1. Introduction

Consider the oscillatory singular integral operator $T$:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i(Bx,y)} K(x-y) f(y) \, dy,$$

where $(Bx, y)$ is a real bilinear form, and $K$ is a Calderón-Zygmund kernel, i.e. $K$ is $C^1$ away from the origin, has mean-value zero on each sphere centered at the origin and satisfies

$$|K(x)| \leq C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leq C|x|^{-n-1}.$$

It is proved by D. H. Phong and E. M. Stein in [PS], that $T$ is a bounded operator on $L^p$ spaces, with bound independent of $B$. They also introduced some variants of the $H^1$ and BMO spaces (denoted by $H^1_{B}$ and BMO$_{B}$, to avoid the confusion with the standard $H^1$ and BMO). Analogous to the fact that the classical singular integral operators are bounded from $H^1$ to $L^1$, Phong and Stein showed that $T$ extends as a bounded operator from $H^1_{B}$ to $L^1$. This fact was then used to prove the $L^p$ boundedness by interpolating between $L^2$ and $L^\infty$, (see [PS]).

The object of our study is a more general class of oscillatory singular integral operators. An operator in this class is obtained when the bilinear form
in (1) is replaced by some real-valued polynomial in $x$ and $y$. These operators have arisen in the study of Hilbert transform along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. F. Ricci and E. M. Stein have proved in [RS] that an operator of this kind is bounded on $L^p$ spaces, with bound depending only on the total degree, not on the coefficients of the polynomial. The fact that these operators are of weak-type $(1,1)$ was subsequently proved by S. Chanillo and M. Christ ([CCJ]).

It is our goal in this paper to establish a Hardy space theory for the class of oscillatory singular integral operators with polynomial phase functions. Given such an operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)}K(x-y)f(y)\,dy,$$

where $P(x,y)$ is a real-valued polynomial, we will define the space $H^1_E$ as some variant of the standard $H^1$ space, and this space $H^1_E$ is closely associated with the given polynomial $P(x,y)$. First let us give the definition of the "atoms":

**Definition.** Let $Q$ be a cube with center $x_Q$, an atom is a function $a(x)$ which is supported in $Q$, so that

$$|a(x)| \leq \frac{1}{|Q|},$$

and

$$\int_Q e^{iP(x,y)}a(y)\,dy = 0.$$

The space $H^1_E$ consists of the subspace of $L^1$ of functions $f$ which can be written as $f = \sum \lambda_j a_j$, where $a_j$ are atoms, and $\lambda_j \in \mathbb{C}$, with $\sum |\lambda_j| < \infty$. Consequently, we define $\text{BMO}_E$ as the dual space of $H^1_E$. Our main result is

**Theorem 1.** Suppose $H^1_E$ and $T$ are defined as above. Then $T$ is a bounded operator from $H^1_E$ to $L^1$. The bound of this operator can be taken to depend only on the total degree of $P$, (not on the coefficients of $P$).

We notice that in the paper of Phong and Stein, the fact that the phase function is a real bilinear form makes it possible to apply the Plancherel's theorem to the Fourier transform (or partial Fourier transform) associated with $B$. When $(Bx, y)$ is replaced by the polynomial $P(x, y)$, we no longer have this advantage. So we have to take a different approach, using some $L^2$ estimates of certain oscillatory integrals. This will become clear in our proof.

For $p < 1$, the Calderón-Zygmund singular integral operators are still bounded from $H^p$ to $L^p$. However, this is no longer the case for the oscillatory
singular integral operators. At the end of this article, we will present a simple example which shows that this fails even in the bilinear phase function case.

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2. Proof of Theorem 1

PROOF. Let us assume that \(a\) is a function supported in the cube \(Q_0\), which is centered at the origin, and has sidelength 1, and \(a\) satisfies

\[
|a| \leq 1, \quad \int_{Q_0} a(y) \, dy = 0.
\]

First we shall prove that if \(P(x, y)\) is a polynomial in \(x, y\), and \(P(0, y) = 0\), then

\[
\left\| \text{p.v.} \int_{\mathbb{R}^d} e^{iP(x, y)} K(x - y) a(y) \, dy \right\|_{L^1} \leq C,
\]

where \(C\) depends only on the total degree of \(P\), and is otherwise independent of the coefficients of \(P\).

To prove (3), we shall use induction on the degree \(l\) of \(y\) in \(P(x, y)\).

If \(l = 0\), then \(e^{iP(x, y)}\) is only a function of \(x\), therefore can be taken out of the integral sign, and (3) follows from the classical result of the standard \(H^1\) theory. (See, for example [CW].)

Next we assume \(l > 0\), and (3) is true for \(l - 1\). By the Ricci-Stein theorem on the \(L^2\) boundedness of \(T\), we have

\[
\int_{|x| \leq 2} |T(a)(x)| \, dx \leq C \left( \int_{|x| \leq 2} |T(a)(x)|^2 \, dx \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}^d} |a|^2 \, dx \right)^{1/2} \leq C.
\]

Write

\[
P(x, y) = \sum_{|\alpha| + |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),
\]

where \(Q(x, y)\) is a polynomial with degree in \(y\) less than or equal to \(l - 1\), and still satisfies \(Q(0, y) = 0\). For any \(r > 0\), we have

\[
\int_{2 < |x| \leq r} |T(a)(x)| \, dx \leq \int_{2 < |x| \leq r} \left| (e^{iP(x, y)} - e^{iQ(x, y)} K(x - y) a(y) \, dy \right| \, dx \\
+ \int_{2 < |x| \leq r} \left| \int_{\mathbb{R}^d} e^{iQ(x, y)} K(x - y) a(y) \, dy \right| \, dx.
\]

(If \(r \leq 2\), all the above integrals are 0.)
By our inductive hypothesis, the second term is bounded. Also \(|x - y| \geq |x|/2, \text{ if } |x| > 2, |y| \leq 1.\) So we have

\[
\int_{2 < |x| \leq r} |T(a)(x)| \, dx \leq C + C \int_{|x| \leq r} \int_{\mathbb{R}^n} \left| \exp \left( \sum_{\alpha, \beta} a_{\alpha\beta} x^\alpha y^\beta \right) - 1 \right| \frac{|a(y)|}{|x|^n} \, dy
\]

\[
\leq C + C \sum_{\alpha \geq 1} |a_{\alpha 0}| \int_{|x| \leq r} |x|^{|\alpha| - n} \, dx
\]

\[
\leq C + C \sum_{\alpha \geq 1} |a_{\alpha \beta}| r^{|\alpha|}.
\]

Now, there exists \((\alpha_0, \beta_0)\) such that \(|\alpha_0| \geq 1, |\beta_0| = 1,\) and

\[
|a_{\alpha_0 \beta_0}|^{1/|\alpha_0|} = \max_{\alpha \geq 1} \left| \frac{|a_{\alpha \beta}|}{|x|^n} \right|^{1/|\alpha|}.
\]

Put \(r = |a_{\alpha_0 \beta_0}|^{-1/|\alpha_0|},\) we have

\[
\int_{2 < |x| \leq r} |T(a)(x)| \, dx \leq C,
\]

where \(C\) depends only on the total degree of \(P(x, y)\). Now we turn to the estimate of the remaining part

\[
\int_{|x| > 2, |x| > r} |T(a)(x)| \, dx.
\]

We shall need the following lemmas:

**Lemma 1.** Suppose

\[
\phi(x) = \sum_{|x| = k} a_x x^x
\]

is a real-valued polynomial in \(\mathbb{R}^n\) of degree \(k, \) and \(\psi \in C_0^\infty.\) Then for any \(r,\)

\(\left| |x| = k, a_x \neq 0, we have\)

\[
(4) \quad \left| \int_{\mathbb{R}^n} e^{i\phi(x)} \psi(x) \, dx \right| \leq C |a_r|^{-1/k} (|\psi|_{L^\infty} + \|\nabla \psi\|_{L^1})
\]

To see this, simply let \(\xi\) be an unit vector, such that

\[
|\langle \xi \cdot \nabla \rangle^k \phi(x) \rangle \geq c |a_r|.
\]
This is possible because
\[ \frac{\partial^r \phi(x)}{\partial x^r} = \nu! a_r. \]

(See [ST], page 317.) Without loss of generality, we may assume
\[ \xi = (1, 0, \ldots, 0). \]

Hence
\[ \left| \frac{\partial^k \phi(y)}{\partial y_1^k} \right| \geq c |a_k|. \]

Now apply the one-dimensional Van der Corput’s lemma to obtain (4). See also [ST].

**Lemma 2.** Let
\[ P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha \]

denote a polynomial in \( \mathbb{R}^n \) of degree \( d \). Suppose \( \epsilon < 1/d \), then
\[ \int_{|x| \leq 1} |P(x)|^{-\epsilon} \, dx \leq A_\epsilon \left( \sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\epsilon}. \]

The bound \( A_\epsilon \) depends on \( \epsilon \) (and the dimension \( n \)), but not on the coefficients \( \{a_\alpha\} \).

This is a result of Ricci and Stein. See [RS], page 182.

Now we continue our proof of Theorem 1. Let
\[ R_j = \{ x \in \mathbb{R}^n : 2^j \leq |x| < 2^{j+1} \}, \]

for \( j \geq 0 \), and let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) satisfy
\[ \varphi(x) = 1 \text{ for } |x| \leq 1, \quad \varphi(x) = 0 \text{ for } |x| \geq 2. \]

Define \( T_j \) by
\[ (T_j f)(x) = \chi_{R_j}(x) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y) \, dy, \]

and consider the operator \( T_j T_j^* \):
\[ T_j T_j^*(f)(x) = \int_{\mathbb{R}^n} L_j(x, z) f(z) \, dz, \]
where

\[ L_j(x,z) = \chi_{R_j}(x)\chi_{R_j}(z) \int_{\mathbb{R}^n} e^{iP(y,z) - P(x,y)}|\varphi(y)|^2 \, dy. \]

Write

\[ P(x,y) - P(z,y) = \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha \beta} \gamma^\alpha (x^\alpha - z^\alpha) + (Q(x,y) - Q(z,y)), \]

where the degree of \( y \) in \( Q(x,y) - Q(z,y) \) is less than or equal to \( l - 1 \).

Applying Lemma 1, with \( \nu = \beta_0 \), we obtain

\[ |L_j(x,z)| \leq C \left| \sum_{|\alpha| \leq 1} a_{\alpha 0} (x^\alpha - z^\alpha) \right|^{-1/\nu} \chi_{R_j}(x)\chi_{R_j}(z). \]

On the other hand, it is obvious that \( |L_j(x,z)| \leq C \), so let \( N > 0 \) be a large number (to be chosen later), we have

\[ |L_j(x,z)| \leq C \left| \sum_{|\alpha| \geq 1} a_{\alpha 0} (x^\alpha - z^\alpha) \right|^{-1/N} \chi_{R_j}(z). \]

By rescaling we would obtain the same norm if we were to replace \( L_j(x,z) \) by \( L_j'(x,z) = 2^{|j|}L_j(2^l x, 2^l z) \), so we have

\[ |L_j'(x,z)| \leq C 2^{|j| \nu} \left| \sum_{|\alpha| \geq 1} (a_{\alpha 0} 2^{|\alpha|}) (x^\alpha - z^\alpha) - \sum_{|\alpha| \geq 1} a_{\alpha 0} 2^{l(|\alpha| - |\alpha|)} \right|^{-1/|\alpha|} \chi_{R_0}(x)\chi_{R_0}(z). \]

Choosing \( N \) sufficiently large and applying Lemma 2, we get

\[ \sup_x \int_{\mathbb{R}^n} |L_j'(x,z)| \, dx \leq C 2^{|j| \nu} \sup_x \left( \sum_{|\alpha| \geq 1} |a_{\alpha 0} 2^{|\alpha|}| + \left| \sum_{|\alpha| \geq 1} a_{\alpha 0} 2^{l(|\alpha| - |\alpha|)} \right|^{-1/|\alpha|} \right)^{1/|\alpha|} \]

\[ \leq C 2^{|j| \nu} |a_{\alpha 0}|^{-1/2} N^{1 - |\alpha|/|\alpha|}. \]

Similar estimate hold: \( \sup \int_{\mathbb{R}^n} |L'(x,z)| \, dz \), therefore we obtain

\[ \| T_j T_j^* \| \leq C 2^{|j| \nu} |a_{\alpha 0}|^{-1/2} N^{1 - |\alpha|/|\alpha|}, \]

so

\[ \| T_j \|_{L^2 \rightarrow L^2} \leq C 2^{|j|/2} |a_{\alpha 0}|^{-1/2} N^{1 - |\alpha|/2|\alpha|}. \]

Now we have

\[ \int_{|x| > r, |y| > r} |T(a)(x)| \, dx \leq \int_{|x| > r, |y| > r} \int_{\mathbb{R}^n} |K(x,y) - K(x)| |a(y)| \, dy \]

\[ + \int_{|x| > r, |x| > r} |K(x)| \, dx \int_{\mathbb{R}^n} e^{iP(x,y)} a(y) \, dy = I_1 + I_2. \]
The estimate for $I_1$ is easy
\[
I_1 \leq C \int_{|x| > 2, |x| > r} dx \left( \int_{|y| < |x|^{n+1}} \frac{|y|}{|x|^{n+1}} dy \right) \leq C \int_{|x| > 2} \frac{dx}{|x|^{n+1}} < C.
\]

As for $I_2$, using our estimate on $T_j$ and assuming $2^{l_0} \leq r < 2^{j_0+1}$, for some $j_0$, we have
\[
I_2 \leq C \int_{|x| > 2, |x| > r} \frac{1}{|x|^n} \left( \int |e^{ip(x,y)}a(y)| dy \right) dx
\]
\[
\leq C \sum_{j \geq j_0} \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^n} |T_j(a)(x)| dx
\]
\[
\leq C \sum_{j \geq j_0} \left( \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^{2n}} dx \right)^{1/2} |T_j(a)|_{L^2}
\]
\[
\leq C \sum_{j \geq j_0} 2^{-n|j/2|} 2^{n|j/2|} |a_0|_{-1/2} |a_{j_0}|_{2N} \leq C,
\]
because $2^{l_0} \geq (1/2) |a_0|_{-1/2}$, and (3) is proved.

To prove the theorem, we only need to prove that $|T(a)|_{L^1} \leq C$, for all atoms $a$, and $C$ is a constant which depends only on the total degree of $P(x,y)$.

Let $a$ be an atom associated to the cube $Q$, and the center and sidelength of $Q$ are $x_Q$ and $\delta$ respectively. We observe that
\[
\delta^n(T(a))(\delta x + x_Q) \equiv \text{p.v.} \int |e^{ip(\delta x + y + x_Q)} K(x-y)\delta^n a(\delta y + x_Q) dy.
\]

Write
\[
P(\delta x + x_Q, \delta y + x_Q) = R(x, y) + P(x_Q, \delta y + x_Q),
\]
where $R(x, y)$ is a polynomial which satisfies $R(0, y) = 0$, and the total degree of $R$ is not greater than that of $P$. Let
\[
b(y) = e^{iP(x_Q, \delta y + x_Q)} \delta^n a(\delta y + x_Q),
\]
by the definition of the atom, we have
\[
\text{supp}(b) \subset Q_0 \quad \text{and} \quad |b(y)| \leq 1,
\]
also
\[
\int_{Q_0} b(y) dy = \int_{Q} e^{iP(x_Q, y)} a(y) dy = 0.
\]
Now invoking (3), we have

\[ \| T(a) \|_{L^1} = \left\| \text{p.v.} \int_{\mathbb{R}^n} e^{iR(x,y)} K(x-y) b(y) \, dy \right\|_{L^1} \leq C. \]

This completes the proof of Theorem 5.

3. An Extension

In [RS], Ricci and Stein pointed out that the \( L^p \) boundedness still holds, if the Calderón-Zygmund kernel in the operator is replaced by some more general distribution. For \( H^1_E \), the same thing is true, i.e.

**Theorem 2.** If \( K(x,y) \) is a distribution and \( C^1 \) away from the diagonal \( \{x = y\} \), and satisfies:

(i) \( |K(x,y)| \leq C|x-y|^{-n} \) and \( |\nabla K(x,y)| \leq C|x-y|^{-n-1} \).

(ii) The operator

\[ f \rightarrow \int K(x,y) f(y) \, dy \]

extends as a bounded operator on \( L^2(\mathbb{R}^n) \).

Then the operator

\[ Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) \, dy \]

is bounded from \( H^1_E \) to \( L^1 \).

The proof of Theorem 2 is essentially the same as Theorem 1.

4. The Dual Space \( \text{BMO}_E \)

We define the sharp function \( f^E \) to be

\[ (f^E(x)) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q(x)| \, dx, \]

where

\[ f_Q(x) = e^{-iP(x,y)} \left( \frac{1}{|Q|} \int_Q e^{-iP(x,y)} f(y) \, dy \right) \]
and as the dual space of $H^1_E$, BMO$_E$ is given by

$$\text{BMO}_E = \{ f \in L^1_{\text{loc}} : f^*_E \in L^\infty \}$$

and

$$\| f \|_{\text{BMO}_E} = \| f^*_E \|_{L^\infty}.$$ 

The dual statement of Theorem 2 is

**Theorem 3.** The operator $T^*$ ($T$ given by (5)) extends as a bounded operator from $L^\infty$ to BMO$_E$.

## 5. A Counterexample

In this section, we shall give a simple example to show that the $H^1$ theory on the oscillatory singular integral operators cannot be extended to the $H^p$ case, if $p < 1$.

Let $T$ be defined as

$$(T\omega)(x) = \text{p.v.} \int \mathbb{R} \frac{1}{x-y} a(y) \, dy.$$  

Take $\delta > 0$, $\delta$ is very small, and $a$ is a function supported on $I_\delta = [-\delta, \delta]$, given by

$$a(y) = \begin{cases} (2\delta)^{-1/p} & \text{if } y \in [\delta/2, \delta], \\ -(2\delta)^{-1/p} & \text{if } y \in [-\delta, -\delta/2], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $a$ satisfies

$$|a| \leq |I_\delta|^{-1/p}, \quad \int_{I_\delta} a(y) \, dy = 0.$$  

Therefore, we have

$$\text{Im} (T\omega)(x) = (2\delta)^{-1/p} \left( \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x-y} \, dy + \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x+y} \, dy \right).$$

Let $x \in (\pi/4\delta, \pi/3\delta)$, then $x - y > 0$, $x + y > 0$ for $y \in [\delta/2, \delta]$. Also $\pi/8 < xy < \pi/3$. 
Hence
\[
\text{Im } (Ta)(x) > c_0 (2\delta)^{-1/p} \left( \int_{\delta/2}^{\delta} \frac{1}{x-y} \, dy + \int_{\delta/2}^{h} \frac{1}{x+y} \, dy \right) \\
= c_0 (2\delta)^{-1/p} \log \left( 1 + \frac{\delta x}{(x^2 - \delta x/2 - \delta^2/2)} \right) \\
> c'_0 \delta^{1-1/p} x^{-1},
\]
for some constant $c'_0 > 0$. Then, we have
\[
\int_{\delta/2}^{h} |Ta(x)|^p \, dx \geq c'^p_0 \int_{\delta/2}^{h} (\delta^{1-1/p} x^{-1})^p \, dx = c_0^p \delta^{2(p-1)}.
\]
This is unbounded as $\delta \to 0$ and $p < 1$.

References


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