [2]).

**Proof of Corollary 4.** The proof of this fact is standard and well known. Let us prove that if $z_0 \in B(X^{**}) \setminus X$ and $0 < b < d(z_0, X)$, then

$$0 \in \overline{S(B(X^*), z_0, b)}^{(X^*, X)}.$$  

Find $\psi \in S(X^\perp) \subset X^{***}$ such that $\psi(z_0) > b$. As $B(X^*)$ is $w^*$-dense in $B(X^{***})$ and $\psi(z_0) > b$, then

$$\psi \in \overline{S(B(X^*), z_0, b)}^{(X^{***}, X^{**})},$$

whence we obtain

$$0 \in \overline{S(B(X^*), z_0, b)}^{(X^*, X)},$$

because $\psi \in X^\perp$. Finally, it is enough to apply the fact that $(B(X^*), w^*)$ is angelic.

Now we prove some auxiliary facts. If $X$ is a Banach space, let $I_X : X \to X$ denote the identity map of $X$, $J_X : X \to X^{**}$ the canonical embedding of $X$ into $X^{**}$ and $R_X : X^{***} \to X^*$ the canonical restriction map such that $\langle R_X(z), x \rangle = \langle z, J_X(x) \rangle$, for every $z \in X^{***}$ and every $x \in X$. Notice that $R_X = (J_X)^*$ and that $R_X \circ J_{X^*} = I_{X^*}$.

It is well-known that $J_X^*(X^*)$ is $1$-complemented into $X^{***}$, by means of the projection $P_X : X^{***} \to X^{***}$ such that $P_X = J_X^* \circ R_X$. Since $\ker(P_X) = \{z \in X^{***} : \langle z, J_X(x) \rangle = 0, \forall x \in X\} = X^\perp$, we have the decomposition $X^{***} = X^\perp \oplus J_X^*(X^*)$. The subspace $X^\perp$ is complemented in $X^{***}$ by means of the projection $Q_X : X^{***} \to X^{***}$ such that $Q_X = I_{X^{***}} - P_X$. Observe that $1 \leq \|Q_X\| \leq 2$ and that:

$$B(X^\perp) \subset Q_X(B(X^{**})) \subset \|Q_X\| \cdot B(X^\perp) \subset 2B(X^\perp).$$

**Lemma 11** Let $X$ be a Banach space and $Q_X : X^{***} \to X^{***}$ be the canonical projection onto $X^\perp$. Assume that $Y \subset X$ is a closed subspace. Then, for every $u \in Y^{**}$ (considered $Y^{**}$ as a subspace of $X^{**}$) we have:

$$d(u, X) \leq d(u, Y) \leq \|Q_X\| \cdot d(u, X) \leq 2d(u, X).$$
Proof. First, it is clear that \( d(u, X) \leq d(u, Y) \), because \( Y \subset X \).

In the following we distinguish \( X \) from \( J_X(X) \), \( Y \) from \( J_Y(Y) \), etc. Let \( i : Y \to X \) denote the inclusion map. Then \( i^* : X^* \to Y^* \) is a quotient map, \( i^{**} : Y^{**} \to X^{**} \) is an inclusion map such that \((i^{**})_Y = i\), and \( i^{**} : X^{***} \to Y^{***} \) is a quotient map such that \((i^{**})_{X^*} = i^*\). Observe that \( i^{**}(B(X^{***})) = B(Y^{***}) \). It is easy to see that \( J_X \circ i = i^{**} \circ J_Y \) and that \( J_Y \circ i^* = i^{***} \circ J_X \), whence we obtain

\[
i^* \circ R_X = i^* \circ (J_X)^* = (J_X \circ i)^* = (i^{**} \circ J_Y)^* = (J_Y)^* \circ i^{***} = R_Y \circ i^{***}.
\]

Claim. \( Q_Y \circ i^{***} = i^{***} \circ Q_X \).

Indeed, we have

\[
Q_Y \circ i^{***} = (I_Y - J_Y \circ R_Y) \circ i^{***} = i^{***} - J_Y \circ R_Y \circ i^{***} = i^{***} - J_Y \circ i^* \circ R_X = i^{***} - i^{***} \circ J_X \circ R_X = i^{***} \circ (I_X - J_X \circ R_X).
\]

From the Claim we obtain \( ||Q_Y|| \leq ||Q_X|| \) and

\[
B(Y^\perp) \subset Q_Y(B(Y^{***})) = Q_Y(i^{***}(B(X^{***}))) = i^{***}(Q_X(B(X^{***}))) \subset i^{***}(||Q_X|| \cdot B(X^\perp)).
\]

Thus, if \( u \in Y^{**} \), we finally get

\[
d(u, J_Y(Y)) = \sup \{ z, u : z \in B(Y^{\perp}) \}
\leq \sup \{ i^{***}(w), u : w \in ||Q_X|| \cdot B(X^\perp) \}
= ||Q_X|| \cdot \sup \{ w, i^*(u) : w \in B(X^\perp) \}
= ||Q_X|| \cdot d(i^*(u), J_X(X))
\leq 2d(i^*(u), J_X(X)).
\]

Let us prove our extension of the Krein-Šmulian Theorem.

Proof of Theorem 5. Suppose that there exist a closed subspace \( Z \subset X \) and a \( w^* \)-compact subset \( K \subset B(X^{**}) \) such that

\[
d(\overline{w}^*(K), Z) > 5d(K, Z).
\]

Then we can find \( z_0 \in \overline{w}^*(K) \) and \( a, b > 0 \) such that

\[
d(z_0, Z) > b > 5a > 5d(K, Z).
\]

Pick \( \psi \in S(Z^\perp(X^{***})) \) with \( \psi(z_0) > b \).
Step 1. Since \( \psi(z_0) > b \), there exists \( x^*_1 \in S(X^*) \) such that \( x^*_1(z_0) > b \).

So, as \( z_0 \in \overline{w^*}w^*(K) \) we can find \( \eta_1 \in \text{co}(K) \) with

\[
\eta_1 = \sum_{i=1}^{n_1} \lambda_{i1} \eta_{i1}, \quad \eta_{i1} \in K, \quad \lambda_{i1} \geq 0, \quad \sum_{i=1}^{n_1} \lambda_{i1} = 1,
\]

such that \( x^*_1(\eta_1) > b \). Since \( d(\eta_{i1}, Z) < a \) we have the decomposition

\[
\eta_{i1} = \eta^1_{i1} + \eta^2_{i1}, \quad \eta^1_{i1} \in Z \quad \text{and} \quad \eta^2_{i1} \in aB(X^{**}).
\]

Step 2. Let \( Y_1 = \left\{ \{ \eta^1_{i1} : 1 \leq i \leq n_1 \} \right\} \subset Z \). Since \( \dim(Y_1) \leq n_1 < \infty \), \( \psi(z_0) > b \) and \( \psi \in Y_1^\perp(X^{**}) \), there exists \( x^*_2 \in S(X^*) \) such that \( x^*_2(z_0) > b \) and \( x^*_2|Y_1 = 0 \). So, as \( x^*_1(z_0) > b, \ i = 1, 2 \), and \( z_0 \in \overline{w^*}w^*(K) \), we can find \( \eta_2 \in \text{co}(K) \) with

\[
\eta_2 = \sum_{i=1}^{n_2} \lambda_{2i} \eta_{2i}, \quad \eta_{2i} \in K, \quad \lambda_{2i} \geq 0, \quad \sum_{i=1}^{n_2} \lambda_{2i} = 1,
\]

such that \( x^*_2(\eta_2) > b, \ i = 1, 2 \). Since \( d(\eta_{2i}, Z) < a \) we have the decomposition

\[
\eta_{2i} = \eta^1_{2i} + \eta^2_{2i}, \quad \eta^1_{2i} \in Z \quad \text{and} \quad \eta^2_{2i} \in aB(X^{**}).
\]

By reiteration, we obtain the sequences \( \{ x^*_n \}_{n \geq 1} \subset S(X^*), \ \eta_k \in \text{co}(K) \) with

\[
\eta_k = \sum_{i=1}^{n_k} \lambda_{ki} \eta_{ki}, \quad \eta_{ki} \in K, \quad \lambda_{ki} \geq 0, \quad \sum_{i=1}^{n_k} \lambda_{ki} = 1,
\]

\[
\eta_{ki} = \eta^1_{ki} + \eta^2_{ki} \quad \text{with} \quad \eta^1_{ki} \in Z \quad \text{and} \quad \eta^2_{ki} \in aB(X^{**}), \ k \geq 1,
\]

such that \( x^*_k(\eta_k) > b, \ i = 1, \ldots, k \), and \( x^*_k|Y_k = 0 \), where

\[
Y_k = \left\{ \{ \eta^1_{ji} : i = 1, \ldots, k; 1 \leq j_i \leq n_i \} \right\} \subset Y_{k+1} \subset Z.
\]

Let \( Y = \bigcup_{k \geq 1} Y_k \subset Z \) and \( K_1 = (K + aB(X^{**})) \cap Y^{**} \). Then \( Y \) is a closed separable subspace of \( Z \) and \( K_1 \) is a \( w^* \)-compact subset of \( Y^{**} \) (considered \( Y^{**} \) canonically embedded into \( Z^{**} \subset X^{**} \)). Observe that \( \{ \eta^1_{ji} : i \geq 1, 1 \leq j_i \leq n_i \} \subset K_1 \). By Lemma 11, since \( K_1 \subset Y^{**} \) and \( d(K_1, Z) \leq 2a \), we have \( d(K_1, Y) \leq 4a \) (in fact, \( d(K_1, Y) \leq 2\|Q_Z\|a \leq 2\|Q_X\|a \leq 4a \)). As \( Y \) has property \( J \) (because \( Y \) is separable and, so, WCG, see Corollary 4), we get

\[
d(\overline{w^*w^*}(K_1), Y) = d(K_1, Y), \quad \text{whence} \quad d(\overline{w^*w^*}(K_1), Y) \leq 4a.
\]

Let \( \eta_0 \) be a \( w^* \)-limit point of \( \{ \eta_k \}_{k \geq 1} \) in \( X^{**} \).

Claim 1. \( d(\eta_0, Y) \leq 5a \).

Indeed, first

\[
\eta_0 \in \overline{w^*w^*}(\{ \eta_{ji} : i \geq 1, 1 \leq j_i \leq n_i \}) \subset \overline{w^*w^*}(K_1) + aB(X^{**}).
\]

On the other hand, \( d(\overline{w^*w^*}(K_1), Y) \leq 4a \). Hence, \( d(\eta_0, Y) \leq 5a \).
**Claim 2.** $d(\eta_0, Y) \geq b$.

Indeed, let $\phi \in B(X^{**})$ be a $w^*$-limit point of $\{x_n^*\}_{n \geq 1}$. Since $x_n^*(\eta_0) > b$ if $k \geq n$, then $x_n^*(\eta_0) \geq b$, $\forall n \geq 1$, whence $\phi(\eta_0) \geq b$. Moreover, $\phi \in Y^\perp(X^{**})$ because $x_{n+1|Y_n} = 0$ and $Y_n \subset Y_{n+1}$. Hence, $d(\eta_0, Y) \geq \phi(\eta_0) \geq b$.

Since $b > 5a$ we get a contradiction and this completes the proof. ■

**Proof of Theorem 6.** Suppose that there exist a closed subspace $Z \subset X$ and a $w^*$-compact subset $K \Subset B(X^{**})$, with $Z \cap K$ $w^*$-dense in $K$, such that $d(\overline{co}^w(K), Z) > 2d(K, Z)$. Then we can find $z_0 \in \overline{co}^w(K)$ and $a, b > 0$ such that $d(z_0, Z) > b > 2a > 2d(K, Z)$. Pick $\psi \in S(Z^\perp(X^{**}))$ such that $\psi(z_0) > b$. We follow the argument of Theorem 5 with the following changes:

(i) As $Z \cap K$ is $w^*$-dense in $K$ we choose $\eta_k \in co(Z \cap K)$ with $\eta_k = \sum_{i=1}^{n_k} \lambda_{ki}e_{ki}$, $\sum_{i=1}^{n_k} \lambda_{ki} = 1$; $i \geq 1, 1 \leq j_i \leq n_i$.

(ii) Define $Y_k = [\{e_{kj_i} : i = 1, \ldots, k; 1 \leq j_i \leq n_i\}]$, $Y = \bigcup_{k \geq 1} Y_k \subset Z$ and $K = w^*\text{-cl}\{e_{kj} : i \geq 1, 1 \leq j \leq n_i\} \subset Y^{**} \cap K$.

Clearly, $d(K_1, Z) \leq d(K, Z) < a$, whence $d(K_1, Y) \leq 2d(K_1, Z) \leq 2a$ (in fact, $d(K_1, Y) \leq \|Q_X\|a \leq \|Q_X\|a \leq 2a$). Since $Y$ is separable, we have $d(\overline{co}^w(K_1), Y) = d(K_1, Y) \leq 2a$. Finally, every $w^*$-limit point $\eta_0$ of $\{\eta_k\}_{k \geq 1}$ in $X^{**}$ satisfies $\eta_0 \in \overline{co}^w(K_1)$, $d(\eta_0, Y) \leq 2a$ and $d(\eta_0, Y) \geq b$, a contradiction. ■

**Remarks.** (1) The argument of Theorem 5 in fact yields the following

$$d(\overline{co}^w(K), Z) \leq 2\|Q_Z\| + 1)d(K, Z) \leq 2\|Q_X\| + 1)d(K, Z) \leq 5d(K, Z).$$

In Theorem 6 we also obtain

$$d(\overline{co}^w(K), Z) \leq \|Q_Z\|d(K, Z) \leq \|Q_X\|d(K, Z) \leq 2d(K, Z).$$

(2) Let $Y \subset X$ be a subspace of the Banach space $X$ and assume that $d(\overline{co}^w(K), X) \leq Md(K, X)$ for some $1 \leq M < \infty$ and every $w^*$-compact subset $K \subset X^{**}$. Then using the fact that $d(z, X) \leq d(z, Y) \leq \|Q_X\|d(z, X) \leq 2d(z, X)$, for every $z \in Y^{**}$, it can be proved easily that $d(\overline{co}^w(K), Y) \leq M\|Q_X\|d(K, Y) \leq 2Md(K, X)$, for every $w^*$-compact subset $K \subset Y^{**}$. A subset $A \subset X^*$ is said to be fragmented by the norm of $X^*$ (see [6, p. 81], [10]) if for every subset $B \subset A$ and every $\varepsilon > 0$ there exists a $w^*$-open subset $V \subset X^*$ such that $V \cap B \neq \emptyset$ and $\text{diam}(V \cap B) \leq \varepsilon$, where $\text{diam}(V \cap B)$ means the diameter of $V \cap B$. In order to prove Corollary 13 and Theorem 7 we need the following lemma.
Lemma 12 Let $X$ be a Banach space, $Z \subset X^*$ a subspace and $K \subset B(X^*)$ a $w^*$-compact subset such that there exist $a, b > 0$ with:

$$d(K, Z) < a < b < d(\overline{w^*}(K), Z).$$

Then there exist $z_0 \in \overline{w^*}(K)$ and $\psi \in S(\overline{Z^+}(X^*))$ with $\psi(z_0) > b$ such that, if $\mu$ is a Radon Borel probability measure on $K$ with barycentre $r(\mu) = z_0$, then: (a) $\mu$ is atomless; (b) if $H = \text{supp}(\mu)$, for every $w^*$-open subset $V$ of $X^*$ with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{w^*}(V \cap H)$ such that $\psi(\xi) > b$; and (c) $H$ is not fragmented by the norm of $X^*$.

Proof. Pick $z \in \overline{w^*}(K)$ and $\psi \in S(\overline{Z^+}(X^*))$ such that $\psi(z) > b + \epsilon$ for some $\epsilon > 0$. By the Bishop-Phelps theorem, there exists $\phi \in S(X^*)$ with $\|\psi - \phi\| \leq \epsilon/4$ such that $\phi$ attains its maximum value on $\overline{w^*}(K)$ in some $z_0 \in \overline{w^*}(K)$. So:

(2.2) $\phi(z_0) \geq \phi(z) = \psi(z) + (\phi - \psi)(z) > b + \epsilon - \frac{1}{4}\epsilon = b + \frac{3}{4}\epsilon$,

(2.3) $\psi(z_0) = \phi(z_0) + (\psi - \phi)(z_0) > b + \frac{3}{4}\epsilon - \frac{1}{4}\epsilon = b + \frac{1}{2}\epsilon$ and

(2.4) $\forall k \in K, \phi(k) = \psi(k) + (\phi - \psi)(k) < a + \frac{1}{4}\epsilon < b + \frac{3}{4}\epsilon < \phi(z_0)$.

In particular, observe that $z_0 \notin K$ by (2.4).

(a) Let $\mu$ be a Radon Borel probability on $K$ with barycentre $r(\mu) = z_0$ and suppose that $\mu$ has some atom, that is, there exist $0 < \lambda \leq 1$ and $k_0 \in K$ such that $\mu = \lambda \cdot \delta_{k_0} + \mu_1$, $\mu_1 \geq 0$. If $\lambda = 1$ then $\mu = \delta_{k_0}$, whence $r(\mu) = k_0 \in K$, which is impossible because $r(\mu) = z_0 \notin K$ by (2.4). So, $0 < \lambda < 1$, i.e., $\mu_1 \neq 0$ and $\|\mu_1\| = 1 - \lambda > 0$. Then $\mu = \lambda \cdot \delta_{k_0} + (1 - \lambda)\frac{\mu_1}{\|\mu_1\|}$ and

$$z_0 = r(\mu) = \lambda k_0 + (1 - \lambda)r(\frac{\mu_1}{\|\mu_1\|}),$$

whence, since $\phi(k_0) < \phi(z_0)$ (by (2.4)) and $\phi(r(\frac{\mu_1}{\|\mu_1\|})) \leq \phi(z_0)$ (because $r(\frac{\mu_1}{\|\mu_1\|}) \in \overline{w^*}(K)$), we get

$$\phi(z_0) = \lambda \phi(k_0) + (1 - \lambda)\phi(r(\frac{\mu_1}{\|\mu_1\|})) < \lambda \phi(z_0) + (1 - \lambda)\phi(z_0) = \phi(z_0),$$

a contradiction.

(b) Let $H = \text{supp}(\mu)$ and suppose that there exists a $w^*$-open subset $V$ of $X^*$ with $V \cap H \neq \emptyset$ such that $\psi(\xi) \leq b$, for every $\xi \in \overline{w^*}(V \cap H)$. Let $\mu_1 = \mu|_{V \cap H}$ denote the restriction of $\mu$ to $V \cap H$ (that is, $\mu_1(B) = \mu(B \cap V \cap H)$, for every Borel subset $B \subset K$) and $\mu_2 := \mu - \mu_1$. Observe that $\mu_1, \mu_2$ are
positive measures such that \( \mu_1 \neq 0 \) (because \( \emptyset \neq V \cap H = V \cap \text{supp}(\mu) \)) and \( \mu_2 \neq 0 \) (if \( \mu_2 = 0 \), i.e., \( \mu = \mu_1 = \mu_{1|V \cap H} \), then \( z_0 = r(\mu) \in \overline{\text{co}}^w(V \cap H) \) and \( \psi(z_0) \leq b \), a contradiction to (2.3)). Thus, we have the decomposition \( \mu = \mu_1 + \mu_2 \) and so:

\[
z_0 = r(\mu) = \|\mu_1\| \cdot r\left(\frac{\mu_1}{\|\mu_1\|}\right) + \|\mu_2\| \cdot r\left(\frac{\mu_2}{\|\mu_2\|}\right).
\]

Since \( r\left(\frac{\mu_1}{\|\mu_1\|}\right) \in \overline{\text{co}}^w(V \cap H) \), then \( \psi(r\left(\frac{\mu_1}{\|\mu_1\|}\right)) \leq b \), whence \( \phi(r(\frac{\mu_1}{\|\mu_1\|})) \leq b + \frac{\epsilon}{4} \) (because \( \|\psi - \phi\| \leq \epsilon/4 \)). Therefore, taking into account that \( r\left(\frac{\mu_2}{\|\mu_2\|}\right) \in \overline{\text{co}}^w(K) \) and (2.2) we get

\[
\phi(z_0) = \|\mu_1\|\phi(r\left(\frac{\mu_1}{\|\mu_1\|}\right)) + \|\mu_2\|\phi(r\left(\frac{\mu_2}{\|\mu_2\|}\right)) \leq \|\mu_1\| (b + \frac{\epsilon}{4}) + \|\mu_2\||\phi(z_0) < \|\mu_1\|\phi(z_0) + \|\mu_2\|\phi(z_0) = \phi(z_0),
\]

a contradiction.

(c) Let \( \eta = b - a \) and suppose that \( H \) is fragmented by the norm of \( X^* \). Then there exists a \( w^* \)-open subset \( V \) such that \( V \cap H \neq \emptyset \) and \( \text{diam}(V \cap H) < \frac{\eta}{2} \). Therefore, if \( h_0 \in V \cap H \), then \( \overline{\text{co}}^w(V \cap H) \subset B(h_0; \eta/2) \) (closed ball with center \( h_0 \) and radius \( \eta/2 \)). Hence, for every \( \xi \in \overline{\text{co}}^w(V \cap H) \) we have

\[
\psi(\xi) \leq \psi(h_0) + \frac{\eta}{2} \leq d(h_0, Z) + \frac{\eta}{2} < a + \frac{\eta}{2} < b,
\]

a contradiction to (b).

\[\blacksquare\]

**Corollary 13** Let \( X \) be a Banach space, \( Z \subset X^* \) a subspace and \( K \subset X^* \) a \( w^* \)-compact subset which is fragmented by the norm of \( X^* \). Then \( d(\overline{\text{co}}^w(K), Z) = d(K, Z) \).

**Proof.** This follows immediately from Lemma 12. It also follows from [10, Theorem 2.3] where it is proved that \( \overline{\text{co}}(K) = \overline{\text{co}}^w(K) \) whenever \( K \subset X^* \) is \( w^* \)-compact subset such that \( (K, w^*) \) is fragmented by the norm of \( X^* \). \[\blacksquare\]

Now we prove Theorem 7. Observe that we cannot apply Theorem 3 because we do not know whether \( \ell_1(I) \) has property \( J \) when \( I \) is uncountable (if \( I \) is countable it has because \( \ell_1(I) \) is separable in this case). In fact, if we assume that there exists an uncountable measurable cardinal \( \alpha \) (see [4, p. 186, 196] for definitions) and \( I \) is a set with \( |I| = \alpha \), then it is easy to prove that \( \ell_1(I) \) fails to have property \( J \).
Proof of Theorem 7. First, observe that $X^* = \ell_\infty(I)$ and $X^{**}$ is the space $M_R(\beta I)$ of Radon Borel measures on $\beta I$. Thus, $X^{**}$ has the decomposition

$$X^{**} = \ell_1(I) \oplus M_R(\beta I \setminus I).$$

Notice that the subspace $\ell_1(I)$ of this decomposition coincides with the space $J(X)$, $J : X \to X^{**}$ being the canonical inclusion. If $\mu \in M_R(\beta I)$, we write $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \ell_1(I)$ and $\mu_2 = \mu|_{\beta I \setminus I} \in M_R(\beta I \setminus I)$. So, $d(\mu, X) = \|\mu_2\|$.

Suppose that there exist a $w^*$-compact subset $K \subset B(X^{**})$ and two numbers $a, b > 0$ such that:

$$d(K, X) < a < b < d(\overline{co}^{w^*}(K), X).$$

By Lemma 12 we have the following Fact:

**Fact.** There exist $\psi \in S(X^\perp)$ and a $w^*$-compact subset $\emptyset \neq H \subset K$ such that for every $w^*$-open subset $V$ with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{co}^{w^*}(V \cap H)$ with $\psi(\xi) > b$.

**Step 1.** By the Fact we can pick $\xi_1 \in \overline{co}^{w^*}(H)$ with $\psi(\xi_1) > b$ and $x_1^* \in S(X^*)$ with $x_1^*(\xi_1) > b$. Now we choose

$$\eta_1 = \sum_{i=1}^{n_1} \lambda_{i1} \eta_{i1} \in co(H), \quad \eta_{i1} \in H, \quad \lambda_{i1} \geq 0, \quad \sum_{i=1}^{n_1} \lambda_{i1} = 1,$$

such that $x_1^*(\eta_1) > b$. If $\eta_1 = \eta_1^1 + \eta_1^2$, with $\eta_1^1 \in \ell_1(I)$ and $\eta_1^2 \in M_R(\beta I \setminus I)$, then

$$\|\eta_1^2\| = d(\eta_1, X) \leq d(K, X) < a,$$

whence $\|\eta_1\| = \|\eta_1^1\| + \|\eta_1^2\| > b - a$, because $\|\eta_1\| \geq x_1^*(\eta_1) > b$. So, we can find $y_1 \in B(X^*) = B(\ell_\infty)$ with finite support $\text{supp}(y_1) = \{\gamma_{11}, \ldots, \gamma_{1p_1}\} \subset I$ such that $y_1(\eta_1^1) > b - a$. Since $y_1(\eta_1^2) = 0$, we have

$$y_1(\eta_1) = y_1(\eta_1^1) > b - a,$$

whence it follows that $y_1(\eta_{i1}) > b - a$ for some $1 \leq i \leq n_1$.

**Step 2.** Let $V_1 = \{u \in X^*: y_1(u) > b - a\}$, which is a $w^*$-open subset of $X^*$ with $V_1 \cap H \neq \emptyset$, because $\eta_{i1} \in V_1 \cap H$ for some $1 \leq i \leq n_1$. By the Fact there exists $\xi_2 \in \overline{co}^{w^*}(V_1 \cap H)$ with $\psi(\xi_2) > b$. Since $\psi(\xi_2) > b$ and $\psi(e_{\gamma_{i1}}) = 0$, $1 \leq i \leq p_1$ (where $e_{\gamma_{i1}} \in \ell_1(I)$ is the unit vector such that $e_{\gamma_{i1}}(\gamma) = 1$, if $\gamma = \gamma_{i1}$, and $e_{\gamma_{i1}}(\gamma) = 0$, if $\gamma \neq \gamma_{i1}$), there exists $x_2^* \in B(X^*)$ with $x_2^*(\xi_2) > b$ and $x_2^*(e_{\gamma_{i1}}) = 0$, $1 \leq i \leq p_1$. Clearly, we can choose

$$\eta_2 = \sum_{i=1}^{n_2} \lambda_{2i}\eta_{2i} \in co(V_1 \cap H), \quad \eta_{2i} \in V_1 \cap H, \quad \lambda_{2i} \geq 0, \quad \sum_{i=1}^{n_2} \lambda_{2i} = 1,$$

such that $x_2^*(\eta_2) > b$. 
As $y_i(\eta_i) > b - a$, $1 \leq i \leq n_2$, we get $y_1(\eta_2) > b - a$. Let $\eta_2 = \eta_1^2 + \eta_2^2$, with $\eta_1^2 \in \ell_1(I)$, $\eta_2^2 \in M_R(\beta I \setminus I)$ and $\|\eta_2^2\| = d(\eta_2, X) \leq d(K, X) < a$. Since

$$\|\eta_2^2\| \geq |x^*_2(\eta_2)| = |x^*_2(\eta_2) - x^*_2(\eta_2^2)| \geq |x^*_2(\eta_2)| - |x^*_2(\eta_2^2)| > b - a,$$

and $x^*_2 = 0$ on $\text{supp}(y_1)$, we can find $y_2 \in B(X^*)$ with finite support $\text{supp}(y_2) = \{\gamma_{21}, \ldots, \gamma_{2p_2}\} \subset I \setminus \text{supp}(y_1)$ such that $y_2(\eta_2) > b - a$. Hence, $y_2(\eta_2) = y_2(\eta_2^2) > b - a$ and this implies $y_2(\eta_2) > b - a$ for some $1 \leq i \leq n_2$.

By reiteration, we obtain the sequence $\{y_k\}_{k \geq 1} \subset B(X^*)$ with pairwise disjoint supports and the sequence $\{\eta_k\}_{k \geq 1} \subset \text{co}(H) \subset B(X^{**})$ such that $y_n(\eta_k) > b - a$ for $k \geq n$.

Since $\|\sum_{i=1}^n y_i\| \leq 1$ (because the vectors $\{y_k\}_{k \geq 1} \subset B(\ell_\infty)$ have pairwise disjoint supports) and $\left(\sum_{i=1}^n y_i(\eta_n)\right) > n(b - a), \forall n \geq 1$, we get $\|\eta_n\| > n(b - a), \forall n \geq 1$, a contradiction, because $\|\eta_n\| \leq 1$.

**Acknowledgements.** The author would like to thank the referee for many suggestions which helped to improve this paper.

**References**


Recibido: 6 de febrero de 2003
Revisado: 30 de abril de 2004

Antonio S. Granero
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040-Madrid, Spain
AS_granero@mat.ucm.es

Supported in part by DGICYT grant BFM2001-1284.