On the First Initial-Boundary Value Problem of the Generalized Burgers' Equation

By
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§1. Introduction and Notations

E. Hopf discussed in details on the Cauchy problem of Burgers' equation in his famous paper [5]. Since then, many papers on the equation and its related topics have been published. However, they have not treated the initial-boundary value problem of it. The author previously discussed on the first initial-boundary value problem of this equation in [17]. Recently, N. Itaya has shown the existence and the uniqueness, in a certain sense, of the temporally global solution of the Cauchy problem of the following generalized Burgers' equation:

\[
\begin{align*}
(1.1)^1 & \quad \frac{\partial v}{\partial t}(x, t) = -\frac{\mu}{\rho(x, t)} \frac{\partial^2}{\partial x^2} v(x, t) - v(x, t) \frac{\partial}{\partial x} v(x, t), \\
(1.1)^2 & \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0, \quad (\mu \text{ is a positive constant})
\end{align*}
\]

in [11] (cf. [9], [10]). Stimulated by his work, the author attempts to discuss on the first initial-boundary value problem of (1.1) in \([0, X] \subset \mathbb{R}^1\), especially from the view-point of the temporally global behavior of the solution of (1.1).

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Notations. The functions considered in this paper should be understood to be defined in \([0, X]\) or \([0, X] \times [0, T]\) (\(0 < X < +\infty, 0 < T < +\infty\)) and continuously differentiable as many times as necessary.

\[
\begin{aligned}
\Omega &= (0, X), \quad \bar{\Omega} = [0, X], \quad S^0_\Omega = \{0\} \times [0, T], \\
S^T_\Omega &= \{X\} \times [0, T], \quad S_T = S^0_\Omega \cup S^T_\Omega, \quad \Gamma_T = S_T \cup \Omega \times \{0\}, \\
Q_T &= \Omega \times (0, T), \quad \bar{Q}_T = \bar{\Omega} \times [0, T].
\end{aligned}
\]

\[
|u(x)|^{(0)} = \sup_{\Omega} |u(x)|,
\]

\[
|u(x)|^{(s)} = \sup_{\Omega, x \neq x'} \frac{|u(x) - u(x')|}{|x - x'|^s}.
\]

\[
\|u(x)\|^{(n)} = \sum_{i=0}^{n} \|D^i_x u(x)\|^{(0)},
\]

\[
\|u(x)\|^{(n+a)} = \|u(x)\|^{(n)} + |D^n_x u(x)|^{(a)} \quad (n = 0, 1, \ldots).
\]

\[
|v(x, t)|^{(0)} = \sup_{\bar{Q}_T} |v(x, t)|,
\]

\[
|v(x, t)|^{(s)} = \sup_{\bar{Q}_T, x \neq x'} \frac{|v(x, t) - v(x', t)|}{|x - x'|^s},
\]

\[
|v(x, t)|^{(s/2)} = \sup_{\bar{Q}_T, t \neq t'} \frac{|v(x, t) - v(x, t')|}{|t - t'|^{s/2}},
\]

\[
|v(x, t)|^{(s/2)} = |v(x, t)|^{(s/2)} + \|v(x, t)|^{(s/2)}.
\]
where \( r \) and \( s \) are non-negative integers.

\[
\begin{align*}
H^n & = \{ u(x) \| u \|^{(n)} < +\infty \}, \\
H^{n+s} & = \{ u(x) \| u \|^{(n+s)} < +\infty \}, \\
H^{n+s} & = \{ v(x, t) \| v \|^{(n+s)} < +\infty \}, \\
\hat{H}^{n+s} & = \{ v(x, t) \| v \|^{(n+s)} < +\infty \}, \\
B^n_T & = \{ v(x, t) \sum_{r+s=0}^{n} \| D^r_x D^s_t v \|^{(0)} < +\infty \}, \\
B^{n+s}_T & = \{ v(x, t) \sum_{r+s=0}^{n} \| D^r_x D^s_t v \|^{(0)} + \sum_{r+s=n} \| D^r_x D^s_t v \|^{(s)} < +\infty \}.
\end{align*}
\]

Other notations, not described above, will be explained where they appear.

\section*{§ 2. Preliminaries}

We assume for (1.1) the following initial-boundary conditions:

\begin{align}
(2.1) & \quad v(x, 0) = v_0(x) \in H^{2+s}, \quad \rho(x, 0) = \rho_0(x) \in H^1, \\
(0 < \bar{\rho}_0 & \equiv \inf_{\mathcal{D}} \rho_0(x) \leq \rho_0(x) \leq \bar{\rho}_0 \equiv |\rho_0(x)|^{(0)}) \notag \\
(2.2) & \quad v(0, t) = v(X, t) = 0, \\
\end{align}

and for \( v(x, t) \) the following compatible condition:

\begin{align}
(2.3) & \quad v_{xx}(x, t)\big|_{S^T_0} = v_{xx}(x, t)\big|_{S^T_X} = 0.
\end{align}

Let \((v, \rho)\) be a solution in \(H^{2+s}_T \times B^1_T\) of (1.1) satisfying the initial-boundary conditions (2.1) and (2.2), and \(\bar{x}(\tau; x, t)\) be the solution curve of the characteristic equation for (1.1)\(^2\) as a linear equation in \(\rho\):

\begin{align}
\frac{d}{d\tau} \bar{x}(\tau; x, t) = v(\bar{x}(\tau; x, t), \tau) \quad (0 \leq \tau \leq t \leq T), \\
\bar{x}(t; x, t) = x.
\end{align}

Since \(v \in H^{2+s}_T\), the solution curve for (2.4) starting at an arbitrary point
(x, t) \in \bar{Q}_T \text{ is unique. By (2.4) we have}

\begin{equation}
\frac{\partial x}{\partial \tau}(\tau; x, t) = \exp \left\{ - \int_{\tau}^{t} v_x(x(\tau'; x, t), \tau') d\tau' \right\} .
\end{equation}

If v(x, t) \in H^{2+a}_1 is given in (1.1)^2, then \( \rho(x, t) \) or \( \rho_0(x, t) \) is uniquely determined by the formula:

\begin{equation}
\rho(x, t) = \rho_0(x, t) = \rho_0(x(0; x, t)) x_0(0; x, t) = \rho_0(x(0; x, t)) \exp \left\{ - \int_{0}^{t} v_x(x(\tau'; x, t), \tau') d\tau' \right\} .
\end{equation}

For simplicity, we put

\begin{equation}
\begin{cases}
\bar{\rho}(\tau; x, t) = \rho(x(\tau; x, t), \tau), \\
\bar{\nu}(\tau; x, t) = v(x(\tau; x, t), \tau),
\end{cases}
\end{equation}

By (1.1)^2, (2.4) and (2.6), the following fundamental lemma holds (cf. [11]).

**Lemma 2.1.** If \((v, \rho)\) is a solution of (1.1) in \( H^{2+a}_1 \times B_1^1 \) with (2.1) and (2.2), then the following equation holds:

\begin{equation}
\mu \int_{0}^{t} \bar{v}_{xx}(\tau; x, t) \bar{x}_x(\tau; x, t) d\tau = \rho(x, t)\{v(x, t) - v_0(x_0(x, t))\},
\end{equation}

where

\begin{equation}
x_0(x, t) = x(0; x, t).
\end{equation}

Concerning \( \rho(x, t) \), we have, by (2.6) and simple calculations,

**Lemma 2.2.** If \((v, \rho_v)\) and \((w, \rho_w)\) are solutions of (1.1) with (2.1) and (2.2) in \( H^{2+a}_1 \times B_1^1 \) (cf. (2.6)). then

\begin{equation}
\left| \frac{1}{\rho_v} - \frac{1}{\rho_w} \right|^{(0)}_{\bar{Q}_T} \leq C_1(T_0; v, w) \|v - w\|^{(1)}_{\bar{Q}_T} (0 \leq T_0 \leq T),
\end{equation}

where \( C_1(T_0; v, w) \downarrow 0 \) as \( T_0 \downarrow 0 \).

From (2.6), \((v, \rho) \in H^{2+a}_1 \times B_1^1 \) implies \( \mu |\rho| \in H^{2}_1 \).

Let
(2.11) \[ Z^0(x - \xi, t; \xi, \tau; \mu/\rho) = \frac{1}{2\sqrt{\pi}} (\mu/\rho(\xi, \tau))^{-1/2} (t - \tau)^{-1/2} \times \]
\[ \times \exp \left\{ -\frac{(x - \xi)^2}{4} \frac{\mu}{\rho}(\xi, \tau)(t - \tau) \right\} \quad (0 \leq \tau \leq t \leq T) \]

be the parametrix of the linear parabolic equation:

(2.12) \[ \frac{\partial w}{\partial t}(x, t) = \frac{\mu}{\rho}(\xi, \tau) \frac{\partial^2 w}{\partial x^2}(x, t). \]

Then the parametrix of (2.12) with (2.2) is given by

(2.13) \[ Z(x - \xi, t; \xi, \tau; \mu/\rho) = \sum_{n=-\infty}^{\infty} \left\{ Z^0(x - \xi + 2nX, t; \xi, \tau; \mu/\rho) - \right\]
\[ - Z^0(x + \xi + 2nX, t; \xi, \tau; \mu/\rho) \right\}. \]

As is well known, \( Z^0 \) has the following properties:

**Lemma 2.3.**

(i) \[ |D_x^m Z^0(x - \xi, t; \xi, \tau; \mu/\rho)| \leq C^m(t - \tau)^{-\frac{1+m}{2}} \times \]
\[ \times \exp \left\{ -\frac{(x - \xi)^2}{8} \frac{\mu}{\rho} \right\}^{(0)} (t - \tau), \]

(ii) \( \text{for } t > t' > \tau, \)
\[ |D_x^m Z^0(x - \xi, t; \xi, \tau; \mu/\rho) - D_x^m Z^0(x - \xi, t'; \xi, \tau; \mu/\rho)| \leq C^m(t - t') \times \]
\[ \times (t' - \tau)^{-\frac{3+m}{2}} \exp \left\{ -\frac{(x - \xi)^2}{8} \frac{\mu}{\rho} \right\}^{(0)} (t - \tau), \]

(iii) \( \text{for } \xi > \xi' > \xi'' \)
\[ |D_x^m Z^0(x - \xi, t; \xi, \tau; \mu/\rho) - D_x^m Z^0(x'' - \xi, t; \xi, \tau; \mu/\rho)| \leq C^m|x - x''| \times \]
\[ \times (t - \tau)^{-\frac{2+m}{2}} \exp \left\{ -\frac{(x'' - \xi)^2}{8} \frac{\mu}{\rho} \right\}^{(0)} (t - \tau), \]

where

\[ x'' = \begin{cases} x & \text{(if } |x - \xi| < |x' - \xi|) \\ x' & \text{(otherwise)} \end{cases} \]

(iv) \[ |D_x^k D_x^l Z^0(z; \xi, \tau; \mu/\rho) - D_x^k D_x^l Z^0(z; \xi', \tau; \mu/\rho)| \leq C^{k,m} |\xi - \xi'|^k \times \]
\[ \times (t - \tau)^{-\frac{1+m+2k}{2}} \exp \left\{ -\frac{z^2}{8} \frac{\mu}{\rho} \right\}^{(0)} (t - \tau). \]
Using the relation (2.13) and Lemma 2.3, $Z$ is estimated as follows:

**Lemma 2.4.**

$$|D_n^m Z(x - \xi, t; \xi, \tau; \mu/\rho)| \leq C_2^{(m)}(t - \tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp \left\{ -\frac{(x - \xi)^2}{16} \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} .$$

**Proof.** In general, $-X \leq x \leq 2X$ implies $\frac{x^2}{2} + \frac{n^2 X^2}{2} \leq (x + 2nX)^2$ except the case that $X < x \leq 2X$ and $n = -1$. We have for any $n$,

$$|D_n^m Z^0(x - \xi + 2nX, t; \xi, \tau; \mu/\rho)| \leq C_2^{(m)}(t - \tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp \left\{ -\frac{(x - \xi)^2}{16} \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} \times$$

$$\times \exp \left\{ -X^2 n^2 / 16 \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} ,$$

and for any $n(\neq -1)$, $|D_n^m Z^0(x + \xi + 2nX, t; \xi, \tau; \mu/\rho)|$ has the same bound. Hence we have

$$(2.14) \sum_{n=-\infty}^{\infty} |D_n^m Z^0(x - \xi + 2nX, t; \xi, \tau; \mu/\rho)| \leq C_2^{(m)} \left\{ 1 + \frac{4(\pi |\mu/\rho|_{T}^{(0)} T)^{1/2}}{X} \right\} \times$$

$$\times (t - \tau)^{-\frac{1+m}{2}} \exp \left\{ -\frac{(x - \xi)^2}{16} \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} ,$$

$$(2.15) \left( \sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty} \right) |D_n^m Z^0(x + \xi + 2nX, t; \xi, \tau; \mu/\rho)| \leq$$

$$\leq C_2^{(m)} \left\{ 1 + \frac{4(\pi |\mu/\rho|_{T}^{(0)} T)^{1/2}}{X} \right\} (t - \tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp \left\{ -\frac{(x - \xi)^2}{16} \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} .$$

In a direct way for $n = -1$, we obtain

$$(2.16) |D_n^m Z^0(x + \xi - 2X, t; \xi, \tau; \mu/\rho)| \leq C_2^{(m)}(t - \tau)^{-\frac{1+m}{2}} \times$$

$$\times \exp \left\{ -\frac{(x - \xi)^2}{16} \left| \frac{\mu}{\rho} \right|_{T}^{(0)} (t - \tau) \right\} .$$
Thus, by (2.14), (2.15) and (2.16), we have finally
\[
|D^n_x Z(x, \xi, t; \xi, \tau; \mu/\rho)| \leq C'_0(t-\tau)^{-1+m/2} \times \\
\times \exp\left\{ -\left(\frac{\xi}{2}\right)^2 / 16 \left| \frac{\mu}{\rho} \right|^{(0)} T (t-\tau) \right\},
\]
where \( C'_0 = C'_2 \left\{ 3 + \frac{8(\pi |\mu/\rho| \mu \nu)}{X(t)} \right\} \).

Making use of the same procedure as in the proof of Lemma 2.4, we have

**Lemma 2.5.**
(i) For \( t > t' > \tau \),
\[
|D^n_x Z(x, \xi, t; \xi, \tau; \mu/\rho)| \leq C'_0(t-t')^{-1+m/2} \times \\
\times \left( \frac{\xi}{2}, t-t' \right)^{1/2} \exp\left\{ -\left(\frac{\xi}{2}\right)^2 / 16 \left| \frac{\mu}{\rho} \right|^{(0)} (t-t') \right\},
\]
(ii) \( |D^n_x Z(x, \xi, t; \xi, \tau; \mu/\rho)| \leq C'_0 |x-x'| \times \\
\times \left( \frac{\xi}{2}, t-t' \right)^{1/2} \exp\left\{ -\left(\frac{\xi}{2}\right)^2 / 16 \left| \frac{\mu}{\rho} \right|^{(0)} (t-t') \right\},
\]
(iii) \( |D^n_x Z(x-\xi, t; \xi, \tau; \mu/\rho)| \leq C'_0 |\xi-\xi'| \times \\
\times \left( \frac{\xi}{2}, t-t' \right)^{1/2} \exp\left\{ -\left(\frac{\xi}{2}\right)^2 / 16 \left| \frac{\mu}{\rho} \right|^{(0)} (t-t') \right\}.
\]
Furthermore, by Lemma 2.5, we have

**Lemma 2.6.**
\[ \left| \int_0^X D^n_x D^n_x Z(x-\xi, \xi, \tau; \mu/\rho) d\xi \right| \leq C_{10} (t-\tau)^{-\frac{2k+m-n}{2}}, \quad \text{for} \ 2k+m>n. \]

It is easily seen that the fundamental solution \( \Gamma(x, t; \xi, \tau; \mu/\rho) \) of
\[
\frac{\partial \Gamma}{\partial t} (x, t) = \frac{\mu}{\rho(x, t)} \frac{\partial^2 \Gamma}{\partial x^2} (x, t) \quad (T \geq t > 0)
\]
with (2.2) is given in the form
(2.18) \[ \Gamma(x, t; \xi, \tau; \mu/\rho) = Z(x - \xi, t; \xi, \tau; \mu/\rho) + \]
\[ + \int_{\tau}^{t} d\sigma \int_{0}^{x} K(x, t; y, \sigma; \mu/\rho) \Phi(y, \sigma; \xi, \tau; \mu/\rho) dy. \]

The function \( \Phi \) satisfies a Volterra-type integral equation:

(2.19) \[ \Phi(x, t; \xi, \tau; \mu/\rho) = K(x, t; \xi, \tau; \mu/\rho) + \]
\[ + \int_{\tau}^{t} d\sigma \int_{0}^{x} K(x, t; y, \sigma; \mu/\rho) \Phi(y, \sigma; \xi, \tau; \mu/\rho) dy, \]

where

(2.20) \[ K(x, t; \xi, \tau; \mu/\rho) = \left\{ \frac{\mu}{\rho(x, t)} - \frac{\mu}{\rho(\xi, \tau)} \right\} D_{x}^{2} Z(x - \xi, t; \xi, \tau; \mu/\rho). \]

The function \( \Phi \) is given in the form

(2.21) \[ \Phi(x, t; \xi, \tau; \mu/\rho) = \sum_{m=0}^{\infty} K_{m}(x, t; \xi, \tau; \mu/\rho), \]

where

(2.22) \[ \left\{ \begin{array}{l}
K_{0}(x, t; \xi, \tau; \mu/\rho) = K(x, t; \xi, \tau; \mu/\rho), \\
K_{m}(x, t; \xi, \tau; \mu/\rho) = \int_{\tau}^{t} d\sigma \int_{0}^{x} K(x, t; y, \sigma; \mu/\rho) K_{m-1}(y, \sigma; \xi, \tau; \mu/\rho) dy.
\end{array} \right. \]

We shall prove the convergence of the series in (2.21) and estimate \( \Gamma \) similarly to the way in which we did \( Z \) in Lemmas 2.5 and 2.6.

**Lemma 2.7.** \[ |K(x, t; \xi, \tau; \mu/\rho)| \leq C_{11} (t - \tau)^{-\frac{3 - \frac{\mu}{\rho}}{2}} \times \]
\[ \times \exp \left\{ x(-\xi)^{2}/32 \left| \frac{\mu}{\rho} \right|^{(0)} (t - \tau) \right\}. \]

**Proof.** This follows directly from the Hölder continuity of \( \frac{\mu}{\rho} \) and Lemma 2.4 \((m=2)\). Q.E.D.

Proceeding similarly to evaluate \( K_{1}, K_{2}, \) etc., for any integer \( m \geq 0 \) we have

**Lemma 2.8.**
\[ |K_m(x, t; \xi, \tau; \frac{\mu}{\rho})| \leq \left[ \frac{2(2\pi |\frac{\mu}{\rho}|(0)_{x/2}^{1/2} C_{11} \Gamma(x/2) T^{x/2})^{m+1}}{2(2\pi |\frac{\mu}{\rho}|(0)_{x/2}^{1/2} \Gamma((m+1)x/2)} T^{-x/2} \right] \times \\
\times (t-\tau)^{-3-x/2} \exp \left\{ -(x-\xi)^2/32 |\frac{\mu}{\rho}|(0)_{x} (t-\tau) \right\}.
\]

From Lemma 2.8, it follows that the series expansion of \( \Phi(x, t; \xi, \tau; \frac{\mu}{\rho}) \) is uniformly convergent for \( T \geq t \geq 0 \) and \( \Phi \) is evaluated as follows:

**Lemma 2.9.** \[ |\Phi(x, t; \xi, \tau; \frac{\mu}{\rho})| \leq C_{12} (t-\tau)^{-3-x/2} \times \\
\times \exp \left\{ -(x-\xi)^2/32 |\frac{\mu}{\rho}|(0)_{x} (t-\tau) \right\}, \]

where \( C_{12} = \sum_{m=0}^{\infty} \cdots \) in Lemma 2.8.

Thus, using Lemmas 2.4 and 2.9, we have

**Lemma 2.10.** \[ |D^x \Gamma(x, t; \xi, \tau; \frac{\mu}{\rho})| \leq C_{13}^x (t-\tau)^{-1-x} \times \\
\times \exp \left\{ -(x-\xi)^2/32 |\frac{\mu}{\rho}|(0)_{x} (t-\tau) \right\}. \]

In order to study \( \Gamma \) in more detail, we shall need the following lemmas.

**Lemma 2.11.** \[ |K(x, t; \xi, \tau; \frac{\mu}{\rho}) - K(x', t; \xi, \tau; \frac{\mu}{\rho})| \leq C_{14} (t-\tau)^{-3/2} \times \\
\times |x-x'| \exp \left\{ -(x-\xi)^2/32 |\frac{\mu}{\rho}|(0)_{x} (t-\tau) \right\}. \]

**Proof.** From (2.20), it follows that the lemma holds, by using the Hölder continuity of \( \frac{\mu}{\rho} \), Lemmas 2.4 and 2.5. Q.E.D.

By induction, we obtain

**Lemma 2.12.** \[ |\Phi(x, t; \xi, \tau; \frac{\mu}{\rho}) - \Phi(x', t; \xi, \tau; \frac{\mu}{\rho})| \leq C_{15} (t-\tau)^{-3/2} \times \\
\times |x-x'| \exp \left\{ -(x-\xi)^2/32 |\frac{\mu}{\rho}|(0)_{x} (t-\tau) \right\}. \]

As a result, by using Lemmas 2.4, 2.5, 2.6, 2.9 and 2.12, we have, after
Lemma 2.13.

(i) \(|\Gamma(x, t; \xi, \tau; \mu/\rho) - \Gamma(x, t'; \xi, \tau; \mu/\rho)| \leq C_0^{(0)}(t-t')(t-\tau)^{-\frac{3}{2}} \times \exp\{-\frac{(x-\xi)^2}{32}\frac{\mu}{\rho}(t-\tau)\} ,

(ii) \(|D_2^{\alpha}\Gamma(x, t; \xi, \tau; \mu/\rho) - D_2^{\alpha}\Gamma(x, t'; \xi, \tau; \mu/\rho)| \leq C_0^{(0)}(t-t')(t-\tau)^{-\frac{3}{2}} + \frac{(t-t')^{\frac{2-\alpha}{2}}}{\rho}(t-\tau)^{-\frac{3}{2}} \exp\{-\frac{(x-\xi)^2}{32}\frac{\mu}{\rho}(t-\tau)\} .

§ 3. The Existence of a Temporally Local Solution of (1.1), (2.1) and (2.2)

In the first place, we construct the sequence \(\{v^n(x, t)\}\) such that

\[
\begin{cases}
  v^0(x, t) = v_0(x) \in H^{\frac{n}{2}+\delta}, \\
  v^n(x, t) = v_0(x) + \int_0^t \int_0^x \Gamma(x, t; \xi, \tau; \mu/\rho) N_{n-1}(\xi, \tau) d\xi
\end{cases}
\]

\((0 \leq t \leq T)\), where \(\rho_{n-1} = \rho v_{n-1}\) and \(N_{n-1} = \frac{\mu}{\rho_{n-1}} v_0^n + v_{n-1}v_{n-1}^{n-1}\) (cf. (2.6)).

We also assume (2.3) for \(t=0\), i.e.,

\[v_0^n(x)|_{x=0} = v_0^n(x)|_{x=x} = 0.\]

The functions \(v^n\) (\(n = 1, 2, \ldots\)) satisfy

\[
v^n_t = \frac{\mu}{\rho_{n-1}} v^n_{xx} - v^n_{n-1}v_{n-1}^{n-1}
\]

and

\[
v^n(0, 0) = v_0(x), \quad v^n(0, t) = v^n(x, t) = 0.
\]

By using the lemmas obtained in §2, especially Lemmas 2.10 and 2.13, we have

Lemma 3.1.
(i) \[ \left\| \nu^n \right\|_{L^1 \rightarrow T} \leq C_{17,1} \left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T, \left\| N_{n-1} \right\|_T^{(0)} + \left\| \nu_0 \right\|_T^{(1)} \right) \]

where \[ \left\| \frac{\mu}{\rho_{n-1}} \right\|_T = \left\| \frac{\mu}{\rho_{n-1}} \right\|_T^{(0)} + \left\| \mu \right\|_T^{(2)} \]

(ii) \[ \left\| \nu^n \right\|_{L^2 \rightarrow T} \leq C_{17,2} \left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T, \left\| N_{n-1} \right\|_T^{(2)} + \left\| \nu_0 \right\|_T^{(0)} \right) \]

(iii) \[ \left\| \nu^n \right\|_{L^2 \rightarrow T} \leq C_{17,3} \left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T, \left\| N_{n-1} \right\|_T^{(0)} \right) \]

(iv) \[ \left\| \nu^n \right\|_{L^2 \rightarrow T} \leq 2 \left\| \nu_0 \right\|_T^{(0)} + \left\| \nu_0 \right\|_T^{(0)} + C_{17,4} \left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T, \left\| N_{n-1} \right\|_T^{(0)} \right) \]

Remark. The constants \( C_{17,i} \) \((i = 1, 2, 3, 4, 5)\) increase monotonically as each argument increases and \( C_{17,i} \to 0 \) as \( T \to 0 \).

It is easy to see that \( \nu^n \in H^{2+z} \) implies \( \frac{\mu}{\rho_{n-1}} \in H^{2+z} \) and \( N_{n-1} \in H^{2+z} \). Thus, by the above lemma, we see clearly that \( \nu^n \in H^{2+z} \) and also \( \nu^n \in H^{2+z} \). Hence by induction we obtain

Lemma 3.2. \( \nu^n(x, t) \in H^{2+z} \).

Now, we take an arbitrary constant \( M_0 \) such that

\[ \left\| \nu_0 \right\|_2 < M_0 < + \infty \] \hspace{1cm} (3.4)

As for \( \left\| \frac{\mu}{\rho_{n-1}} \right\|_T \), it holds that

\[ \left\| \frac{\mu}{\rho_{n-1}} \right\|_T \leq \frac{\bar{\rho}_0}{\mu} \exp \{ T \left| \nu_x^{n-1} \right|_T^{(0)} \} + (\bar{\rho}_0)^{-1} \exp \{ T \left| \nu_x^{n-1} \right|_T^{(0)} \} + 2 \mu \left[ (\bar{\rho}_0)^{-2} \left| \nu_x^{n-1} \right|_T^{(0)} + (\bar{\rho}_0)^{-1} T \left| \nu_x^{n-1} \right|_T^{(0)} \exp \{ 2T \left| \nu_x^{n-1} \right|_T^{(0)} \} \right] + (\bar{\rho}_0)^{-1} \exp \{ T \left| \nu_x^{n-1} \right|_T^{(0)} \} \right] + 2 \mu \left[ (\bar{\rho}_0)^{-1} (1 + \left| \nu_x^{n-1} \right|_T^{(0)} + \left| \nu_x^{n-1} \right|_T^{(0)} \right) \times \exp \{ T \left| \nu_x^{n-1} \right|_T^{(0)} \} + \left| \nu_x^{n-1} \right|_T^{(0)} \right] (\bar{\rho}_0)^{-2} \left| \rho_0 \right|_T^{(0)} + \right)
\]
+ (\bar{\rho}_0)^{-1} T |v_{n-1}^x| \exp\{2T |v_{n-1}^x| f(0)\}\].

If we assume that \(\|u_n\|_{L^2} < M_0\), then we have

\[
\left\| \frac{\mu}{\rho_{n-1}} \right\|_{L^2(T)} \leq \left\{ \frac{\bar{\rho}_0}{\mu} + \mu(5 + 2M_0)(\bar{\rho}_0)^{-1} \right\} \exp\{M_0 T\} + 2\mu(1 + M_0) \left\{ (\bar{\rho}_0)^{-2} |\rho_0|^{(0)} + (\bar{\rho}_0)^{-1} M_0 T e^{2M_0 T} \right\} \equiv A(T, M_0).
\]

By Lemma 3.1, we have \(\|v^n\|_{L^2} \leq \|v_0\|_{L^2} + (C_{17,1} + C_{17,2}) \|N_{n-1}\|_{L^2}\). Furthermore, \(\|v^{n-1}\|_{L^2} < M_0\) implies

\[
\|N_{n-1}\|_{L^2} \leq C_{18} \left( T, M_0, \left\| \frac{\mu}{\rho_{n-1}} \right\|_{L^2(T)} \right),
\]

where \(C_{18}\) is monotonically increasing in each argument and \(C_{18} \downarrow 'a certain positive constant' as \(T \downarrow 0\). Therefore, we have

\[
\|v^n\|_{L^2(T)} \leq \|v_0\|_{L^2} + (C_{17,1}^* + C_{17,2}^*) C_{18} (T, M_0, A(T, M_0)),
\]

where \(C_{17,i}^* = C_{17,i}(T, A(T, M_0))\) \((i = 1, 2)\). Hence, for a sufficiently small \(T_1 \in (0, T]\)

\[
\|v^n\|_{L^2(T_1)} \leq M_0.
\]

By induction, for some \(T_2 \in (0, T]\)

\[
\|v^n\|_{L^2(T_2)} \leq M_0 \quad (n = 1, 2, 3,...),
\]

For simplicity we choose \(T\) from the beginning in such a way that \(T = T_2\).

In the next place, by (3.2) the differences \(v^n - v^{n-1}\) satisfy the equation:

\[
(v^n - v^{n-1})_t = \frac{\mu}{\rho_{n-1}} (v^n - v^{n-1})_{xx} + \bar{N}_{n-1}, \quad (n = 1, 2, 3,...),
\]

where \(\bar{N}_{n-1} = \left( \frac{\mu}{\rho_{n-1}} - \frac{\mu}{\rho_{n-2}} \right) v_{n-1}^{n-1} + v^{n-1} (v^{n-1} - v^{n-2})_x + (v^{n-1} - v^{n-2}) v_{n-2}^{n-2} \in H_T^n\) \((n = 2, 3, 4,...)\), and by (3.3) it also satisfies the initial-boundary conditions:
(3.10) \[(v^n - v^{n-1})(x, 0) = 0, \quad (v^n - v^{n-1})(0, t) = (v^n - v^{n-1})(X, t) = 0.\]

In the same way as we did in §2, we can construct the fundamental solution of (3.9) and (3.10), and the solution of (3.9) and (3.10) is uniquely expressed by

\[(3.11) \quad (v^n - v^{n-1})(x, t) = \int_0^t d\tau \int_0^x \Gamma(x, t; \xi, \tau; \mu/\rho_{n-1}) \tilde{N}_{n-1}(\xi, \tau) d\xi.\]

Similarly to Lemma 3.1, we have the following lemma.

**Lemma 3.3.** \[\|v^n - v^{n-1}\|_1 \leq C_{19} \left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T \right) |\tilde{N}_{n-1}|_T^{(0)}\] \[\quad (n = 1, 2, 3, \ldots), \] where \(C_{19}\) has the same property as \(C_{17,1}\).

Directly by the above lemma, we have

\[(3.12) \quad \|v^n - v^{n-1}\|_T \leq C_{19}(T, A(T, M_0)) |\tilde{N}_{n-1}|_T^{(0)}.\]

**Lemma 3.4.**

\[|\tilde{N}_{n-1}|_T^{(0)} \leq C_{20}\left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T + \left\| \frac{\mu}{\rho_{n-2}} \right\|_T \right) \|v^{n-1} - v^{n-2}\|_T^{(1)},\]

where \(C_{20}\) has the same property as \(C_{18}\).

**Proof.**

\[|\tilde{N}_{n-1}|_T^{(0)} \leq \left| \frac{\mu}{\rho_{n-1}} - \frac{\mu}{\rho_{n-2}} \right|_T^{(0)} |\tilde{v}^{n-1}_x|_T^{(0)} + |v^{n-1}|_x^{(0)} \left| (v^{n-1} - v^{n-2})_x \right|_x^{(0)} + |v^{n-1} - v^{n-2}|_x^{(0)} |\tilde{v}^{n-2}_x|_x^{(0)}.\]

Using Lemma 2.2, we have

\[|\tilde{N}_{n-1}|_T^{(0)} \leq C_{20}\left( T, \left\| \frac{\mu}{\rho_{n-1}} \right\|_T + \left\| \frac{\mu}{\rho_{n-2}} \right\|_T \right) \|v^{n-1} - v^{n-2}\|_T^{(1)}.\]

Q.E.D.

Hence we have

\[(3.13) \quad |\tilde{N}_{n-1}|_T^{(0)} \leq C_{20}(T, A(T, M_0)) \|v^{n-1} - v^{n-2}\|_T^{(1)}.\]

Combining (3.12) and (3.13), we obtain
\begin{equation}
\|v^n - v^{n-1}\|_{\mathcal{H}_I} \leq C_{21} \|v^{n-1} - v^{n-2}\|_{\mathcal{H}_I},
\end{equation}

where $C_{21}$ has the same property as $C_{19}$.

By induction and (3.14), we have
\begin{equation}
\|v^n - v_0\|_{\mathcal{H}_I} \leq C_{21}^{-1} \|v^1 - v^0\|_{\mathcal{H}_I}.
\end{equation}

Since $C_{21} \downarrow 0$ as $T \downarrow 0$, it holds, for some $T_0 \in (0, T]$, that $C_{21}(T_0, A(T_0, (M_0)) < 1$, whereas by Lemma 3.1
\begin{equation}
\|v^1 - v^0\|_{\mathcal{H}_I} \leq C_{17,1}(T_0, A(T_0, M_0)|N_0|^0)
\end{equation}

and
\begin{equation}
|N_0|^0 \leq (\bar{\rho}_0)^{-1} M_0 \exp \{M_0 T_0\} + M_0^2 < +\infty.
\end{equation}

Thus
\begin{equation}
\sum_{n=1}^{\infty} C_{21}^{-1} \|v^n - v^0\|_{\mathcal{H}_I} < +\infty.
\end{equation}

Therefore, \{v^n\} converges to an element $v$ of $H^{2+\varepsilon}_I$ as $n \to \infty$. As is known the expression (2.5), \{\rho^n\} converges to an element $\rho_v$ of $B_1^1$ as $n \to \infty$. $N_n$ also converges to $N = \frac{H}{\rho_v} v_0^0 + v_0^x$. Hence, by the formula (2.16), (2.18), (2.25), (2.26), (2.27), (2.28) and (2.29), $Z^0(x - \xi, t; \xi, \tau; \rho/\rho_{n-1})$, $Z(x - \xi, t; \xi, \tau; \rho/\rho_{n-1})$, $K_m(x, t; \xi, \tau; \rho/\rho_{n-1})$, $K(x, t; \xi, \tau; \rho/\rho_{n-1})$, $K(x, t; \xi, \tau; \rho/\rho_{n-1})$ converge to $Z^0(x - \xi, t; \xi, \tau; \rho/\rho_v)$, $Z(x - \xi, t; \xi, \tau; \rho/\rho_v)$, respectively, as $n \to \infty$. Thus by (3.1), it holds, for $0 \leq t \leq T$, that
\begin{equation}
v(x, t) = v_0^0(x) + \int_0^t d\tau \int_0^x \Gamma(x, t; \xi, \tau; \rho_v) \left\{ \frac{H}{\rho_v} v_0^0(\xi) - v_0^x(\xi, \tau) \right\} d\xi.
\end{equation}

As a result, we have

**Theorem 3.1.** For some $T \in (0, \infty)$, there exists a solution of (1.1), (2.1) and (2.2) in $H^{2+\varepsilon}_I \times B_1^1$.

**Remark.** For $v$, Lemma 3.1 also holds.
§4. The Uniqueness of the Solution of (1.1), (2.1) and (2.2) in $H^k_x \times B_1^T$.

Now let us direct ourselves towards the problem of uniqueness concerning the system (1.1) of differential equations. (cf. [11]). We assume that there exist two solutions $(v, \rho_v)$ and $(w, \rho_w)$ of (1.1) in $H^k_x \times B_1^T$ satisfying one and the same initial-boundary conditions (2.1) and (2.2). The difference $v - w$ satisfies the equation (3.9) and the initial-boundary condition (3.10) as $v^n$ and $v^{n-1}$ are replaced by $v$ and $w$ respectively. Then $v - w$ can be uniquely expressed in the form (3.11) as $v^n$ and $v^{n-1}$ are replaced by $v$ and $w$ respectively, i.e.,

$$ (v - w)(x, t) = \int_0^t \int_0^x \Gamma(x, t; \xi, \tau; \mu/\rho_v) \tilde{N}(\xi, \tau) d\xi d\tau, $$

where $\tilde{N}(x, t) = \begin{pmatrix} \frac{\mu}{\rho_v} - \frac{\mu}{\rho_w} \end{pmatrix}(x, t) - v(x, t)(v-w)_x(x, t) + (v-w)(x, t)v_x(x, t)$.

As for $v - w$, in a way analogous to that used in the preceding section for $v^n - v^{n-1}$, we obtain

**Lemma 4.1.**

(i) $\|v - w\|_{T_0}^{(1)} \leq C_{22} \left( T_0, \left\| \frac{\mu}{\rho_v} \right\|_T + \left\| \frac{\mu}{\rho_w} \right\|_T \right) \|\tilde{N}\|_{T_0}^{(1)}$,

(ii) $\|\tilde{N}\|_{T_0}^{(1)} \leq C_{23} \left( T_0, \left\| \frac{\mu}{\rho_v} \right\|_T + \left\| \frac{\mu}{\rho_w} \right\|_T \right) \|v - w\|_{T_0}^{(1)}$,

$(0 < T_0 \leq T)$, where $C_{22}$ and $C_{23}$ have the same property as $C_{19}$ and $C_{20}$, respectively.

Finally, we have an inequality similar to (3.14):

$$ \|v - w\|_{T_0}^{(1)} \leq C_{24}(T_0; v, w) \|v - w\|_{T_0}^{(1)}, $$

where $C_{24}(T_0; \cdots)$ has the same property as $C_{22}$.

Since $C_{24}(T_0; \cdots) \downarrow 0$ as $T_0 \downarrow 0$, it holds for a sufficiently small $T_1 \in (0, T]$, that
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Hence, we obtain \( \|v - w\|_{T_1} = 0 \), i.e., \( v(x, t) = w(x, t) \) \((0 \leq t \leq T_1 \leq T)\). According to the assumption, we can continue this procedure again by starting at \( t = T_1 \). After a finite number of repetitions of this procedure, it is shown in a conventional way that the following assertion holds.

**Theorem 4.1.** If \((v, \rho)\) and \((w, \rho^*)\) \(\in H_0^{2+a} \times B_\Gamma^1\) satisfy (1.1), (2.1) and (2.2), then \((v, \rho) = (w, \rho^*)\) \((\rho = \rho^* = \rho_0)\).

§ 5. An a priori Estimate for \( |\rho|_{T}^{(0)} \)

We begin with the following well known lemmas. (see, e.g., [5], [15]).

**Lemma 5.1.** If \( u(x, t) \) satisfies regularly the equation:

(5.1) \[
\frac{\partial u}{\partial t} - a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u, \quad (0 < t \leq T)
\]

where \( a(x, t) \), \( b(x, t) \) and \( c(x, t) \) are continuous in \( \bar{Q}_T \) and satisfy

(5.1)' \[
0 \leq a(x, t) \leq |a|_{T}^{(0)} < +\infty, \quad c(x, t) \leq 0,
\]

then it holds that

(5.2) \[
\max_{Q_T} |u| \leq \max_{\Gamma_T} |u|.
\]

**Lemma 5.2.** If \( u(x, t) \) satisfies regularly the equation:

(5.3) \[
\frac{\partial u}{\partial t} = a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u + f(x, t),
\]

where \( a(x, t) > 0 \) in \( \bar{Q}_T \) and if

\[
D_t^k D_x^m a, D_t^k D_x^m b, D_t^k D_x^m c, D_t^k D_x^m f \quad (0 \leq m + 2k \leq p, k \leq q)
\]

belong to \( H_\Gamma^q \), then \( D_t^k D_x^m u \) \((0 \leq m + 2k \leq p + 2, k \leq q + 1)\) exist and are Hölder continuous (exponent \( q \)) in \( \bar{Q} \times \left[T', T\right] \) for an arbitrary \( T' \in (0, T) \).
Directly by the above lemmas, we have

**Lemma 5.3.** If \((v, \rho) \in H^{1+\varepsilon}_1 \times B_1^1\) satisfies (1.1), (2.1) and (2.2), then it holds that

\[
|v|_{L}^{(0)} \leq |v_0|^{(0)}.
\]

**Lemma 5.4.** If \((v, \rho) \in H^{2+\varepsilon}_1 \times B_1^1\) satisfies (1.1), (2.1), (2.2) and an additional condition:

\[
\rho_0 \in H^{1+\varepsilon},
\]

then \(v \in H^{2+\varepsilon}_1 \times B_1^1\) where the suffix \([T', T]\) denotes that \(\bar{q}_T\) in (1.4) and (1.4)' is replaced by \(\bar{q} \times [T', T]\). [We note that (5.5) implies \(\rho \in B_1^{1+\varepsilon}\).]

**Lemma 5.5.** If \((v, \rho) \in H^{2+\varepsilon}_1 \times B_1^1\) satisfies (1.1), (2.1), (2.2) and (5.5), then it holds that

\[
\rho_0 \exp \left\{ -\frac{1}{\mu} \left| \rho_0 v_0 \right|^{(0)} X \right\} \leq \rho(x, t) \leq \rho_0 \exp \left\{ \frac{1}{\mu} \left| \rho_0 v_0 \right|^{(0)} X \right\}.
\]

**Proof.** By (2.5), we need to estimate \(\int_0^T \tilde{v}_x (\tau; x, t) \, dt\). Since

\[
\frac{\partial}{\partial x} \int_0^T \tilde{v}_x (\tau; x, t) \, d\tau = \int_0^T \tilde{v}_x (\tau; x, t) \, d\tau = \frac{\rho(x, t)}{\mu} \left\{ v(x, t) - v_0(x_0(x, t)) \right\},
\]

\[
\int_0^T \tilde{v}_x (\tau; x, t) \, d\tau = \left\{ \frac{1}{\mu} \int_0^x \rho v \, dx + \int_0^T \tilde{v}_x (\tau; 0, t) \, d\tau \right\} - \frac{1}{\mu} \int_0^x \rho(x, t) v_0(x_0(x, t)) \, dx.
\]

The second term of the right-hand side of (5.7) is transformed, by using (2.6), as follows:

\[
\int_0^x \rho(x, t) v_0(x_0) \, dx = \int_0^{x_0} \rho_0(x_0') v_0(x_0') \, dx_0.
\]

We denote the first term of the right-hand side of (5.7) by \(\psi(x, t)\). Then, we have
\[
\psi_t(x, t) = \frac{1}{\mu} \int_0^x (\rho_x v_x + \rho v_x^2)dx + v_x(0, t) = \frac{1}{\mu} (-\rho v^2 + \mu v_x).
\]

On the other hand, \(\psi_x(x, t) = \frac{1}{\mu} \rho v, \quad \left(\frac{\mu}{\rho} \psi_x\right)_x = v_x.\)
Hence, we have

\[
(5.8) \quad \psi_t = \frac{\mu}{\rho} \psi_{xx} - \left(\frac{\mu \rho_x}{\rho^2} + v\right)\psi_x,
\]

\[
(5.9) \quad \psi(x, 0) = \frac{1}{\mu} \int_0^x \rho_0 v_0 dx, \quad \psi_x(0, t) = \psi_x(X, t) = 0 \quad (T \geq t \geq 0).
\]

Therefore, \(\psi(x, t)\) is to be expressed by utilizing the fundamental solution of the linear parabolic equation (5.8) in the following way:

\[
(5.10) \quad \psi(x, t) = \frac{1}{\mu} \int_0^x \bar{\psi}(x, t; \xi, 0)\psi(\xi, 0) d\xi = \frac{1}{\mu} \int_0^x \bar{\psi}(x, t; \xi, 0) \left(\int_0^\xi \rho_0 v_0 dy\right) d\xi
\]

where \(\bar{\psi}\) is the fundamental solution of (5.8).

Hence, we have \(\left|\psi - \frac{1}{\mu} \int_0^x \rho(x, t)v_0(x_0(x, t))dx\right|_T^{(0)} \leq \frac{1}{\mu} \left|\rho_0 v_0\right|^{(0)} X.\)
As a result, we obtain (5.6).

Q.E.D.

If \((v, \rho) \in H^{2+\varepsilon}_T \times B^{1+\varepsilon}_T\) satisfies (1.1), (2.1), (2.2), and (5.5), then by Lemmas 2.1 and 5.2, we have

\[
(5.11) \quad \left\|\frac{\mu}{\rho}\right\|_T^{(\varepsilon)} \leq 5 \left\|\frac{\mu}{\rho}\right\|_T^{(0)} + 2 \left|\left(\frac{\mu}{\rho}\right)_x\right|_T^{(0)} + 2 \left|\left(\frac{\mu}{\rho}\right)_t\right|_T^{(0)}
\]

\[\leq (5 + 2 |v_x|^{(0)} \mu(\rho_0)^{-1} \exp \{1\} \int_0^t \bar{v}_x(\tau, x) d\tau \|v_0\|^{(0)}) + 4 |v_0|^{(0)}(1 + |v_0|^{(0)}) + 2\mu(\rho_0)^{-2} |\rho_0|^{(0)}(1 + |v_0|^{(0)}),\]

since

\[
\begin{cases}
\left(\frac{\mu}{\rho}\right)_x = -\frac{\mu \rho_x}{\rho^2} = -\frac{\rho_x}{\rho} + v(x, t) - v_0(x_0(x, t)), \\
\left(\frac{\mu}{\rho}\right)_t = -\frac{\mu \rho_t}{\rho^2} = \mu\left\{\frac{v_x}{\rho} - v\left(\frac{1}{\rho}\right)_x\right\}.
\end{cases}
\]

By (5.11), we know that, in order to have an a priori estimate for \(\left\|\frac{\mu}{\rho}\right\|_T^{(\varepsilon)}\), we have to obtain beforehand one for \(|v_x|_T^{(0)}\). Hereafter in §6, we shall endeavor to have an a priori estimate for \(|v_x|_T^{(0)}\).
§ 6. An *a priori* Estimate for $|v_x|_T^{(0)}$

Lemma 6.1. Under the initial-boundary conditions (2.1), (2.2) and (5.5), $|v_x|_T^{(0)}$ is bounded by a constant depending only on the quantities appearing in (2.1), (2.2) and (5.5) but independent of $T$.

*Proof.* The procedure of the demonstration is divided into three steps.

(1-st step). First of all, we note that (5.6) holds by Lemma 5.5. Now we define $v_\lambda(x, t)$ by

\begin{equation}
(6.1)

v_\lambda(x, t) \equiv v(x, t)^2 + \lambda v(x, t)^2,
\end{equation}

where $\lambda$ is a constant to be determined later.

Since $v \in H_{T}^{3+q}$ by Lemma 5.4, $v_x$ satisfies the equation:

\begin{equation}
(6.2)

(v_x)_t = \frac{\mu}{\rho}(v_x)_{xx} + \left\{\left(\frac{\mu}{\rho}\right)_x - v\right\}(v_x)_x - (v_x)^2.
\end{equation}

Let $\mathcal{L}$ be defined by

\begin{equation}
(6.3)

\mathcal{L} \equiv \frac{\partial}{\partial t} - \frac{\mu}{\rho} \frac{\partial^2}{\partial x^2} + v \frac{\partial}{\partial x}.
\end{equation}

Then, we have for any $\varepsilon > 0$

\begin{equation}
(6.4)

\mathcal{L} v_\lambda = 2 \left(\frac{\mu}{\rho}\right)_x v_x v_{xx} - 2v_x^3 - 2 \frac{\mu}{\rho} v_x^2 - 2 \frac{\mu \lambda}{\rho} v_x^2 +
\end{equation}

\begin{equation}
\leq 2 \left\{((\bar{\rho}_0))^{-2} |\rho_0^{(0)}| + 2 |v_0^{(0)}| e - \frac{2\mu}{|\rho_1^{(0)}|} \right\} v_x^2 +
\end{equation}

\begin{equation}
+ \left[\frac{1}{2\varepsilon} \left\{((\bar{\rho}_0))^{-2} |\rho_0^{(0)}| + 2 |v_0^{(0)}| \right\} + 2 |v_x|_T^{(0)} -
\end{equation}

\begin{equation}
- \frac{2\mu \lambda}{|\rho_1^{(0)}|} \right\} v_x^2.
\end{equation}

We choose $\varepsilon = \varepsilon_0$ in such a way that
For such a fixed number $\varepsilon_0 (> 0)$, it holds that

\begin{equation}
\mathcal{L} v_\lambda \leq \left[ \frac{1}{2\varepsilon_0} \{(\bar{\rho}_0)^{-2} |\rho_0|^{(0)} + 2 |v_0|^{(0)}\} + 2 |v_x|^{(0)} - \frac{2\mu\lambda}{|\rho|^{(0)}} v_x^2 \right].
\end{equation}

If we take $\lambda = \lambda_0 \equiv \frac{|\rho|^{(0)}}{2\mu} \left[ \frac{1}{2\varepsilon_0} \{(\bar{\rho}_0)^{-2} |\rho_0|^{(0)} + 2 |v_0|^{(0)}\} + 2 |v_x|^{(0)} \right]$, then we have an inequality

\begin{equation}
\mathcal{L} v_{\lambda_0} \leq 0.
\end{equation}

By (6.7) and the maximum principle, it holds that

\begin{equation}
\max_{\partial T} v_{\lambda_0} \leq \max_{T} (v_x^2 + \lambda_0 v^2) \leq (|v_0|^{(0)})^2 + \lambda_0 (|v_0|^{(0)})^2 + \max_{S_T} v_x^2.
\end{equation}

(2-nd step). To evaluate the last term of (6.8), it is clear the case $|v_0|^{(0)}=0$. Then, suppose $|v_0|^{(0)} \neq 0$ and consider $v = \phi(w)$, where $\phi$ is a smooth function to be determined later. Thus, we get

\begin{equation}
\mathcal{L} v = \phi \left[ w_t - \frac{\mu}{\rho} w_{xx} - \frac{\mu}{\rho} \frac{\phi''}{\phi'} w_x^2 + \frac{1}{\phi'} v v_x \right] = 0.
\end{equation}

If $\phi'>0$ and $\phi''<0$, then it follows that

\begin{equation}
\begin{aligned}
w_t - \frac{\mu}{\rho} w_{xx} &\leq \frac{\phi''}{\phi'} \frac{\mu}{\rho} w_x^2 - \frac{1}{\phi'} vv_x \\
&\leq \frac{\mu}{|\rho|^{(0)}} \left[ \frac{\phi''}{\phi'} + \frac{|\rho|^{(0)} |v_0|^{(0)} v}{2\mu} \right] w_x^2 + \\
&\quad + \frac{|\rho|^{(0)} |v_0|^{(0)}}{2\mu} \left( \frac{1}{\phi'} \right).
\end{aligned}
\end{equation}
Furthermore, we choose $\phi$ in such a way that
\[ \phi'' + \frac{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right| \phi'}{2\mu} \leq 0 \quad \text{and} \quad \phi(0) = 0, \]
that is to say, for example,
\[ \phi(w) = \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \log(1 + w), \]
or
\[ w = -1 + \exp \left\{ \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \right\}. \]

By (6.11), it is clear that $w_{|x=0} = 0$.

For such a function $\phi$, it follows from (6.10) that
\[ \frac{d}{dt} \frac{w_{xx}}{\rho} \leq \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \exp \left\{ \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \right\} = C_{25}. \]

Differentiating both sides of (6.11) once in $x$ and putting $t=0$, we get
\[ \max_{\tilde{0}} \left| w_x(x, 0) \right| \leq \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \left| \frac{v_0'}{(0)} \right| \exp \left\{ \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \right\}. \]

Define the constant $C_{26}$ by $\max \left\{ \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \left| \frac{v_0'}{(0)} \right| \exp \left\{ \frac{2\mu}{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|} \right\}, 1 \right\}$, then
\[ \max_{\tilde{0}} \left| w_x(x, 0) \right| \leq C_{26}. \]

Now, consider the function $w(x, t) + ve^{-x}$. For $v \geq C_{26}e^x$, $w(x, t) + ve^{-x} \geq 0$ and
\[ \max_{\tilde{0}} \{w + ve^{-x}\} = \max_{\tilde{0}} \{w + ve^{-x}\} = v, \]
since $\frac{\partial}{\partial x} \{w(x, 0) + ve^{-x}\} \leq C_{26} - ve^{-x} \leq 0$.

Next, if we take $v = v_0 \equiv e^x \max \left\{ C_{26}, \frac{\rho \frac{\partial}{\partial x} \left| \frac{(0)}{v_0} \right|}{\mu} C_{25} \right\}$, then
\[ \frac{\partial}{\partial t} \{w + v_0 e^{-x}\} - \frac{\mu}{\rho} \frac{\partial^2}{\partial x^2} \{w + v_0 e^{-x}\} \leq 0, \]
because it holds that

\[ w_t - \frac{\mu}{\rho} (w + v_0 e^{-x})_{xx} \leq C_{25} - \frac{\mu v_0}{|\rho|^o} e^{-x} \leq 0. \]

Hence, by (6.15) and (6.16), the maximum of \( w + v_0 e^{-x} \) in \( \bar{Q}_T \) is attained at all points of \( S^o_T \). Thus

(6.17) \[ \frac{\partial w}{\partial x} \big|_{S^o_T} \leq v_0. \]

Directly, from (6.17), we have

(6.18) \[ \frac{\partial v}{\partial x} \big|_{S^o_T} \leq \frac{2\mu v_0}{|\rho|^o |v_0|^o}. \]

In order to obtain the estimate for \( \frac{\partial v}{\partial x} \big|_{S^o_T} \) from below, it is sufficient to apply the above one to the solution \(-v(x, t)\) of the equation:

\[ (-v)_t - \frac{\mu}{\rho} (-v)_{xx} - vv_x = 0. \]

As a result, we have

(6.19) \[ \left| \frac{\partial v}{\partial x} \big|_{S^o_T} \right| \leq \frac{2\mu v_0}{|\rho|^o |v_0|^o}. \]

As for \( \frac{\partial v}{\partial x} \big|_{S^o_T} \), consider the function \( \tilde{v}(x, t) = v(X - x, t) \), and repeat the same argument for \( \tilde{v}_x \) on \( S^o_T \) as for \( v_x \) on \( S^o_T \). Finally we get

(6.20) \[ \max_{S^o_T} |v_x| \leq \frac{2\mu v_0}{|\rho|^o |v_0|^o}. \]

(3-rd step). By (6.8) and (6.20), we have

(6.21) \[ (|v_x|^o)_T^2 \leq (|v_0|^o)_T^2 + \lambda_0 (|v_0|^o)_T^2 + \]

\[ + \left( \frac{2\mu v_0}{|\rho|^o |v_0|^o} \right)^2 \]

\[ \leq (|v_0|^o)_T^2 + (|v_0|^o)_T^2 \frac{|\rho|^o}{2\mu} \left( \frac{1}{2\varepsilon_0} \right)^2 + \]
\[ + 2 |v_0|^{(0)} + 2 |v_x|^{(0)} + \left( \frac{2\mu v_0}{|\rho|^{(0)} v_0|^{(0)}} \right)^2 \]

\[ \equiv a_0 + b_0 |v_x|^{(0)}. \]

where

\[ a_0 = \left( |v_0|^{(0)} \right)^2 + \left( \frac{|\rho|^{(0)}}{2\mu} - \frac{1}{2\varepsilon_0} (\overline{\rho}_0)^{-2} |\rho_0|^{(0)} + 2 |v_0|^{(0)} \right) \times \]

\[ \left( |v_0|^{(0)} \right)^2 + \left( \frac{2\mu v_0}{|\rho|^{(0)} v_0|^{(0)}} \right)^2, \]

\[ b_0 = \frac{|\rho|^{(0)} (|v_0|^{(0)})^2}{\mu}. \]

Thus, it holds that

\[ (6.22) \quad |v_x|^{(0)} \leq b_0 + (b_0^2 + 4a_0)^{1/2}. \]

Q.E.D.

From (5.11), it follows that

\[ (6.23) \quad \left\| \frac{\mu}{\rho} \right\|_T \leq C_{27}(v_0, \rho_0). \]

§7. An *a priori* Estimate for \( \|v\|_T^{(2+\alpha)} \) and the Main Theorem

By Lemma 3.1 for \( v \) instead of \( v^n \), we have

**Lemma 7.1.** \( \|v\|_T^{(1+\alpha)} \leq C_{28}(T; v_0, \rho_0), \)

where \( C_{28}(T; \cdots) \) increases monotonically as \( T \) increases.

Hence, from this, it follows that

\[ \|N\|_T^{(\alpha)} \leq C_{29}(T; v_0, \rho_0), \]

where \( C_{29}(T; \cdots) \) has the same property as \( C_{28} \). Thus, we have

**Lemma 7.2.** \( \|v_{xx}\|_T^{(\alpha)} \leq C_{30}(T; v_0, \rho_0). \)

From the discussions made thus far follows:

**Lemma 7.3.** *Under the initial-boundary conditions* (2.1), (2.2)
and (5.5), if there exists a solution \((v, \rho) \in H_t^{2+s} \times B_t^1\) of (1.1), then 
\[ \|v\|^{(2+s)} + [\rho]_T^{(1)} \] has a priori bounds in \(T\), where

\[ [\rho]_T^{(n)} = \sum_{r+s=0}^{n} |D^r_tD^s_x \rho|^{(0)} \quad (r \text{ and } s, \text{ integers} \geq 0). \]

**Proof.** We have only to note that

\[ \|v\|^{(2+s)} \leq \|v\|^{(1+s)} + \left(1 + \left| \frac{\mu}{\rho} \right|_T \right) \|v_{xx}\|^{(s)} + \|v_{x}\|^{(s)}. \]

The results obtained since §5 guarantee that each term of the right-hand side of the above inequality has a priori bounds in \(T\).

Q.E.D.

Combining Theorems 3.1, 4.1 and Lemma 7.3, we have the following main theorem on the existence of a temporally global solution of (1.1), (2.1), (2.2) and (5.5).

**Theorem 7.1.** Under the initial-boundary conditions (2.1), (2.2) and (5.5), there uniquely exists a regular temporally global solution and it holds that

\[
\begin{align*}
|v(x, t)| & \leq |v_0|^{(0)}, \\
0 & \leq \tilde{\rho}_0 \exp\left\{-\frac{1}{\mu} |\rho_0v_0|^{(0)}X\right\} \leq \rho(x, t) \leq \\
& \leq \tilde{\rho}_0 \exp\left\{\frac{1}{\mu} |\rho_0v_0|^{(0)}X\right\}, \\
|v_x(x, t)| & \leq K\left(\|v_0\|^{(1)}, \|\rho_0\|^{(1)}, \frac{1}{\tilde{\rho}_0}\right) \leq +\infty,
\end{align*}
\]

(7.1)

where \(K\) increases as each argument increases.

**Remark.** (i) The word "regular" means, exactly speaking, regular up to the boundary.

(ii) If there exists a regular solution \((v, \rho)\) defined in \([0, X] \times [0, \infty)\), then \((v, \rho) \in H_t^{2+s} \times B_t^{1+s}\) for an arbitrary \(T \in (0, \infty)\).
References
