Abstract Potential Operators on Hilbert Space

(Dedicated to Professor Yasuo Akizuki on his 70th birthday)

By

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Let $X$ be a (real or complex) Hilbert space. A linear operator $V$ with its domain $D(V)$ and range $R(V)$ both strongly dense in $X$ is called an abstract potential operator (see K. Yosida [2], p. 412) if the inverse $V^{-1}$ exists in such a way that

$$A = -V^{-1}$$

is the infinitesimal generator of a one-parameter semi-group of class $(C_0)$ of linear contraction operators on $X$ into $X$. The purpose of the present note is to prove the following existence theorem. (Hereafter, we shall denote by $S^*$ the strong closure of a subset $S$ of $X$.)

**Theorem.** Let $U$ be a linear operator satisfying three conditions:

(2) $D(U)^* = X$,
(3) $R(U)^* = X$,
(4) $U$ is accretive, that is, $Re(Uf, f) \geq 0$ for every $f \in D(U)$.

Then there exists at least one abstract potential operator $V$ which is a closed linear accretive extension of $U$; $V$ might coincide with $U$.

**Proof.** The proof is given in two steps. The first is to construct a maximal accretive extension $V$ of $U$ by virtue of R.S. Phillips' theory of
The first step. For every \( \lambda > 0 \) and \( f \in D(V) \), we have, by (4),

\[
\|\lambda U f + f\|^2 = (\lambda U f + f, \lambda U f + f) = \|\lambda U f\|^2 + 2Re(\lambda U f, f) + \|f\|^2 \\
\geq \|\lambda U f\|^2 + \|f\|^2 \geq \|\lambda U f\|^2 - 2Re(\lambda U f, f) + \|f\|^2 = \|\lambda U f - f\|^2.
\]

Hence the inverse \((\lambda U + I)^{-1}\) exists and moreover, the Cayley transform \( C \) defined through

\[
C(Uf + f) = (Uf - f)
\]

is a contraction operator mapping \( R(U + I) \) onto \( R(U - I) \). Let us define a bounded linear extension \( \hat{C} \) of \( C \):

(7) through continuity on \( R(U + I)^* \), and through putting \( \hat{C} \cdot g = 0 \) on the orthogonal complement of \( R(U + I) \).

This everywhere defined contraction operator \( \hat{C} \) cannot admit eigenvalue one. Assume the contrary and let \( \hat{C} \cdot f_0 = f_0 \) with \( \|f_0\| = 1 \). Then its adjoint operator \( \hat{C}^* \), which is also a contraction, must satisfy \( \hat{C}^* \cdot f_0 = f_0 \) because

\[
\|\hat{C}^* \cdot f_0 - f_0\|^2 = \|\hat{C}^* \cdot f_0\|^2 - 2Re(\hat{C}^* \cdot f_0, f_0) + \|f_0\|^2 \\
\leq \|f_0\|^2 - 2Re(f_0, \hat{C} \cdot f_0) + \|f_0\|^2 = 1 - 2 + 1 = 0.
\]

Thus we obtain, by (6) and (7),

\[
(f_0, (U - I)f) = (f_0, \hat{C} \cdot (U + I)f) = (\hat{C}^* \cdot f_0, (U + I)f) = (f_0, (U + I)f),
\]

hence \((f_0, f) = 0\) and so \( f_0 = 0 \) by (2).

Therefore the inverse \((I - \hat{C})^{-1}\) exists and so we can define a linear operator \( V \) through

(8) \( V \cdot (I - \hat{C}) f = (I + \hat{C}) f \).
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\( V \) is an extension of \( U \). In fact, we have, by (6), \( (I-C)=I-(U-I)(U+I)^{-1}=2(U+I)^{-1} \), that is, \( U=(I+C)(I-C)^{-1} \), proving that \( V \) is an extension of \( U \). Here the existence of \( (I-C)^{-1} \) is assured by that of \( (I-\hat{C})^{-1} \). We can prove that \( V \) is accretive. For, by putting \( f=(I-\hat{C})^{-1}g \) and observing (8) and the contraction property of \( \hat{C} \), we obtain

\[
Re(Vg, g) = Re((I+\hat{C})f, (I-\hat{C})f) = ||f||^2 - ||\hat{C} \cdot f||^2 \geq 0.
\]

We can also prove, by (8) and the boundedness of the operator \( \hat{C} \), that \( V \) is a closed linear operator. Moreover, by (8), we have \( (I+V) = I+(I+\hat{C})(I-\hat{C})^{-1}=2(I-\hat{C})^{-1} \), and so we obtain the existence theorem

\[
R(V+I)=D(I-\hat{C})=X \quad \text{(and also} \quad R(\lambda V+I)=X \quad \text{whenever} \quad \lambda > 0). \]

Hence the accretive extension \( V \) is maximal as regards its range \( R(\lambda V+I) \) for \( \lambda > 0 \).

The Second Step. We will show that \( V \) is an abstract potential operator following after the proof of Theorem 2 on p. 414–415 in K. Yosida [2].

\( V \) being accretive, we have, as in (5), \( ||\lambda Vf+f|| \geq ||\lambda Vf|| \) for every \( f \in D(V) \) and \( \lambda > 0 \). Hence, by (9), we can define a bounded linear operator

\[
J_\lambda=V(\lambda V+I)^{-1}
\]

satisfying

\[
||\lambda J_\lambda|| \leq 1.
\]

It is easy to see that \( J_\lambda \) is a pseudo-resolvent, i.e.,

\[
J_\lambda-J_\mu=(\mu-\lambda)J_\lambda J_\mu.
\]

Therefore, by (11), we can apply the abelian ergodic theorem to the effect that

\[
R(J_\mu)=\{x \in X; s\lim\limits_{\lambda \uparrow \infty} \lambda J_\lambda x = x\} \quad \text{for all} \quad \mu > 0,
\]
By $R(V) = R(U)$, we have $R(J_\lambda)^a = X$ by (10) and so, by (11) and (12), the null space of $J_\lambda$ consists of zero vector only, independently of $\lambda > 0$. Hence $J_\lambda$ is the resolvent of a linear operator, i.e.,

$$J_\lambda = (\lambda I - A)^{-1}, \quad \text{where } A = \lambda I - J_\lambda^{-1} \text{ is independent of } \lambda > 0.$$  

We have thus $D(A)^a = [R(J_\lambda)]^a = X$ and so, by (11), the operator $A$ is the infinitesimal generator of a contraction semi-group of class $(C_0)$. We can also prove that $R(A)^a = X$. For, we have, by (10) and (15),

$$(\lambda I - A)J_\lambda(\lambda Vf + f) = \lambda Vf + f = (\lambda I - A)Vf = \lambda Vf - AVf,$$

that is, 

$$-AVf = f \quad \text{whenever } f \in D(V),$$

proving that $R(A)^a = D(V)^a = D(U)^a = X$. Thus, by (14) and $AJ_\mu = (\mu J_\mu - I)$, we obtain $s\lim_{\lambda \downarrow 0} \lambda J_\lambda f = 0$ for all $f \in X$. This implies that the inverse $A^{-1}$ exists. In fact, the condition $Af_0 = 0$ is equivalent to $\lambda(\lambda I - A)^{-1}f_0 = f_0$ and hence $f_0 = s\lim_{\lambda \downarrow 0} \lambda J_\lambda f_0 = 0$.

Thus $-A^{-1}$ is an abstract potential operator. On the other hand, (16) shows that the inverse $V^{-1}$ exists. Hence, by $J_\lambda = (\lambda V + I)V^{-1} = \lambda I + V^{-1}$, we obtain $-A = V^{-1}$, completing the proof of our Theorem.

**Remark.** We shall verify (2), (3) and (4) for Newtonian and logarithmic potentials

$$(Uf)(y) = \int_{\mathbb{R}^n} K_n(|y - z|)f(z)dz \quad (n \geq 2),$$

$$K_n(r) = r^{2-n} \quad \text{for } n \geq 3, \quad \text{and } K_0(r) = \log r^{-1}.$$

The proof of $D(U)^a = R(U)^a = X = L^2(\mathbb{R}^n)$ can be obtained by making use of the fact that, for $0 < \delta_1 < \delta_2$, 

\[ (14) \quad R(I - \mu J_\mu)^a = \{ x \in X; s\lim_{\lambda \downarrow 0} \lambda J_\lambda x = 0 \} \quad \text{for all } \mu > 0. \]
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\[ u_{x, \delta_1, \delta_2}(y) = (K_n(\delta_1) - K_2(\delta_2))^{-1} \int_{R^n} K_n(|y - z|) \left( d\nu_{x, \delta_1}(z) - d\nu_{x, \delta_2}(z) \right) \]

is continuous in \( y \) satisfying

\[
\begin{align*}
u_{x, \delta_1, \delta_2}(y) &= 1 & \text{if } |y - x| \leq \delta_1, \\
&= 0 & \text{if } |y - x| \geq \delta_2, \\
0 < \nu_{x, \delta_1, \delta_2}(y) < 1 & \text{if } \delta_1 < |y - x| < \delta_2.
\end{align*}
\]

Here \( \nu_{x, \delta} \) is the unit measure uniformly distributed over the hypersurface of the sphere of centre \( x \) and radius \( \delta \) in \( R^n \).

The Gauss-Frostmann energy inequality

\[ \int_{R^n} (Uf)(y) \cdot f(y) \, dy \geq 0 \quad (n \geq 2) \]

holds good whenever \( f \in L^2(R^n) \) is of compact support satisfying \( \int_{R^n} f(y) \, dy = 0 \). It is easy to prove that such \( f \)'s constitute a strongly dense subset of \( L^2(R^n) \).

ANOTHER TREATMENT OF THE SECOND STEP (Added on 20 April, 1972). As in the above proof of the non-existence of the eigenvalue 1 for the operator \( \hat{C} \), we can show that \( \hat{C} \cdot f_0 = -f_0 \) implies \( \hat{C}^* \cdot f_0 = -f_0 \) and hence \( (f_0, Uf) = 0 \), proving by (3) the non-existence of the eigenvalue \(-1\) for \( \hat{C} \). Thus \( V = (I + \hat{C})(I - \hat{C})^{-1} \) given by (8) admits the inverse \( V^{-1} = (I - \hat{C})(I + \hat{C})^{-1} \). Hence we can prove that \( V \) is an abstract potential operator without appealing to the abelian ergodic theorem.

Remark (added during the proof). On reading the pre-print, Prof. K. Sato gave interesting comments and extensions. See his paper to appear.

References

