Regular Holonomic $\mathcal{D}[[h]]$-modules

Dedicated to Professor Mikio Sato on the occasion of his 80th birthday with our deep admiration and warmest regards

by

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Abstract

We describe the category of regular holonomic modules over the ring $\mathcal{D}[[h]]$ of linear differential operators with a formal parameter $h$. In particular, we establish the Riemann–Hilbert correspondence and discuss the additional $t$-structure related to $h$-torsion.

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Introduction

On a complex manifold $X$, we will be interested in the study of holonomic modules over the ring $\mathcal{D}_X[[h]]$ of differential operators with a formal parameter $h$. Such modules naturally appear when studying deformation quantization modules (DQ-modules) along a smooth Lagrangian submanifold of a complex symplectic manifold (see [13, Chapter 7]).

In this paper, after recalling the tools from [13] that we shall use, we explain some basic notions of $\mathcal{D}_X[[h]]$-modules theory. For example, it follows easily from general results on modules over $\mathbb{C}[[h]]$-algebras that given two holonomic $\mathcal{D}_X[[h]]$-

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modules $\mathcal{M}$ and $\mathcal{N}$, the complex $\mathcal{R}\text{Hom}_{\mathcal{D}_{X}[\hbar]}(\mathcal{M},\mathcal{N})$ is constructible over $\mathbb{C}[\hbar]$ and that the microsupport of the solution complex $\mathcal{R}\text{Hom}_{\mathcal{D}_{X}[\hbar]}(\mathcal{M},\mathcal{O}_{X}[\hbar])$ coincides with the characteristic variety of $\mathcal{M}$.

Then we establish our main result, the Riemann–Hilbert correspondence for regular holonomic $\mathcal{D}_{X}[\hbar]$-modules, an $\hbar$-variant of Kashiwara’s classical theorem. In other words, we show that the solution functor with values in $\mathcal{O}_{X}[\hbar]$ induces an equivalence between the derived category of regular holonomic $\mathcal{D}_{X}[\hbar]$-modules and that of constructible sheaves over $\mathbb{C}[\hbar]$. A quasi-inverse is obtained by constructing the “sheaf” of holomorphic functions with temperate growth and a formal parameter $\hbar$ in the subanalytic site. This needs some care since the literature on this subject is written in the framework of sheaves over a field and does not immediately apply to the ring $\mathbb{C}[\hbar]$.

We also discuss the $t$-structure related to $\hbar$-torsion. Indeed, as we work over the ring $\mathbb{C}[\hbar]$ and not over a field, the derived category of holonomic $\mathcal{D}_{X}[\hbar]$-modules (or, equivalently, that of constructible sheaves over $\mathbb{C}[\hbar]$) has an additional $t$-structure related to $\hbar$-torsion. We will show how the duality functor interchanges it with the natural $t$-structure.

We end this paper by describing some natural links between the ring $\mathcal{D}_{X}[\hbar]$ and deformation quantization algebras, as mentioned above.

**Historical remark.** As is well-known, holonomic modules play an essential role in mathematics. They appeared independently in the work of M. Kashiwara [4] and J. Bernstein [1], but they were first invented by Mikio Sato in a series of (unfortunately unpublished) lectures at Tokyo University in the 60’s. (See [17] for a more detailed history.)

**Notation and conventions**

We shall mainly follow the notation of [12]. In particular, if $\mathcal{C}$ is an abelian category, we denote by $D(\mathcal{C})$ the derived category of $\mathcal{C}$ and by $D^{*}(\mathcal{C})\ (\ *=+,-,b)$ the full triangulated subcategory consisting of objects with cohomology bounded from below (resp. bounded from above, resp. bounded).

For a sheaf $\mathcal{R}$ of rings on a topological space $X$, or more generally on a site, we denote by Mod($\mathcal{R}$) the category of left $\mathcal{R}$-modules and we write $D^{*}(\mathcal{R})$ instead of $D^{*}(\text{Mod}(\mathcal{R}))\ (\ *=\emptyset,+,-,b)$. We denote by $\text{Mod}_{\text{coh}}(\mathcal{R})$ the full abelian subcategory of $\text{Mod}(\mathcal{R})$ of coherent objects, and by $D^{b}_{\text{coh}}(\mathcal{R})$ the full triangulated subcategory of $D^{b}(\mathcal{R})$ of objects with coherent cohomology groups.

If $R$ is a ring (a sheaf of rings over a point), we write for short $D^{b}_{\text{coh}}(R)$ instead of $D^{b}_{\text{coh}}(\mathcal{R})$. 

§1. Formal deformations (after [13])

We review here some definitions and results from [13] that we shall use in this paper.

**Modules over \( \mathbb{Z}[h]\)-algebras.** Let \( X \) be a topological space. One says that a sheaf of \( \mathbb{Z}_X[h] \)-modules \( \mathcal{M} \) has no \( h \)-torsion if \( h: \mathcal{M} \to \mathcal{M} \) is injective; and one says that \( \mathcal{M} \) is \( h \)-complete if \( \mathcal{M} \to \lim_{\rightarrow n} \mathcal{M} / h^n \mathcal{M} \) is an isomorphism.

Let \( \mathcal{R} \) be a sheaf of \( \mathbb{Z}[h] \)-algebras, and assume that \( \mathcal{R} \) has no \( h \)-torsion. Set \( \mathcal{R}^{\text{loc}} := \mathbb{Z}[h,h^{-1}] \otimes_{\mathbb{Z}[h]} \mathcal{R} \), \( \mathcal{R}_0 := \mathcal{R} / h \mathcal{R} \), and consider the functors

\[
(\bullet)^{\text{loc}}: \text{Mod}(\mathcal{R}) \to \text{Mod}(\mathcal{R}^{\text{loc}}), \quad \mathcal{M} \mapsto \mathcal{M}^{\text{loc}} := \mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M},
\]

\[
\text{gr}_h: \text{D}(\mathcal{R}) \to \text{D}(\mathcal{R}_0), \quad \mathcal{M} \mapsto \text{gr}_h(\mathcal{M}) := \mathcal{R}_0^L \text{gr}_h \mathcal{M}.
\]

Note that \((\bullet)^{\text{loc}}\) is exact and that for \( \mathcal{M}, \mathcal{N} \in \text{D}^h(\mathcal{R}) \) and \( \mathcal{P} \in \text{D}^h(\mathcal{R}_0) \) one has isomorphisms

\[
(1.1) \quad \text{gr}_h(\mathcal{P} \otimes_{\mathcal{R}_0} \mathcal{M}) \simeq \text{gr}_h \mathcal{P}^L \otimes_{\mathcal{R}_0} \text{gr}_h \mathcal{M},
\]

\[
(1.2) \quad \text{gr}_h(\text{RHom}_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})) \simeq \text{RHom}_{\mathcal{R}_0}(\text{gr}_h(\mathcal{M}), \text{gr}_h(\mathcal{N})).
\]

Here, the functor \( \text{gr}_h \) on the left hand side acts on \( \mathbb{Z}_X[h] \)-modules.

**Cohomologically \( h \)-complete sheaves**

**Definition 1.1.** One says that an object \( \mathcal{M} \) of \( \text{D}(\mathcal{R}) \) is cohomologically \( h \)-complete if \( \text{RHom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}, \mathcal{M}) = 0 \).

Hence, the full subcategory of cohomologically \( h \)-complete objects is triangulated. In fact, it is the right orthogonal complement to the full subcategory \( \text{D}(\mathcal{R}^{\text{loc}}) \) of \( \text{D}(\mathcal{R}) \).

Remark that \( \mathcal{M} \in \text{D}(\mathcal{R}) \) is cohomologically \( h \)-complete if and only if its image in \( \text{D}(\mathbb{Z}_X[h]) \) is cohomologically \( h \)-complete.

**Proposition 1.2.** Let \( \mathcal{M} \in \text{D}(\mathcal{R}) \). Then \( \mathcal{M} \) is cohomologically \( h \)-complete if and only if

\[
\lim_{\rightarrow U \ni x} \text{Ext}^i_{\mathbb{Z}[h]}(\mathbb{Z}[h,h^{-1}], H^j(U; \mathcal{M})) = 0
\]

for any \( x \in X \), any integer \( i \in \mathbb{Z} \) and any \( j = 0, 1 \). Here, \( U \) ranges over an open neighborhood system of \( x \).
Corollary 1.3. Let $\mathcal{M} \in \text{Mod}(\mathcal{R})$. Assume that $\mathcal{M}$ has no $h$-torsion, is $h$-complete and there exists a base $\mathfrak{B}$ of open subsets such that $H^i(U; \mathcal{M}) = 0$ for any $i > 0$ and any $U \in \mathfrak{B}$. Then $\mathcal{M}$ is cohomologically $h$-complete.

The functor $\text{gr}_h$ is conservative on the category of cohomologically $h$-complete objects:

Proposition 1.4. Let $\mathcal{M} \in D(\mathcal{R})$ be a cohomologically $h$-complete object. If $\text{gr}_h(\mathcal{M}) = 0$, then $\mathcal{M} = 0$.

Proposition 1.5. If $\mathcal{M} \in D(\mathcal{R})$ is cohomologically $h$-complete, then the object $R\text{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in D(\mathbb{Z}_X[h])$ is cohomologically $h$-complete for any $\mathcal{N} \in D(\mathcal{R})$.

Proposition 1.6. Let $f : X \rightarrow Y$ be a continuous map, and $\mathcal{M} \in D(\mathbb{Z}_X[h])$. If $\mathcal{M}$ is cohomologically $h$-complete, then so is $Rf_*\mathcal{M}$.

Reductions to $h = 0$. Now we assume that $X$ is a Hausdorff locally compact topological space.

By a basis $\mathfrak{B}$ of compact subsets of $X$, we mean a family of compact subsets such that for any $x \in X$ and any open neighborhood $U$ of $x$, there exists $K \in \mathfrak{B}$ such that $x \in \text{Int}(K) \subset K \subset U$.

Let $\mathcal{A}$ be a $\mathbb{Z}[h]$-algebra, and recall that we set $\mathcal{A}_0 = \mathcal{A} / h\mathcal{A}$. Consider the following conditions:

(i) $\mathcal{A}$ has no $h$-torsion and is $h$-complete,

(ii) $\mathcal{A}_0$ is a left Noetherian ring,

(iii) there exists a basis $\mathfrak{B}$ of compact subsets of $X$ and a prestack $U \mapsto \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$ ($U$ open in $X$) such that

(a) for any $K \in \mathfrak{B}$ and any open subset $U$ such that $K \subset U$, there exists $K' \in \mathfrak{B}$ such that $K \subset \text{Int}(K') \subset K' \subset U$,

(b) $U \mapsto \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$ is a full subprestack of $U \mapsto \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$,

(c) for any $K \in \mathfrak{B}$, any open set $U$ containing $K$, any $j > 0$ and any $\mathcal{M} \in \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$, one has $H^j(K; \mathcal{M}) = 0$,

(d) for any open subset $U$ and any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$, if $\mathcal{M}|_V$ belongs to $\text{Mod}_{\text{good}}(\mathcal{A}_0|_V)$ for any relatively compact open subset $V$ of $U$, then $\mathcal{M}$ belongs to $\text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$,

(e) for any $U$ open in $X$, $\text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$ is stable under subobjects, quotients and extensions in $\text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$,

(f) for any $U$ open in $X$ and any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathcal{M}|_{U_i} \in \text{Mod}_{\text{good}}(\mathcal{A}_0|_{U_i})$,

(g) $\mathcal{A}_0 \in \text{Mod}_{\text{good}}(\mathcal{A}_0)$. 

(iii)' there exists a basis \( B \) of open subsets of \( X \) such that for any \( U \in B \), any \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{O}_X|_U) \) and any \( j > 0 \), one has \( H^j(U; \mathcal{M}) = 0 \).

We will suppose that \( \mathcal{A} \) and \( \mathcal{O}_0 \) satisfy either Assumption 1.7 or Assumption 1.8 below.

**Assumption 1.7.** \( \mathcal{A} \) and \( \mathcal{O}_0 \) satisfy conditions (i)--(iii) above.

**Assumption 1.8.** \( \mathcal{A} \) and \( \mathcal{O}_0 \) satisfy conditions (i), (ii) and (iii)' above.

**Theorem 1.9.**

(i) \( \mathcal{A} \) is a left Noetherian ring.

(ii) Any coherent \( \mathcal{A} \)-module \( \mathcal{M} \) is \( \hbar \)-complete.

(iii) Let \( \mathcal{M} \in D_{\text{coh}}^b(\mathcal{A}) \). Then \( \mathcal{M} \) is cohomologically \( \hbar \)-complete.

**Corollary 1.10.** The functor \( \text{gr}_\hbar : D_{\text{coh}}^b(\mathcal{A}) \to D_{\text{coh}}^b(\mathcal{O}_0) \) is conservative.

**Theorem 1.11.** Let \( \mathcal{M} \in D^+(\mathcal{A}) \) and assume:

(a) \( \mathcal{M} \) is cohomologically \( \hbar \)-complete,

(b) \( \text{gr}_\hbar(\mathcal{M}) \in D_{\text{coh}}^+(\mathcal{O}_0) \).

Then \( \mathcal{M} \in D_{\text{coh}}^+(\mathcal{A}) \) and for all \( i \in \mathbb{Z} \) we have the isomorphism

\[
H^i(\mathcal{M}) \cong \lim_{\rightarrow n} H^i(\mathcal{A}/\hbar^n \mathcal{A} \otimes_{\mathcal{A}} \mathcal{M}).
\]

**Theorem 1.12.** Assume that \( \mathcal{O}_0^{\text{op}} = \mathcal{O}_0^{\text{op}}/h\mathcal{O}_0^{\text{op}} \) is a Noetherian ring and the flabby dimension of \( X \) is finite. Let \( \mathcal{M} \) be an \( \mathcal{A} \)-module. Assume the following conditions:

(a) \( \mathcal{M} \) has no \( h \)-torsion,

(b) \( \mathcal{M} \) is cohomologically \( h \)-complete,

(c) \( \mathcal{M}/h\mathcal{M} \) is a flat \( \mathcal{O}_0 \)-module.

Then \( \mathcal{M} \) is a flat \( \mathcal{A} \)-module.

If moreover \( \mathcal{M}/h\mathcal{M} \) is a faithfully flat \( \mathcal{O}_0 \)-module, then \( \mathcal{M} \) is a faithfully flat \( \mathcal{A} \)-module.

**Theorem 1.13.** Let \( d \in \mathbb{N} \). Assume that \( \mathcal{O}_0 \) is \( d \)-syzygic, i.e., any coherent \( \mathcal{O}_0 \)-module locally admits a projective resolution of length \( \leq d \) by free \( \mathcal{O}_0 \)-modules of finite rank. Then

(a) \( \mathcal{A} \) is \( (d+1) \)-syzygic.
Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{A}$-modules concentrated in degrees $[a, b]$ and with coherent cohomology groups. Then locally there exists a quasi-isomorphism $L^\bullet \to \mathcal{M}^\bullet$ where $L^\bullet$ is a complex of free $\mathcal{A}$-modules of finite rank concentrated in degrees $[a-d-1, b]$.

**Proposition 1.14.** Let $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{A})$ and let $a \in \mathbb{Z}$. The conditions below are equivalent:

(i) $H^a(\text{gr}_h(\mathcal{M})) \simeq 0$,

(ii) $H^a(\mathcal{M}) \simeq 0$ and $H^{a+1}(\mathcal{M})$ has no $h$-torsion.

**Cohomologically $h$-complete sheaves on real manifolds.** Let now $X$ be a real analytic manifold. Recall from [9] that the microsupport of $F \in D^b(\mathcal{Z}_X)$ is a closed involutive subset of the cotangent bundle $T^*X$ denoted by $\text{SS}(F)$. The microsupport is additive on $D^b(\mathcal{Z}_X)$ (cf. Definition 3.3(ii) below). Considering the distinguished triangle $F \xrightarrow{h} F \to \text{gr}_h F \xrightarrow{+1}$, one gets

\[(1.3) \quad \text{SS(\text{gr}_h(F))} \subset \text{SS}(F).\]

**Proposition 1.15.** Let $F \in D^b(\mathcal{Z}_X[h])$ and assume that $F$ is cohomologically $h$-complete. Then

\[(1.4) \quad \text{SS}(F) = \text{SS(\text{gr}_h(F))}.\]

**Proof.** It is enough to show that $\text{SS}(F) \subset \text{SS(\text{gr}_h(F))}$. For $V \subset U$ open subsets, consider the distinguished triangle

\[R\Gamma(U; F) \to R\Gamma(V; F) \to G \xrightarrow{+1}.\]

By Proposition 1.6, $R\Gamma(U; F)$ and $R\Gamma(V; F)$ are cohomologically $h$-complete, and thus so is $G$. One has the distinguished triangle

\[R\Gamma(U; \text{gr}_h F) \to R\Gamma(V; \text{gr}_h F) \to \text{gr}_h G \xrightarrow{+1}.\]

By the definition of microsupport, it is enough to prove that $\text{gr}_h G = 0$ implies $G = 0$. This follows from Proposition 1.4.

For $\mathbb{K}$ a commutative unital Noetherian ring, one denotes by $\text{Mod}_{\mathbb{R},c}(\mathbb{K}_X)$ the full subcategory of $\text{Mod}(\mathbb{K}_X)$ consisting of $\mathbb{R}$-constructible sheaves and by $D^b_{\mathbb{R},c}(\mathbb{K}_X)$ the full triangulated subcategory of $D^b(\mathbb{K}_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology (see [9, §8.4]). In this paper, we shall mainly be interested in the case where $\mathbb{K}$ is either $\mathbb{C}$ or the ring of formal power series in an indeterminate $h$, which we denote by

\[\mathbb{C}^h := \mathbb{C}[[h]].\]
Proposition 1.16. Let \( F \in D_{b\text{-}c}(C^h_X) \). Then \( F \) is cohomologically \( h \)-complete.

Proof. This follows from Proposition 1.2 since for any \( x \in X \) one has \( R\Gamma(U; F) \tilde{\to} F_x \) for \( U \) in a fundamental system of neighborhoods of \( x \).

Corollary 1.17. The functor \( \text{gr}_h : D_{b\text{-}c}(C^h_X) \to D_{b\text{-}c}(C_X) \) is conservative.

Corollary 1.18. For \( F \in D_{b\text{-}c}(C^h_X) \), one has the equality
\[
\text{SS}(\text{gr}_h(F)) = \text{SS}(F).
\]

Proposition 1.19. For \( F \in D_{b\text{-}c}(C^h_X) \) and \( i \in \mathbb{Z} \) one has \( \text{supp} H^i(F) \subset \text{supp} H^i(\text{gr}_h F) \). In particular if \( H^i(\text{gr}_h F) = 0 \) then \( H^i(F) = 0 \).

Proof. We apply Proposition 1.14 to \( F_x \) for any \( x \in X \).

§2. Formal extension

Let \( X \) be a topological space, or more generally a site, and let \( \mathcal{R}_0 \) be a sheaf of rings on \( X \). In this section, we let
\[
\mathcal{R} := \mathcal{R}_0[[h]] = \prod_{n \geq 0} \mathcal{R}_0 h^n
\]
be the formal extension of \( \mathcal{R}_0 \), whose sections on an open subset \( U \) are formal series \( r = \sum_{n=0}^{\infty} r_n h^n \), with \( r_n \in \Gamma(U; \mathcal{R}_0) \). Consider the associated functor
\[
(\bullet)^h : \text{Mod}(\mathcal{R}_0) \to \text{Mod}(\mathcal{R}), \quad N \mapsto N[[h]] = \lim_{\leftarrow n} (\mathcal{R}_n \otimes_{\mathcal{R}_0} N),
\]
where \( \mathcal{R}_n := \mathcal{R}/h^{n+1}\mathcal{R} \) is regarded as an \((\mathcal{R}_0, \mathcal{R})\)-bimodule. Since \( \mathcal{R}_n \) is free of finite rank over \( \mathcal{R}_0 \), the functor \((\bullet)^h\) is left exact. We denote by \((\bullet)^{Rh}\) its right derived functor.

Proposition 2.1. For \( N \in D^b(\mathcal{R}_0) \) one has
\[
N^{Rh} \simeq R\text{Hom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/h\mathcal{R}, N),
\]
where \( \mathcal{R}^{\text{loc}}/h\mathcal{R} \) is regarded as an \((\mathcal{R}_0, \mathcal{R})\)-bimodule.

Proof. It is enough to prove that for \( N \in \text{Mod}(\mathcal{R}_0) \) one has
\[
N^h \simeq \text{Hom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/h\mathcal{R}, N).
\]
Using the right \( \mathcal{R}_0 \)-module structure of \( \mathcal{R}_n \), set \( \mathcal{R}_n^* = \text{Hom}_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0) \). Then \( \mathcal{R}_n^* \) is an \((\mathcal{R}_0, \mathcal{R})\)-bimodule, and
\[
N^h = \lim_{\leftarrow n} (\mathcal{R}_n \otimes_{\mathcal{R}_0} N) \simeq \text{Hom}_{\mathcal{R}_0}(\lim_{\leftarrow n} \mathcal{R}_n^*, N).
\]
Since
\[ R_{\text{loc}} / h R \simeq \lim_{\longrightarrow} (h^{-n} R / h R), \]
it is enough to prove that there is an isomorphism of \((R_0, R)\)-bimodules
\[ \mathcal{H}\text{om}_{R_0}(R_n, R_0) \simeq h^{-n} R / h R. \]
Recalling that \( R_n = R / h^{n+1} R \), this follows from the pairing
\[ (R / h^{n+1} R) \otimes_{R_0} (h^{-n} R / h R) \to R_0, \quad f \otimes g \mapsto \text{Res}_{h=0}(fg dh / h). \]

Note that the isomorphism of \((R, R_0)\)-bimodules
\[ R \simeq (R_0)^{h} = \mathcal{H}\text{om}_{R_0}(R_{\text{loc}} / h R, R_0) \]
induces a natural morphism
\[ (2.2) \quad R \bigotimes_{R_0} N \to N^{Rh} \quad \text{for } N \in D^b(R_0). \]

**Proposition 2.2.** For \( N \in D^b(R_0) \), the formal extension \( N^{Rh} \) is cohomologically \( h \)-complete.

**Proof.** The statement follows from \((R_{\text{loc}} / h R)^{L} \otimes_{R_0} R_{\text{loc}} \simeq 0\) and from the isomorphism
\[ R \mathcal{H}\text{om}_{R}(R_{\text{loc}}, N^{Rh}) \simeq R \mathcal{H}\text{om}_{R_0}((R_{\text{loc}} / h R)^{L} \otimes_{R_0} R_{\text{loc}}, N). \]

**Lemma 2.3.** Assume that \( R_0 \) is an \( S_0 \)-algebra, for \( S_0 \) a commutative sheaf of rings, and let \( I = I_0[[h]] \). For \( M, N \in D^b(I) \) we have an isomorphism in \( D^b(I) \)
\[ R \mathcal{H}\text{om}_{R_0}(M, N)^{Rh} \simeq R \mathcal{H}\text{om}_{R_0}(M, N^{Rh}). \]

**Proof.** Note the isomorphisms
\[ R_{\text{loc}} / h R \simeq R_0 \otimes_{I_0} (I_{\text{loc}} / h I) \simeq R_0 \otimes_{I_0} (I_{\text{loc}} / h I) \]
as \((R_0, I)\)-bimodules. Then one has
\[ R \mathcal{H}\text{om}_{R_0}(M, N)^{Rh} = R \mathcal{H}\text{om}_{I_0}(I_{\text{loc}} / h I, R \mathcal{H}\text{om}_{I_0}(M, N)) \]
\[ \simeq R \mathcal{H}\text{om}_{I_0}(M, R \mathcal{H}\text{om}_{I_0}(I_{\text{loc}} / h I, N)) \]
\[ \simeq R \mathcal{H}\text{om}_{I_0}(M, R \mathcal{H}\text{om}_{I_0}(R_{\text{loc}} / h R, N)) \]
\[ = R \mathcal{H}\text{om}_{I_0}(M, N^{Rh}). \]
Lemma 2.4. Let $f : X \to Y$ be a morphism of sites, and assume that $(f^{-1} \mathcal{A}_0)^h \simeq f^{-1} \mathcal{A}$. Then the functors $Rf_*$ and $(\cdot)^{R\mathcal{H}}$ commute, that is, for $\mathcal{P} \in D^b(f^{-1} \mathcal{A}_0)$ we have $(Rf_*)^\mathcal{P} \simeq Rf_*(\mathcal{P}^{R\mathcal{H}})$ in $D^b(\mathcal{A})$.

Proof. One has the isomorphism

$$Rf_*(\mathcal{P}^{R\mathcal{H}}) = Rf_* R\mathcal{H}om_{f^{-1} \mathcal{A}_0}(f^{-1}(\mathcal{A}_{loc}/h\mathcal{A}), \mathcal{P}) \simeq R\mathcal{H}om_{\mathcal{A}_0}(\mathcal{A}_{loc}/h\mathcal{A}, Rf_\mathcal{P}) = Rf_*(\mathcal{P}^{R\mathcal{H}}).$$

Proposition 2.5. Let $\mathcal{T}$ be either a basis of open subsets of the site $X$ or, assuming that $X$ is a locally compact topological space, a basis of compact subsets. Denote by $J_\mathcal{T}$ the full subcategory of $Mod(\mathcal{A}_0)$ consisting of $\mathcal{T}$-acyclic objects, i.e., sheaves $\mathcal{N}$ for which $H^k(S; \mathcal{N}) = 0$ for all $k > 0$ and all $S \in \mathcal{T}$. Then $J_\mathcal{T}$ is injective with respect to the functor $(\cdot)^h$. In particular, for $\mathcal{N} \in J_\mathcal{T}$, we have $\mathcal{N}^h \simeq \mathcal{N}^{R\mathcal{H}}$.

Proof. (i) Since injective sheaves are $\mathcal{T}$-acyclic, $J_\mathcal{T}$ is cogenerating.

(ii) Consider an exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ in $Mod(\mathcal{A}_0)$. Clearly, if both $\mathcal{N}'$ and $\mathcal{N}$ belong to $J_\mathcal{T}$, then so does $\mathcal{N}''$.

(iii) Consider an exact sequence as in (ii) and assume that $\mathcal{N}' \in J_\mathcal{T}$. We have to prove that $0 \to \mathcal{N}'^h \to \mathcal{N}^h \to \mathcal{N}''^h \to 0$ is exact. Since $(\cdot)^h$ is left exact, it is enough to prove that $\mathcal{N}^h \to \mathcal{N}''^h$ is surjective. Noticing that $\mathcal{N}^h \simeq \prod \mathcal{N}$ as $\mathcal{A}_0$-modules, it is enough to prove that $\prod \mathcal{N} \to \prod \mathcal{N}''$ is surjective.

(iii)-(a) Assume that $\mathcal{T}$ is a basis of open subsets. Any open subset $U \subset X$ has a cover $\{U_i\}_{i \in I}$ by elements $U_i \in \mathcal{T}$. For any $i \in I$, the morphism $\mathcal{N}(U_i) \to \mathcal{N}''(U_i)$ is surjective. The result follows taking the product over $\mathcal{N}$.

(iii)-(b) Assume that $\mathcal{T}$ is a basis of compact subsets. For any $K \in \mathcal{T}$, the morphism $\mathcal{N}(K) \to \mathcal{N}''(K)$ is surjective. Hence, there exists a basis $\mathcal{V}$ of open subsets such that for any $x \in X$ and any $V \ni x$ in $\mathcal{V}$, there exists $V' \in \mathcal{V}$ with $x \in V' \subset V$ and the image of $\mathcal{N}(V') \to \mathcal{N}''(V')$ contains the image of $\mathcal{N}''(V)$ in $\mathcal{N}''(V)$. The result follows as in (iii)-(a) by taking the product over $\mathcal{N}$. 

Corollary 2.6. The following sheaves are acyclic for the functor $(\cdot)^h$:

(i) $\mathcal{R}$-constructible sheaves of $\mathbb{C}$-vector spaces on a real analytic manifold $X$,

(ii) coherent modules over the ring $\mathcal{O}_X$ of holomorphic functions on a complex analytic manifold $X$,

(iii) coherent modules over the ring $\mathcal{D}_X$ of linear differential operators on a complex analytic manifold $X$.

Proof. The statements follow by applying Proposition 2.5 for the following choices of $\mathcal{T}$. 

(i) Let $F$ be an $R$-constructible sheaf. Then for any $x \in X$ one has $F_x \sim \text{R}\Gamma(U_x; F)$ for $U_x$ in a fundamental system of open neighborhoods of $x$. Take for $\mathcal{T}$ the union of these fundamental systems.

(ii) Take for $\mathcal{T}$ the family of open Stein subsets.

(iii) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. The problem being local, we may assume that $\mathcal{M}$ is endowed with a good filtration. Then take for $\mathcal{T}$ the family of compact Stein subsets.

Example 2.7. Let $X = \mathbb{R}$, $\mathcal{R}_0 = \mathbb{C}_X$, $Z = \{1/n : n = 1, 2, \ldots \} \cup \{0\}$ and $U = X \setminus Z$. One has the isomorphisms $(\mathbb{C}^h)_X \simeq (\mathbb{C}_X)^h \simeq (\mathbb{C}_X)^R$ and $(\mathbb{C}^h)_U \simeq (\mathbb{C}_U)^h$. Considering the exact sequences

$$0 \to (\mathbb{C}_U)^h \to (\mathbb{C}_X)^h \to (\mathbb{C}_Z)^h \to H^1(\mathbb{C}_U)^R \to 0,$$

we get $H^1(\mathbb{C}_U)^R \simeq (\mathbb{C}_Z)^h/(\mathbb{C}^h)_Z$, whose stalk at the origin does not vanish. Hence $\mathbb{C}_U$ is not acyclic for the functor $(\cdot)^h$.

Assume now that

$$\mathcal{A}_0 = \mathcal{R}_0 \quad \text{and} \quad \mathcal{A} = \mathcal{R}_0[[h]]$$

satisfy either Assumption 1.7 or Assumption 1.8 (where condition (i) is clear) and that $\mathcal{A}_0$ is syzygic. Note that by Proposition 2.5 one has $\mathcal{A} \simeq (\mathcal{A}_0)^R$.

Proposition 2.8. For $\mathcal{N} \in \text{Db}_{\text{coho}}(\mathcal{A}_0)$:

(i) there is an isomorphism $\mathcal{A}^R \simeq \mathcal{L}^L \otimes_{\mathcal{A}_0} \mathcal{N}$ induced by (2.2),

(ii) there is an isomorphism $\text{gr}_h(\mathcal{N}^R) \simeq \mathcal{N}$.

Proof. Since $\mathcal{A}_0$ is syzygic, we may locally represent $\mathcal{N}$ by a bounded complex $\mathcal{L}^*$ of free $\mathcal{A}_0$-modules of finite rank. Then (i) is obvious. As for (ii), both complexes are isomorphic to the mapping cone of $h: (\mathcal{L}^*)^h \to (\mathcal{L}^*)^h$.

In particular, the functor $(\cdot)^h$ is exact on $\text{Mod}_{\text{coho}}(\mathcal{A}_0)$ and preserves coherence. One thus gets a functor $(\cdot)^R: \text{Db}_{\text{coho}}(\mathcal{A}_0) \to \text{Db}_{\text{coho}}(\mathcal{A})$.

The subanalytic site. The subanalytic site associated to an analytic manifold $X$ has been introduced and studied in [11, Chapter 7] (see also [15] for a detailed and systematic study as well as for complementary results). Denote by $\text{Op}_X$ the category of open subsets of $X$, the morphisms being the inclusion morphisms, and by $\text{Op}_{X_{sa}}$ the full subcategory consisting of relatively compact subanalytic open subsets of $X$. The site $X_{sa}$ is the presite $\text{Op}_{X_{sa}}$ endowed with the Grothendieck
Consider the diagram

\[
\begin{array}{ccc}
\text{D}^b(\mathbb{C}_X) & \overset{\rho_*}{\longrightarrow} & \text{D}^b(\mathbb{C}_{X_{\text{sa}}}) \\
(\ast)^{\text{RH}} & \downarrow & (\ast)^{\text{RH}} \\
\text{D}^b(\mathbb{C}_X^h) & \overset{\rho_*}{\longrightarrow} & \text{D}^b(\mathbb{C}_{X_{\text{sa}}})
\end{array}
\]

**Lemma 2.9.** (i) The functors \(\rho^{-1}\) and \((\ast)^{\text{RH}}\) commute, that is, for \(G \in \text{D}^b(\mathbb{C}_{X_{\text{sa}}})\) we have \((\rho^{-1}G)^{\text{RH}} \simeq \rho^{-1}(G^{\text{RH}})\) in \(\text{D}^b(\mathbb{C}_X^h)\).

(ii) The functors \(\rho_*\) and \((\ast)^{\text{RH}}\) commute, that is, for \(F \in \text{D}^b(\mathbb{C}_X)\) we have \((\rho_*(F))^{\text{RH}} \simeq \rho_*(F^{\text{RH}})\) in \(\text{D}^b(\mathbb{C}_{X_{\text{sa}}})\).

**Proof.** (i) Since it admits a left adjoint, the functor \(\rho^{-1}\) commutes with projective limits. It follows that for \(G \in \text{Mod}(\mathbb{C}_{X_{\text{sa}}})\) one has an isomorphism

\[\rho^{-1}(G^h) \rightarrow (\rho^{-1}G)^h.\]

To conclude, it remains to show that \((\rho^{-1}(\ast))^{\text{RH}}\) is the derived functor of \((\rho^{-1}(\ast))^h\).

Recall that an object \(G\) of \(\text{Mod}(\mathbb{C}_{X_{\text{sa}}})\) is quasi-injective if the functor \(\text{Hom}_\mathbb{C}_{X_{\text{sa}}}(-, G)\) is exact on the category \(\text{Mod}_{\mathbb{C}_{X_{\text{sa}}}(-, G)}\). By a result of [13], if \(G \in \text{Mod}(\mathbb{C}_{X_{\text{sa}}})\) is quasi-injective, then \(\rho^{-1}G\) is soft. Hence, \(\rho^{-1}G\) is injective for the functor \((\ast)^{\text{RH}}\) by Proposition 2.3.

(ii) By (i) we can apply Lemma 2.4.

\[\square\]

### §3. \(\mathcal{D}[[\hbar]]\)-modules and propagation

Let now \(X\) be a complex analytic manifold of complex dimension \(d_X\). As usual, denote by \(\mathbb{C}_X\) the constant sheaf with stalk \(\mathbb{C}\), by \(\mathcal{O}_X\) the structure sheaf and by \(\mathcal{D}_X\) the ring of linear differential operators on \(X\). We will use the notation

- \(D' : \text{D}^b(\mathbb{C}_X)^{\text{op}} \rightarrow \text{D}^b(\mathbb{C}_X), \quad F \mapsto R\text{Hom}_\mathbb{C}_X(F, \mathbb{C}_X)\),
- \(D : \text{D}^b_{\text{coh}}(\mathcal{D}_X)^{\text{op}} \rightarrow \text{D}^b_{\text{coh}}(\mathcal{D}_X), \quad \mathcal{M} \mapsto R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes_{\mathcal{D}_X} \Omega_X^{\leq -1})[d_X]\),
- \(\text{Sol} : \text{D}^b_{\text{coh}}(\mathcal{D}_X)^{\text{op}} \rightarrow \text{D}^b(\mathbb{C}_X), \quad \mathcal{M} \mapsto R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),\)
- \(\text{DR} : \text{D}^b_{\text{coh}}(\mathcal{D}_X) \rightarrow \text{D}^b(\mathbb{C}_X), \quad \mathcal{M} \mapsto R\text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})\),

where \(\Omega_X\) denotes the line bundle of holomorphic forms of maximal degree and \(\Omega_X^{\leq -1}\) the dual bundle.
As shown in Corollary 2.6, the sheaves $\mathcal{C}_X, \mathcal{O}_X$ and $\mathcal{D}_X$ are all acyclic for the functor $(\cdot)^h$. We will be interested in the formal extensions

$$\mathcal{C}_X^h = \mathcal{C}_X[[h]], \quad \mathcal{O}_X^h = \mathcal{O}_X[[h]], \quad \mathcal{D}_X^h = \mathcal{D}_X[[h]].$$

In the following, we shall treat left $\mathcal{D}_X^h$-modules, but all results apply to right modules since the categories $\text{Mod}(\mathcal{D}_X^h)$ and $\text{Mod}(\mathcal{D}_X^{h,\text{op}})$ are equivalent.

**Proposition 3.1.** Assumption 1.7 is satisfied by the $\mathcal{C}^h$-algebras $\mathcal{D}_X^h$ and $\mathcal{D}_X^{h,\text{op}}$.

**Proof.** Assumption 1.7 holds for $\mathcal{A} = \mathcal{D}_X^h$, $\mathcal{A}_0 = \mathcal{D}_X$, $\text{Mod}_{\text{good}}(\mathcal{A}_0|U)$ the category of good $\mathcal{D}_U$-modules (see [7]) and for $\mathcal{B}$ the family of Stein compact subsets of $X$.

In particular, by Theorem 1.9, $\mathcal{D}_X^h$ is right and left Noetherian (and thus coherent). Moreover, by Theorem 1.13 any object of $\mathcal{D}_X^{b,\text{coh}}$ can be locally represented by a bounded complex of free $\mathcal{D}_X^h$-modules of finite rank.

We will use the notation

$$D_h': D^b_\text{coh}(\mathcal{C}_X^h)^\text{op} \to D^b_\text{coh}(\mathcal{C}_X^h), \quad F \mapsto R\mathcal{H}om_{\mathcal{C}_X^h}(F, \mathcal{C}_X^h),$$

$$D_h: D^b_\text{coh}(\mathcal{D}_X^h)^\text{op} \to D^b_\text{coh}(\mathcal{D}_X^h), \quad \mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{D}_X^h \otimes_{\mathcal{D}_X^h} \Omega_{\mathcal{D}_X}^{0,-1}[d_X]),$$

$$\text{Sol}_h: D^b_\text{coh}(\mathcal{D}_X^h)^\text{op} \to D^b(\mathcal{C}_X^h), \quad \mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{D}_X^h),$$

$$\text{DR}_h: D^b_\text{coh}(\mathcal{D}_X^h) \to D^b(\mathcal{C}_X^h), \quad \mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{D}_X^h, \mathcal{M}).$$

By Proposition 2.8 and Lemma 2.3 for $\mathcal{N} \in D^b_\text{coh}(\mathcal{D}_X^h)$ one has

$$\begin{align*}
(3.1) & \quad \mathcal{N}^{\text{DR}_h} \simeq \mathcal{D}_X^h \otimes_{\mathcal{D}_X^h} \mathcal{N}, \\
(3.2) & \quad \text{gr}_h(\mathcal{N}^{\text{DR}_h}) \simeq \mathcal{N}, \\
(3.3) & \quad \text{Sol}_h(\mathcal{N}^{\text{DR}_h}) \simeq \text{Sol}(\mathcal{N})^{\text{DR}_h}.
\end{align*}$$

**Definition 3.2.** For $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$, denote by $\mathcal{M}_{h,\text{tor}}$ its submodule consisting of sections locally annihilated by some power of $h$ and set $\mathcal{M}_{h,\text{tf}} = \mathcal{M} / \mathcal{M}_{h,\text{tor}}$. We say that $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$ is an $h$-torsion module if $\mathcal{M}_{h,\text{tor}} \simeq \mathcal{M}$ and that $\mathcal{M}$ has no $h$-torsion (or is $h$-torsion free) if $\mathcal{M} \simeq \mathcal{M}_{h,\text{tf}}$.

Denote by $n_\mathcal{M}$ the kernel of $h^{n+1}: \mathcal{M} \to \mathcal{M}$. Then $\mathcal{M}_{h,\text{tor}}$ is the sheaf associated with the increasing union of the $n_\mathcal{M}$'s. Hence, if $\mathcal{M}$ is coherent, the increasing family $\{n_\mathcal{M}\}_n$ is locally stationary and $\mathcal{M}_{h,\text{tor}}$ as well as $\mathcal{M}_{h,\text{tf}}$ are coherent.

**Characteristic variety.** Recall the following definition.
Definition 3.3.  (i) For \( \mathcal{C} \) an abelian category, a function \( c: \text{Ob}(\mathcal{C}) \to \text{Set} \) is called \textit{additive} if \( c(M) = c(M') \cup c(M'') \) for any short exact sequence \( 0 \to M' \to M \to M'' \to 0 \).

(ii) For \( \mathcal{T} \) a triangulated category, a function \( c: \text{Ob}(\mathcal{T}) \to \text{Set} \) is called \textit{additive} if \( c(M) = c(M[1]) \) and \( c(M) \subset c(M') \cup c(M'') \) for any distinguished triangle \( M' \to M \to M'' \to +1 \).

Note that an additive function \( c \) on \( \mathcal{C} \) naturally extends to the derived category \( D(\mathcal{C}) \) by setting \( c(M) = \bigcup_i c(H^i(M)) \).

For \( N \) a coherent \( \mathcal{D}_X \)-module, denote by \( \text{char}(N) \) its characteristic variety, a closed involutive subvariety of the cotangent bundle \( T^*X \). The characteristic variety is additive on \( \text{Mod}_{\text{coh}}(\mathcal{D}_X) \). For \( N \in D^b_{\text{coh}}(\mathcal{D}_X) \) one sets \( \text{char}(N) = \bigcup_i \text{char}(H^i(N)) \).

Definition 3.4.  The \textit{characteristic variety} of \( M \in D^b_{\text{coh}}(\mathcal{D}_\hbar X) \) is defined by

\[
\text{char}_\hbar(M) = \text{char}(\text{gr}_\hbar(M)).
\]

To \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_\hbar X) \) one associates the coherent \( \mathcal{D}_X \)-modules

\[
\begin{align*}
\mathcal{M}_0 &= \ker(h: \mathcal{M} \to \mathcal{M}) = H^{-1}(\text{gr}_\hbar(\mathcal{M})), \\
\mathcal{M}_0 &= \coker(h: \mathcal{M} \to \mathcal{M}) = H^0(\text{gr}_\hbar(\mathcal{M})).
\end{align*}
\]

Lemma 3.5.  For \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_\hbar X) \) an \( \hbar \)-torsion module, one has

\[
\text{char}_\hbar(\mathcal{M}) = \text{char}(\mathcal{M}_0) = \text{char}(\mathcal{M}_0).
\]

Proof.  By definition, \( \text{char}_\hbar(\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}(\mathcal{M}_0) \). It is thus enough to prove the equality \( \text{char}(\mathcal{M}_0) = \text{char}(\mathcal{M}_0) \).

Since the statement is local we may assume that \( h^N \mathcal{M} = 0 \) for some \( N \in \mathbb{N} \). We proceed by induction on \( N \).

For \( N = 1 \) we have \( \mathcal{M} \simeq \mathcal{M}_0 \simeq \mathcal{M}_0 \), and the statement is obvious.

Assume that the statement has been proved for \( N - 1 \). The short exact sequence

\[
0 \to h.\mathcal{M} \to \mathcal{M} \to \mathcal{M}_0 \to 0
\]

induces the distinguished triangle

\[
\text{gr}_h h.\mathcal{M} \to \text{gr}_h \mathcal{M} \to \text{gr}_h \mathcal{M}_0 \xrightarrow{+1}.
\]

Noticing that \( \mathcal{M}_0 \simeq (\mathcal{M}_0)_0 \simeq 0.\mathcal{M}_0 \), the associated long exact cohomology sequence gives

\[
0 \to 0(h.\mathcal{M}) \to 0.\mathcal{M} \to (\mathcal{M}_0)_0 \to 0.
\]
By inductive hypothesis we have \( \text{char}(\mathcal{M}_0) = \text{char}((\mathcal{M}_0)_0) \), and we deduce \( \text{char}(\mathcal{M}) = \text{char}((\mathcal{M}_0)_0) \) by additivity of char.

**Proposition 3.6.** (i) For \( \mathcal{M} \in \text{Modcoh}(\mathcal{D}^b_X) \) one has

\[
\text{char}_h(\mathcal{M}) = \text{char}_h(\mathcal{M}_0).
\]

(ii) The characteristic variety \( \text{char}_h \) is additive both on \( \text{Modcoh}(\mathcal{D}^b_X) \) and on \( \text{D}^b_{\text{coh}}(\mathcal{D}^b_X) \).

**Proof.** (i) As \( \text{char}(\text{gr}_h \mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}(0.\mathcal{M}) \), it is enough to prove the inclusion

\[
(3.7) \quad \text{char}(0.\mathcal{M}) \subset \text{char}(\mathcal{M}_0).
\]

Consider the short exact sequence \( 0 \rightarrow \mathcal{M}_{h,\text{tor}} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{h,\text{tf}} \rightarrow 0 \). Since \( \mathcal{M}_{h,\text{tf}} \) has no \( h \)-torsion, \( 0(\mathcal{M}_{h,\text{tf}}) = 0 \). The associated long exact cohomology sequence thus gives

\[
o(\mathcal{M}_{h,\text{tor}}) \simeq 0.\mathcal{M}, \quad 0 \rightarrow (\mathcal{M}_{h,\text{tor}})_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}_{h,\text{tf}})_0 \rightarrow 0.
\]

We deduce

\[
\text{char}(0.\mathcal{M}) = \text{char}(0(\mathcal{M}_{h,\text{tor}})) = \text{char}((\mathcal{M}_{h,\text{tor}})_0) \subset \text{char}(\mathcal{M}_0),
\]

where the second equality follows from Lemma 3.5.

(ii) It is enough to prove the additivity on \( \text{Modcoh}(\mathcal{D}^b_X) \), i.e. the equality

\[
\text{char}_h(\mathcal{M}) = \text{char}_h(\mathcal{M}') \cup \text{char}_h(\mathcal{M}'')
\]

for \( 0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0 \) a short exact sequence of coherent \( \mathcal{D}^b_X \)-modules.

The associated distinguished triangle \( \text{gr}_h \mathcal{M}' \rightarrow \text{gr}_h \mathcal{M} \rightarrow \text{gr}_h \mathcal{M}'' \rightarrow 1 \) induces the long exact cohomology sequence

\[
o(\mathcal{M}'') \rightarrow (\mathcal{M}')_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}'')_0 \rightarrow 0.
\]

By additivity of \( \text{char}(*) \), the exactness of this sequence at the first, second and third term from the right, respectively, gives

\[
\text{char}_h(\mathcal{M}'') \subset \text{char}_h(\mathcal{M}),
\]

\[
\text{char}_h(\mathcal{M}) \subset \text{char}_h(\mathcal{M}') \cup \text{char}_h(\mathcal{M}'')
\]

\[
\text{char}_h(\mathcal{M}') \subset \text{char}(0(\mathcal{M}'')) \cup \text{char}_h(\mathcal{M}).
\]

Finally, note that \( \text{char}(0(\mathcal{M}'')) \subset \text{char}_h(\mathcal{M}'') \subset \text{char}_h(\mathcal{M}). \)
In view of Proposition 3.6 (i), in order to define the characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ one could avoid derived categories considering $\text{char}(\mathcal{M}_0)$ instead of $\text{char}(\text{gr}_h\mathcal{M})$. The next lemma shows that these definitions are still compatible for $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X^h)$.

**Lemma 3.7.** For $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X^h)$ one has

$$\bigcup_i \text{char}(H^i(\text{gr}_h\mathcal{M})) = \bigcup_i \text{char}((H^i\mathcal{M})_0).$$

**Proof.** By additivity of char, the short exact sequence

$$0 \to (H^i\mathcal{M})_0 \to H^i(\text{gr}_h\mathcal{M}) \to 0$$

from [13, Lemma 1.4.2] induces the relations

$$\text{char}((H^i\mathcal{M})_0) \subset \text{char}(H^i(\text{gr}_h\mathcal{M})),
\text{char}(H^i(\text{gr}_h\mathcal{M})) = \text{char}((H^i\mathcal{M})_0) \cup \text{char}(0(H^{i+1}\mathcal{M})).$$

One concludes by noticing that (3.7) gives

$$\text{char}(0(H^{i+1}\mathcal{M})) \subset \text{char}((H^{i+1}\mathcal{M})_0).$$

**Proposition 3.8.** Let $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$ be an $h$-torsion module. Then $\mathcal{M}$ is coherent as a $\mathcal{D}_X^h$-module if and only if it is coherent as a $\mathcal{D}_X$-module, and in this case one has $\text{char}_h(\mathcal{M}) = \text{char}(\mathcal{M})$.

**Proof.** As in the proof of Lemma 3.5 we assume that $h^NM = 0$ for some $N \in \mathbb{N}$. Since coherence is preserved by extension and since the characteristic varieties of $\mathcal{D}_X^h$-modules and $\mathcal{D}_X$-modules are additive, we can argue by induction on $N$ using the exact sequence (3.6). We are thus reduced to the case $N = 1$, where $\mathcal{M} = \mathcal{M}_0$ and the statement becomes obvious.

From (3.2) we obtain

**Proposition 3.9.** For $\mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_X)$ one has $\text{char}_h(\mathcal{N}^h) = \text{char}(\mathcal{N})$.

**Holonomic modules.** Recall that a coherent $\mathcal{D}_X$-module (or an object of the derived category) is called *holonomic* if its characteristic variety is isotropic. We refer e.g. to [7, Chapter 5] for the notion of regularity.

**Definition 3.10.** We say that $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X^h)$ is holonomic, or regular holonomic, if so is $\text{gr}_h(\mathcal{M})$. We denote by $D^b_{\text{hol}}(\mathcal{D}_X^h)$ the full triangulated subcategory of $D^b_{\text{coh}}(\mathcal{D}_X^h)$ of holonomic objects and by $D^b_{\text{hol}}(\mathcal{D}_X^h)$ the full triangulated subcategory of regular holonomic objects.
Note that a coherent $\mathcal{D}_X$-module is holonomic if and only if its characteristic variety is isotropic.

**Example 3.11.** Let $\mathcal{N}$ be a regular holonomic $\mathcal{D}_X$-module. Then

(i) $\mathcal{N}$ itself, considered as a $\mathcal{D}_X$-module, is regular holonomic, as follows from the isomorphism $\text{gr}_h \mathcal{N} \simeq \mathcal{N} \oplus \mathcal{N}[1]$;

(ii) $\mathcal{N}^h$ is a regular holonomic $\mathcal{D}_X^h$-module, as follows from the isomorphism $\text{gr}_h \mathcal{N}^h \simeq \mathcal{N}$. In particular, $\mathcal{O}_X^h$ is regular holonomic.

**Remark 3.12.** We denote by $\text{Mod}_{\text{rh}}(\mathcal{D}_X)$ the category of regular holonomic $\mathcal{D}_X$-modules and by $\text{Mod}_{\text{rh}}(\mathcal{D}_X^h)$ the subcategory of $\text{Mod}(\mathcal{D}_X^h)$ of regular holonomic objects in the above sense. The proofs of Lemma 3.5 and Proposition 3.6 adapt to the notion of regular holonomy and give the following results:

(i) For $M \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$ an $h$-torsion module,

$M \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \Leftrightarrow M_0 \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \Leftrightarrow 0_M \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$.

(ii) For $M \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$,

$M \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \Leftrightarrow M_0 \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$.

In this case, $0_M \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$.

Now for $\mathcal{M} \in \text{D}^b_{\text{coh}}(\mathcal{D}_X^h)$ the exact sequence (3.8) shows that, for any $i$,

$H^i(\text{gr}_h \mathcal{M}) \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \Leftrightarrow (H^i \mathcal{M})_0 \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$.

By the above we deduce that $\mathcal{M} \in \text{D}^b_{\text{rh}}(\mathcal{D}_X^h)$ if and only if $(H^i \mathcal{M})_0 \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$ for all $i$. This is again equivalent to $H^i \mathcal{M} \in \text{Mod}_{\text{rh}}(\mathcal{D}_X^h)$ for all $i$.

**Propagation.** Denote by $\text{D}^b_{\text{C}}(\mathcal{C}_X^h)$ the full triangulated subcategory of $\text{D}^b(\mathcal{C}_X^h)$ consisting of objects with $\mathbb{C}$-constructible cohomology over the ring $\mathbb{C}_h$.

**Theorem 3.13.** Let $\mathcal{M}, \mathcal{N} \in \text{D}^b_{\text{coh}}(\mathcal{D}_X^h)$. Then

$$\text{SS}(R\text{Hom}_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})) = \text{SS}(R\text{Hom}_{\mathcal{D}_X^h}(\text{gr}_h(\mathcal{M}), \text{gr}_h(\mathcal{N}))).$$

If moreover $\mathcal{M}$ and $\mathcal{N}$ are holonomic, then $R\text{Hom}_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})$ is an object of $\text{D}^b_{\text{C}}(\mathcal{C}_X^h)$.

**Proof.** Set $F = R\text{Hom}_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})$. By Theorem 1.9 and Proposition 1.5, $F$ is cohomologically $h$-complete. Hence $\text{SS}(F) = \text{SS}(\text{gr}_h(F))$ by Proposition 1.15. If moreover $\mathcal{M}$ and $\mathcal{N}$ are holonomic, then $\text{gr}_h F$ is $\mathbb{C}$-constructible. The equality $\text{SS}(F) = \text{SS}(\text{gr}_h(F))$ implies that $F$ is weakly $\mathbb{C}$-constructible. Moreover, the
finiteness of the stalks $\text{gr}_h(F)_x \simeq \text{gr}_h(F_x)$ over $\mathbb{C}$ implies the finiteness of $F_x$ over $\mathbb{C}^h$ by Theorem 1.11 applied with $X = \{\text{pt}\}$ and $\mathscr{A} = \mathbb{C}^h$.

Applying Theorem 3.13 and [9, Theorem 11.3.3], we get:

**Corollary 3.14.** Let $\mathcal{M} \in D^b_{\text{coh}}(\mathscr{D}_X^h)$. Then

$$\text{SS} (\text{Sol}_h(\mathcal{M})) = \text{SS} (\text{DR}_h(\mathcal{M})) = \text{char}_h(\mathcal{M}).$$

If moreover $\mathcal{M}$ is holonomic, then $\text{Sol}_h(\mathcal{M})$ and $\text{DR}_h(\mathcal{M})$ belong to $D^b_{\mathbb{C},c}(\mathbb{C}^h_X)$.

**Theorem 3.15.** Let $\mathcal{M} \in D^b_{\text{hol}}(\mathscr{D}_X^h)$. Then there is a natural isomorphism in $D^b_{\mathbb{C},c}(\mathbb{C}^h_X)$

$$\text{Sol}_h(\mathcal{M}) \simeq D'_h(\text{DR}_h(\mathcal{M})).$$

**Proof.** The natural $\mathbb{C}^h$-linear morphism

$$\text{RHom}_{\mathscr{D}_X^h}(\mathcal{O}_X^h, \mathcal{M}) \otimes_{\mathbb{C}^h_X} \text{RHom}_{\mathscr{D}_X^h}(\mathcal{O}_X^h, \mathcal{O}_X^h) \to \text{RHom}_{\mathscr{D}_X^h}(\mathcal{O}_X^h, \mathcal{O}_X^h) \simeq \mathbb{C}^h$$

induces the morphism in $D^b_{\mathbb{C},c}(\mathbb{C}^h_X)$

$$\alpha: \text{RHom}_{\mathscr{D}_X^h}(\mathcal{M}, \mathcal{O}_X^h) \to D'_h(\text{RHom}_{\mathscr{D}_X^h}(\mathcal{O}_X^h, \mathcal{M})).$$

(Note that, choosing $\mathcal{M} = \mathscr{D}_X^h$, this morphism defines the morphism $\mathcal{O}_X^h \to D'_h(\Omega_X^h[-d_X])$.) The morphism (3.10) induces an isomorphism

$$\text{gr}_h(\alpha): \text{RHom}_{\mathscr{D}_X}(\text{gr}_h(\mathcal{M}), \mathcal{O}_X) \to D'(\text{RHom}_{\mathscr{D}_X}(\mathcal{O}_X, \text{gr}_h(\mathcal{M}))).$$

It is thus an isomorphism by Corollary 1.17.

### §4. Formal extension of tempered functions

Let us start by reviewing after [11, Chapter 7] the construction of the sheaves of tempered distributions and of $C^\infty$-functions with temperate growth on the subanalytic site.

Let $X$ be a real analytic manifold, and $U$ an open subset. One says that a function $f \in \mathcal{E}^\infty_X(U)$ has polynomial growth at $p \in X$ if, for a local coordinate system $(x_1, \ldots, x_n)$ around $p$, there exist a sufficiently small compact neighborhood $K$ of $p$ and a positive integer $N$ such that

$$\sup_{x \in K \cap U} (\text{dist}(x, K \setminus U))^N |f(x)| < \infty.$$ 

One says that $f$ is tempered at $p$ if all its derivatives are of polynomial growth at $p$. One says that $f$ is tempered if it is tempered at any point of $X$. One denotes
Lemma 4.1. One has $H^j(U; \sC_X^{\infty,t}) = 0$ for any $U \in \text{Op}_{X_{sa}}$ and any $j > 0$.

This result is well-known (see [10] Chapter 1), but we recall its proof for the reader’s convenience.

Proof. Consider the full subcategory $\mathcal{J}$ of $\text{Mod}(\mathcal{C}_X^{\text{sa}})$ whose objects are sheaves $F$ such that for any pair $U,V \in \text{Op}_{X_{sa}}$, the Mayer–Vietoris sequence

$$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V) \to 0$$

is exact. Let us check that this category is injective with respect to the functor $\Gamma(U; \cdot)$. The only non-obvious fact is that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence and that $F'$ belongs to $\mathcal{J}$, then $F(U) \to F''(U)$ is surjective. Let $t \in F''(U)$. There exist a finite covering $U = \bigcup_{i \in I} U_i$ and $s_i \in F(U_i)$ whose image in $F''(U_i)$ is $t|_{U_i}$. Then the proof goes by induction on the cardinality of $I$ using the property of $F'$ and standard arguments. To conclude, note that $\mathcal{C}_X^{\infty,t}$ belongs to $\mathcal{J}$ thanks to Łojasiewicz’s result (see [14]). 

Let $\mathcal{D}b_X$ be the sheaf of distributions on $X$. For $U \in \text{Op}_{X_{sa}}$, denote by $\mathcal{D}b_X(U)$ the space of tempered distributions on $U$, defined by the exact sequence

$$0 \to \Gamma_{X\setminus U}(X; \mathcal{D}b_X) \to \Gamma(X; \mathcal{D}b_X) \to \mathcal{D}b_X^t(U) \to 0.$$ 

Again, it follows from a theorem of Łojasiewicz that $U \mapsto \mathcal{D}b^t(U)$ is a sheaf on $X_{sa}$. We denote it by $\mathcal{D}b_X^t$ or simply $\mathcal{D}b_X^t$ if there is no risk of confusion. The sheaf $\mathcal{D}b_X^t$ is quasi-injective, that is, the functor $\mathcal{H}om_{\mathcal{C}_X^{\text{sa}}}(\cdot, \mathcal{D}b_X^t)$ is exact in the category $\text{Mod}_{R,c}(\mathcal{C}_X)$. Moreover, for $U \in \text{Op}_{X_{sa}}$, $\mathcal{H}om_{\mathcal{C}_X^{\text{sa}}}(\mathcal{C}_U, \mathcal{D}b_X^t)$ is also quasi-injective and $\mathcal{R}\mathcal{H}om_{\mathcal{C}_X^{\text{sa}}}(\mathcal{C}_U, \mathcal{D}b_X^t)$ is concentrated in degree 0. Note that the sheaf

$$\Gamma_{[U]} \mathcal{D}b_X := \rho^{-1}\mathcal{H}om_{\mathcal{C}_X^{\text{sa}}}(\mathcal{C}_U, \mathcal{D}b_X^t)$$

is an $\mathcal{C}_X^{\infty}$-module, so that in particular $R\Gamma(V; \Gamma_{[U]} \mathcal{D}b_X)$ is concentrated in degree 0 for $V \subset X$ an open subset.

**Formal extensions.** By Proposition 2.5 the sheaves $\mathcal{C}_X^{\infty,t}$, $\mathcal{D}b_X^t$ and $\Gamma_{[U]} \mathcal{D}b_X$ are acyclic for the functor $(\cdot)^h$. We set

$$\mathcal{C}_X^{\infty,t,h} := (\mathcal{C}_X^{\infty,t})^h, \quad \mathcal{D}b_X^{t,h} := (\mathcal{D}b_X^t)^h, \quad \Gamma_{[U]} \mathcal{D}b_X^h := (\Gamma_{[U]} \mathcal{D}b_X)^h.$$ 

Note that, by Lemmas 2.3 and 2.9

$$\Gamma_{[U]} \mathcal{D}b_X^h \simeq \rho^{-1}\mathcal{H}om_{\mathcal{C}_X^{\text{sa}}}(\mathcal{C}_U, \mathcal{D}b_X^{t,h}).$$
By Proposition 2.2 we get:

**Proposition 4.2.** The sheaves $C^\infty_{X, t, \hbar}$, $\mathcal{D}^{L,h}_{X}$ and $\Gamma_U \mathcal{D}^{L,h}_{X}$ are cohomologically $\hbar$-complete.

Now assume $X$ is a complex manifold. Denote by $\overline{X}$ the complex conjugate manifold and by $X^\mathbb{R}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. One defines the sheaf (in fact, an object of the derived category) of tempered holomorphic functions by

$$\mathcal{O}_{t, \hbar} := R \mathcal{H}om_{\rho, \mathcal{D}^X} (\rho \mathcal{O}_X, C^\infty_{\hbar, X}) \sim R \mathcal{H}om_{\rho, \mathcal{D}^X} (\rho \mathcal{O}_X, \mathcal{D}^{L,h}_X).$$

Here and below, we write $C^\infty_{X, t, \hbar}$ and $\mathcal{D}^{L,h}_X$ instead of $C^\infty_{X, t, \hbar}^\mathbb{R}$ and $\mathcal{D}^{L,h}_X^\mathbb{R}$, respectively.

We set

$$\mathcal{O}_{t, \hbar} := (\mathcal{O}_{t, \hbar})^\mathbb{R},$$

a cohomologically $\hbar$-complete object of $D^b(C^\hbar_X)$. By Lemma 2.3,

$$\mathcal{O}_{t, \hbar} \simeq R \mathcal{H}om_{\rho, \mathcal{D}^X} (\rho \mathcal{O}_X, C^\infty_{X, t, \hbar}) \sim R \mathcal{H}om_{\rho, \mathcal{D}^X} (\rho \mathcal{O}_X, \mathcal{D}^{L,h}_X).$$

Note that $gr\mathcal{O}_{t, \hbar} \simeq \mathcal{O}_{t, \hbar}$ in $D^b(C^\hbar_X)$.

**§5. Riemann–Hilbert correspondence**

Let $X$ be a complex analytic manifold. Consider the functors

$$TH : D^b_{\mathcal{C}, c}(\mathcal{D}^h_X) \to D^b_{\mathcal{D}}(\mathcal{D}^h_X)^{op}, \quad F \to \rho^{-1} R \mathcal{H}om_{C^\hbar_X} (\rho_* F, \mathcal{O}_{t, \hbar}^X),$$

$$TH^h : D^b_{\mathcal{C}, c}(\mathcal{D}^h_X) \to D^b_{\mathcal{D}}(\mathcal{D}^h_X)^{op}, \quad F \to \rho^{-1} R \mathcal{H}om_{\mathcal{C}^\hbar_X} (\rho_* F, \mathcal{O}_{t, \hbar}^X).$$

The classical Riemann–Hilbert correspondence of Kashiwara [6] states that the functors $Sol$ and $TH$ are equivalences of categories between $D^b_{\mathcal{C}, c}(\mathcal{D}^h_X)$ and $D^b_{\mathcal{D}}(\mathcal{D}^h_X)^{op}$ quasi-inverse to each other. In order to obtain a similar statement for $\mathcal{C}^h_X$ and $\mathcal{D}^h_X$ replaced with $\mathcal{C}^\hbar_X$ and $\mathcal{D}^{L,h}_X$, respectively, we start by establishing some lemmas.

**Lemma 5.1.** For $M, N \in D^b_{\mathcal{C}, c}(\mathcal{D}^h_X)$ one has a natural isomorphism in $D^b_{\mathcal{C}, c}(\mathcal{D}^h_X)$

$$R \mathcal{H}om_{\mathcal{D}^h_X} (M, N) \sim R \mathcal{H}om_{\mathcal{C}^\hbar_X} (Sol^h(M), Sol^h(N)).$$

**Proof.** Applying the functor $gr\mathcal{h}$ to this morphism, we get an isomorphism by the classical Riemann–Hilbert correspondence. Then the result follows from Corollary 1.17 and Theorem 3.13.

Note that there is an isomorphism in $D^b(\mathcal{D}^h_X)$

$$(5.1) \quad gr\mathcal{h}(TH\mathcal{h}(F)) \simeq TH(gr\mathcal{h}(F)).$$
Lemma 5.2. The functor $\text{TH}_h$ induces a functor

\begin{equation}
(5.2) \quad \text{TH}_h : D^b_{C,c}(\mathbb{C}^h_X) \to D^b_{\text{th}}(\mathcal{D}^h_X)^{\text{op}}.
\end{equation}

Proof. Let $F \in D^b_{C,c}(\mathbb{C}^h_X)$. By [5.1] and the classical Riemann–Hilbert correspondence we know that $\text{gr}_h(\text{TH}_h(F))$ is regular holonomic, and in particular coherent. It is thus left to prove that $\text{TH}_h(F)$ is coherent. Note that our problem is of local nature.

We use the Dolbeault resolution of $\mathcal{O}^{t,h}_X$ with coefficients in $\mathcal{D}^{t,h}_X$ and we choose a resolution of $F$ as given in Proposition [A.2]. We find that $\text{TH}_h(F)$ is isomorphic to a bounded complex $\mathcal{M}^*$, where the $\mathcal{M}^i$ are locally finite sums of sheaves of the type $\Gamma[U,\mathcal{D}^{t,h}_X]$ with $U \in \text{Op}_{X^h}$. It follows from Proposition 4.2 that $\text{TH}_h(F)$ is cohomologically $h$-complete, and we conclude by Theorem 1.11 with $\mathcal{A} = \mathcal{D}^h_X$.

Lemma 5.3. We have $R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^h_X, \mathcal{O}^{t,h}_X) \simeq \mathcal{C}^h_{X^h}$.

Proof. This isomorphism is given by the sequence

$$R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^h_X, \mathcal{O}^{t,h}_X) \simeq R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}_X, \mathcal{O}^{t,h}_X) \simeq R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}_X, \mathcal{O}^{t,h}_X)^{Rh} \simeq (\rho_* R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}_X, \mathcal{O}^{t,h}_X))^Rh \simeq (\mathcal{C}^h_{X^h})^{Rh} \simeq \mathcal{C}^h_{X^h},$$

where the first isomorphism is an extension of scalars, the second follows from Lemma 2.3 and the third is given by the adjunction between $\rho$ and $\rho^{-1}$.

Theorem 5.4. The functors $\text{Sol}_h$ and $\text{TH}_h$ are equivalences of categories between $D^b_{C,c}(\mathbb{C}^h_X)$ and $D^b_{\text{th}}(\mathcal{D}^h_X)^{\text{op}}$ quasi-inverse to each other.

Proof. In view of Lemma 5.1, the functor $\text{Sol}_h$ is fully faithful. It is then enough to show that $\text{Sol}_h(\text{TH}_h(F)) \simeq F$ for $F \in D^b_{C,c}(\mathbb{C}^h_X)$. By Theorem 3.15, this is equivalent to $D\text{R}_{\text{th}}(\text{TH}_h F) \simeq D^h F$. Since we already know by Lemma 5.2 that $\text{TH}_h(F)$ is holonomic, we may use [3.9]. We have the sequence of isomorphisms

$$\rho_* R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^{t,h}_X, \text{TH}_h(F)) = \rho_* R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^h_X, \rho^{-1} R\mathcal{H}\text{om}_{\mathcal{C}^{h}_{X^h}}(\rho_* F, \mathcal{O}^{t,h}_X))$$

$$\simeq R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^h_X, R\mathcal{H}\text{om}_{\mathcal{C}^{h}_{X^h}}(\rho_* F, \mathcal{O}^{t,h}_X))$$

$$\simeq R\mathcal{H}\text{om}_{\mathcal{C}^{h}_{X^h}}(\rho_* F, R\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\rho, \mathcal{O}^h_X, \mathcal{O}^{t,h}_X))$$

$$\simeq R\mathcal{H}\text{om}_{\mathcal{C}^{h}_{X^h}}(\rho_* F, \mathcal{C}^h_X) \simeq R\mathcal{H}\text{om}_{\mathcal{C}^{h}_{X^h}}(\rho_* F, \rho_* \mathcal{C}^h_X) \simeq \rho_* D^h F,$$

where we have used the adjunction between $\rho$ and $\rho^{-1}$, the isomorphism of Lemma 5.3 and the commutation of $\rho_*$ with $R\mathcal{H}\text{om}$. One concludes by recalling the isomorphism of functors $\rho^{-1} \rho_* \simeq \text{id}$.

\]
**t-structure.** Recall the definition of the middle perversity $t$-structure for complex constructible sheaves. Let $K$ denote either the field $\mathbb{C}$ or the ring $\mathbb{C}$. For $F \in D^b_{\text{hol}}(K_X)$, we have $F \in pD^{\geq 0}_{\text{hol}}(K_X)$ if and only if

$$\forall i \in \mathbb{Z} \quad \dim \text{supp} \, H^i(F) \leq d_X - i,$$

and $F \in pD^{\leq 0}_{\text{hol}}(K_X)$ if and only if, for any locally closed complex analytic subset $S \subset X$,

$$H^i_{\text{et}}(F) = 0 \quad \text{for all } i < d_X - \dim(S).$$

One denotes by $\text{Perv}(K_X)$ the heart of this $t$-structure.

With the above convention, the de Rham functor

$$\text{DR}: D^b_{\text{hol}}(\mathcal{X}) \to pD^b_{\text{hol}}(\mathcal{X})$$

is $t$-exact, when $D^b_{\text{hol}}(\mathcal{X})$ is equipped with the natural $t$-structure.

**Theorem 5.5.** The de Rham functor $\text{DR}^h: D^b_{\text{hol}}(\mathcal{X}) \to pD^b_{\text{hol}}(\mathcal{X})$ is $t$-exact, and induces a $t$-exact equivalence between $D^b_{\text{hol}}(\mathcal{X})$ and $pD^b_{\text{hol}}(\mathcal{X})$. In particular, it induces an equivalence between $\text{Mod}_{\text{tor}}(\mathcal{X})$ and $\text{Perv}(\mathcal{X})$.

**Proof.** (i) Let $\mathcal{M} \in D^\leq_{\text{hol}}(\mathcal{X})$. Let us prove that $\text{DR}^h(\mathcal{M}) \in pD^{\geq 0}_{\text{hol}}(\mathcal{X})$. Since $\text{DR}^h(\mathcal{M})$ is constructible, by Proposition 1.19 it is enough to check (5.3) for $\text{gr}^h(\text{DR}^h(\mathcal{M})) \simeq \text{GR}(\text{DR}^h(\mathcal{M}))$. In other words, it is enough to check that $\text{DR}^h(\text{gr}^h(\mathcal{M})) \in pD^{\geq 0}_{\text{hol}}(\mathcal{X})$. Since $\text{gr}^h(\mathcal{M}) \in D^\leq_{\text{hol}}(\mathcal{X})$, this result follows from the $t$-exactness of the functor $\text{DR}$.

(ii) Let $\mathcal{M} \in D^\geq_{\text{hol}}(\mathcal{X})$. Let us prove that $\text{DR}^h(\mathcal{M}) \in pD^{\leq 0}_{\text{hol}}(\mathcal{X})$. We set $\mathcal{N} = (H^0(\mathcal{M}))_{\text{h-af}}$. We have a morphism $u: \mathcal{N} \to \mathcal{M}$ induced by $H^0(\mathcal{M}) \to \mathcal{M}$ and we let $\mathcal{M}'$ be the mapping cone of $u$. We have a distinguished triangle

$$\text{DR}^h(\mathcal{N}) \to \text{DR}^h(\mathcal{M}) \to \text{DR}^h(\mathcal{M}') \xrightarrow{1}$$

so that it is enough to show that $\text{DR}^h(\mathcal{N})$ and $\text{DR}^h(\mathcal{M}')$ belong to $pD^{\geq 0}_{\text{hol}}(\mathcal{X})$.

(ii-a) By Propositions 3.6(ii) and 3.8, $\mathcal{N}$ is holonomic as a $\mathcal{X}$-module. Hence $\text{DR}^h(\mathcal{N}) \simeq \text{DR}^h(\mathcal{N})$ is a perverse sheaf (over $\mathcal{X}$) and satisfies (5.4). Since (5.4) does not depend on the coefficient ring, $\text{DR}^h(\mathcal{N}) \in pD^{\geq 0}_{\text{hol}}(\mathcal{X})$.

(ii-b) We note that $H^0(\mathcal{M}') \simeq (H^0(\mathcal{M}))_{\text{h-af}}$. Hence by Proposition 1.14, $\text{gr}^h(\mathcal{M}') \in D^{\geq 0}_{\text{hol}}(\mathcal{X})$ and $\text{DR}^h(\text{gr}^h(\mathcal{M}')) = pD^{\leq 0}_{\text{hol}}(\mathcal{X})$, that is, $\text{DR}^h(\text{gr}^h(\mathcal{M}'))$ satisfies (5.4). Let $S \subset X$ be a locally closed complex subanalytic subset. We have

$$\text{RG}_S(\text{DR}^h(\mathcal{M}')) \simeq \text{gr}^h(\text{RG}_S(\text{DR}^h(\mathcal{M}')))$$

and it follows from Proposition 1.19 that $\text{DR}^h(\mathcal{M}')$ also satisfies (5.4) and thus belongs to $pD^{\leq 0}_{\text{hol}}(\mathcal{X})$. For

$$\forall i \in \mathbb{Z} \quad \dim \text{supp} \, H^i(F) \leq d_X - i,$$
(iii) Consider the restriction $\text{DR}_h: D^b_{\text{th}}(\mathcal{D}_X) \to \mathcal{D}_{\text{th}}^b(\mathcal{C}_X)$ to regular holonomic complexes. In view of Lemma 3.11, it follows from Theorems 5.4 and 3.15 that the functor $\text{TH}_h \circ D^b_{\text{th}}$ is a quasi-inverse to $\text{DR}_h$. As quasi-inverse to a $t$-exact functor, $\text{TH}_h \circ D^b_{\text{th}}$ is also $t$-exact. Thus $\text{DR}_h$ is a $t$-exact equivalence, and it induces an equivalence between the respective hearts, i.e. between $\text{Mod}_{\text{th}}(\mathcal{D}_X)$ and $\text{Perv}(\mathcal{C}_X)$.

§6. Duality and $h$-torsion

The duality functors $D$ on $D^b_{\text{th}}(\mathcal{D}_X)$ and $D'$ on $\mathcal{D}^b_{\text{th}}(\mathcal{C}_X)$ are $t$-exact. We will discuss here the finer $t$-structures needed in order to obtain a similar result when replacing $\mathcal{C}_X$ and $\mathcal{D}_X$ by their formal extensions $\mathcal{C}_X$ and $\mathcal{D}_X$.

Following [2, Chapter I.2], let us start by recalling some facts related to torsion pairs and $t$-structures. We need in particular Proposition 6.2 below, which can also be found in [3].

Definition 6.1. Let $\mathcal{C}$ be an abelian category. A torsion pair on $\mathcal{C}$ is a pair $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$ of full subcategories such that

(i) for all objects $T$ in $\mathcal{C}_{\text{tor}}$ and $F$ in $\mathcal{C}_{\text{tf}}$, we have $\text{Hom}_{\mathcal{C}}(T, F) = 0$,

(ii) for any object $M$ in $\mathcal{C}$, there are objects $M_{\text{tor}}$ in $\mathcal{C}_{\text{tor}}$ and $M_{\text{tf}}$ in $\mathcal{C}_{\text{tf}}$ and a short exact sequence $0 \to M_{\text{tor}} \to M \to M_{\text{tf}} \to 0$.

Proposition 6.2. Let $\mathcal{D}$ be a triangulated category endowed with a $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let us denote its heart by $\mathcal{C}$ and its cohomology functors by $H^i: \mathcal{D} \to \mathcal{C}$. Suppose that $\mathcal{C}$ is endowed with a torsion pair $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$. Then we can define a new $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ by setting

$$\mathcal{D}^{\leq 0} = \{ M \in \mathcal{D}^{\leq 1}, H^1(M) \in \mathcal{C}_{\text{tor}} \}, \quad \mathcal{D}^{\geq 0} = \{ M \in \mathcal{D}^{\geq 0}, H^0(M) \in \mathcal{C}_{\text{tf}} \}.$$

With the notation of Definition 6.2, there is a natural torsion pair attached to $\text{Mod}(\mathcal{D}_X)$ given by the full subcategories

$$\text{Mod}(\mathcal{D}_X)_{\text{h-tor}} = \{ \mathcal{M}: \mathcal{M}_{\text{h-tor}} \sim \mathcal{M} \}, \quad \text{Mod}(\mathcal{D}_X)_{\text{h-tf}} = \{ \mathcal{M}: \mathcal{M} \sim \mathcal{M}_{\text{h-tf}} \}.$$

Definition 6.3. (a) We call the torsion pair on $\text{Mod}(\mathcal{D}_X)$ defined above, the $h$-torsion pair.

(b) We denote by $(\mathcal{D}^{\leq 0}(\mathcal{D}_X), \mathcal{D}^{\geq 0}(\mathcal{D}_X))$ the natural $t$-structure on $\mathcal{D}(\mathcal{D}_X)$.

(c) We denote by $(\mathcal{D}^{\leq 0}(\mathcal{D}_X), \mathcal{D}^{\geq 0}(\mathcal{D}_X))$ the $t$-structure on $\mathcal{D}(\mathcal{D}_X)$ associated via Proposition 6.2 with the $h$-torsion pair on $\text{Mod}(\mathcal{D}_X)$.
Proposition 1.14 implies the following equivalences for $\mathcal{M} \in \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^h)$:

\begin{align*}
(6.1) & \quad \mathcal{M} \in \mathcal{D}^{\geq 0}(\mathcal{D}_X^h) \iff \text{gr}_h \mathcal{M} \in \mathcal{D}^{\geq 0}(\mathcal{D}_X), \\
(6.2) & \quad \mathcal{M} \in \mathcal{D}^{\leq 0}(\mathcal{D}_X^h) \iff \text{gr}_h \mathcal{M} \in \mathcal{D}^{\leq 0}(\mathcal{D}_X).
\end{align*}

Proposition 6.4. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X^h$-module.

(i) If $\mathcal{M}$ has no $h$-torsion, then $\mathcal{D}_h \mathcal{M}$ is concentrated in degree 0 and has no $h$-torsion.

(ii) If $\mathcal{M}$ is an $h$-torsion module, then $\mathcal{D}_h \mathcal{M}$ is concentrated in degree 1 and is an $h$-torsion module.

Proof. By (1.2) we have $\text{gr}_h(\mathcal{D}_h \mathcal{M}) \simeq \mathcal{D}(\text{gr}_h \mathcal{M})$. Since $\text{gr}_h \mathcal{M}$ is concentrated in degrees 0 and $-1$, with holonomic cohomology, $\mathcal{D}(\text{gr}_h \mathcal{M})$ is concentrated in degrees 0 and 1. By Proposition 1.14 $\mathcal{D}_h \mathcal{M}$ itself is concentrated in degrees 0 and 1 and $H^0(\mathcal{D}_h \mathcal{M})$ has no $h$-torsion.

(i) The short exact sequence

$$0 \to \mathcal{M} \xrightarrow{h} \mathcal{M} \to \mathcal{M}/h \mathcal{M} \to 0$$

induces the long exact sequence

$$\cdots \to H^1(\mathcal{D}_h(\mathcal{M}/h \mathcal{M})) \to H^1(\mathcal{D}_h \mathcal{M}) \xrightarrow{h} H^1(\mathcal{D}_h \mathcal{M}) \to 0.$$

By Nakayama’s lemma $H^1(\mathcal{D}_h \mathcal{M}) = 0$ as required.

(ii) Since $\mathcal{M}$ is locally annihilated by some power of $h$, the cohomology groups $H^i(\mathcal{D}_h \mathcal{M})$ also are $h$-torsion modules. As $H^0(\mathcal{D}_h \mathcal{M})$ has no $h$-torsion, we get $H^0(\mathcal{D}_h \mathcal{M}) = 0$.

Theorem 6.5. The duality functor $\mathcal{D}_h : \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X^h) \to \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X^h)$ is $t$-exact. In other words, $\mathcal{D}_h$ interchanges $\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h)$ with $\mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h)$ and $\mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h)$ with $\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h)$.

Proof. (i) Let us first prove, for $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X^h)$,

\begin{align*}
(6.3) & \quad \mathcal{M} \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h) \iff \mathcal{D}_h(\mathcal{M}) \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X).
\end{align*}

By (1.2) we have $\text{gr}_h(\mathcal{D}_h \mathcal{M}) \simeq \mathcal{D}(\text{gr}_h \mathcal{M})$ and we know that the analog of (6.3) holds true for $\mathcal{D}_X$-modules:

$$\mathcal{N} \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X) \iff \mathcal{D}(\mathcal{N}) \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X).$$

Hence (6.3) follows easily from (6.1) and (6.2).
(ii) We recall the general fact for a $t$-structure $(\mathcal{D}, \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $A \in \mathcal{D}$:

\[ A \in \mathcal{D}^{\leq 0} \Leftrightarrow \text{Hom}(A, B) = 0 \text{ for any } B \in \mathcal{D}^{\geq 1}, \]

\[ A \in \mathcal{D}^{\geq 0} \Leftrightarrow \text{Hom}(B, A) = 0 \text{ for any } B \in \mathcal{D}^{\leq -1}. \]

Since $\mathcal{D}_h$ is an involutive equivalence of categories we deduce from \[6.3\] the dual statement:

\[ \mathcal{M} \in \mathcal{D}^{\leq 0}_{\text{hol}}(\mathcal{D}_X^h) \Leftrightarrow \mathcal{D}_h(\mathcal{M}) \in \mathcal{D}^{\leq 0}_{\text{hol}}(\mathcal{D}_X^h). \]

**Remark 6.6.** The above result can be stated as follows in the language of quasi-abelian categories of [19]. We will follow the notation of [8, Chapter 2]. The category $\mathcal{C} = \text{Mod}(\mathcal{D}_X^b)$ is quasi-abelian. Hence its derived category has a natural generalized $t$-structure $(\mathcal{D}^{\leq s}(\mathcal{C}), \mathcal{D}^{>s-1}(\mathcal{C}))_{s \in \mathbb{Z}}$. Note that $\mathcal{D}^{[-1/2,0]}(\mathcal{C})$ is equivalent to $\text{Mod}(\mathcal{D}_X^b)$, and $\mathcal{D}^{[0,1/2]}(\mathcal{C})$ is equivalent to the heart of $\mathcal{D}_h^b(\mathcal{D}_X^b)$. Then Theorem 6.5 states that the duality functor $\mathcal{D}_h$ is $t$-exact on $\mathcal{D}_h^b(\mathcal{C})$.

Recall that $\text{Perv}(\mathcal{C}_X^h)$ denotes the heart of the middle perversity $t$-structure on $\mathcal{D}_c^b(\mathcal{C}_X^h)$. Consider the full subcategories of $\text{Perv}(\mathcal{C}_X^h)$

\[ \text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}} = \{ F : \text{locally } h^N F = 0 \text{ for some } N \in \mathbb{N} \}, \]

\[ \text{Perv}(\mathcal{C}_X^h)_{h\text{-tf}} = \{ F : F \text{ has no non-zero subobjects in } \text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}} \}. \]

**Lemma 6.7.** (i) Let $F \in \text{Perv}(\mathcal{C}_X^h)$. Then the inductive system of sub-perverse sheaves $\text{Ker}(h^n : F \to F)$ is locally stationary.

(ii) The pair $(\text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}}, \text{Perv}(\mathcal{C}_X^h)_{h\text{-tf}})$ is a torsion pair.

**Proof.** (i) Set $\mathcal{M} = \text{D}_h \text{TH}_h(F)$. By the Riemann–Hilbert correspondence, one has $\text{Ker}(h^n : F \to F) \simeq \text{DR}_h(\text{Ker}(h^n : \mathcal{M} \to \mathcal{M}))$. Since $\mathcal{M}$ is coherent, the inductive system $\text{Ker}(h^n : \mathcal{M} \to \mathcal{M})$ is locally stationary. Hence so is the system $\text{Ker}(h^n : F \to F)$.

(ii) By (i) it makes sense to define, for $F \in \text{Perv}(\mathcal{C}_X^h)$,

\[ F_{h\text{-tor}} = \bigcup_n \text{Ker}(h^n : F \to F), \quad F_{h\text{-tf}} = F/F_{h\text{-tor}}. \]

It is easy to check that $F_{h\text{-tor}} \in \text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}}$ and $F_{h\text{-tf}} \in \text{Perv}(\mathcal{C}_X^h)_{h\text{-tf}}$. Then property (ii) in Definition 6.1 is clear. For property (i) let $u : F \to G$ be a morphism in $\text{Perv}(\mathcal{C}_X^h)$ with $F \in \text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}}$ and $G \in \text{Perv}(\mathcal{C}_X^h)_{h\text{-tf}}$. Then $\text{Im} u$ is also in $\text{Perv}(\mathcal{C}_X^h)_{h\text{-tor}}$ and so it is zero by the definition of $\text{Perv}(\mathcal{C}_X^h)_{h\text{-tf}}$.$\square$

Denote by $(^\pi \mathcal{D}_c^{\leq 0}(\mathcal{C}_X^h), ^\pi \mathcal{D}_c^{>0}(\mathcal{C}_X^h))$ the $t$-structure on $\mathcal{D}_c(\mathcal{C}_X^h)$ induced by the perversity $t$-structure and this torsion pair as in Proposition 6.2. We also set $^\pi \text{Perv}(\mathcal{C}_X^h) = ^\pi \mathcal{D}_c^{\leq 0}(\mathcal{C}_X^h) \cap ^\pi \mathcal{D}_c^{>0}(\mathcal{C}_X^h)$. 

Theorem 6.8. There is a quasi-commutative diagram of $t$-exact functors

$D^b_{hol}(\mathcal{D}^h_X)^{op} \xrightarrow{DR_h} D^b_{c-c}(\mathcal{C}^h_X)^{op}$

where the duality functors are equivalences of categories and the de Rham functors become equivalences when restricted to the subcategories of regular objects.

Example 6.9. Let $X = \mathbb{C}, U = X \setminus \{0\}$ and denote by $j: U \hookrightarrow X$ the embedding. Let $L$ be the local system on $U$ with stalk $\mathbb{C}^h$ and monodromy $1 + h$. The sheaf $Rj_*L \cong D^b_{hol}(j_!(D^h_L))$ is perverse for both $t$-structures, as is the sheaf $H^0(Rj_*L) \cong j^*L$. The sheaf $H^1(Rj_*L) \cong \mathbb{C}\{0\}$ has $h$-torsion. From the distinguished triangle $j^*L \rightarrow Rj_*L \rightarrow \mathbb{C}\{0\}[-1] \rightarrow +1$, one gets the short exact sequences

$0 \rightarrow j_*L \rightarrow Rj_*L \rightarrow \mathbb{C}\{0\}[-1] \rightarrow 0 \text{ in } \text{Perv}(\mathcal{C}^h_X),$

$0 \rightarrow \mathbb{C}\{0\}[-2] \rightarrow j_*L \rightarrow Rj_*L \rightarrow 0 \text{ in } \text{Perv}(\mathcal{C}^h_X).$

§7. $\mathcal{D}(\!(\!(h)\!)\!)$-modules

Denote by

$\mathcal{C}^h,\text{loc} := \mathcal{C}(\!(h)\!) = \mathbb{C}[\![-1, h]\!]$

the field of Laurent series in $h$, that is, the fraction field of $\mathcal{C}^h$. Recall the exact functor

$(\cdot)^{\text{loc}}: \text{Mod}(\mathcal{C}^h_X) \rightarrow \text{Mod}(\mathcal{C}^{h,\text{loc}}_X), \ F \mapsto \mathcal{C}^{h,\text{loc}} \otimes_{\mathbb{C}^h} F,$

and note that by [9, Proposition 5.4.14] one has the inclusion

$(7.2) \quad \text{SS}(F^{\text{loc}}) \subset \text{SS}(F).$

For $G \in D^b(\mathcal{C}_X)$, we write $G^{h,\text{loc}}$ instead of $(G^h)^{\text{loc}}$. We will consider in particular

$\mathcal{O}^{h,\text{loc}}_X = \mathcal{O}(\!(h)\!), \quad \mathcal{D}^{h,\text{loc}}_X = \mathcal{D}(\!(h)\!).$

Lemma 7.1. Let $\mathcal{M}$ be a coherent $\mathcal{D}^{h,\text{loc}}_X$-module. Then $\mathcal{M}$ is pseudo-coherent over $\mathcal{D}^h_X$. In other words, if $\mathcal{L} \subset \mathcal{M}$ is a finitely generated $\mathcal{D}^h_X$-module, then $\mathcal{L}$ is $\mathcal{D}^h_X$-coherent.

Proof. The proof follows from [7, Appendix, A1].

Definition 7.2. A lattice $\mathcal{L}$ of a coherent $\mathcal{D}^{h,\text{loc}}_X$-module $\mathcal{M}$ is a coherent $\mathcal{D}^h_X$-submodule of $\mathcal{M}$ which generates it.
Since $\mathcal{M}$ has no $h$-torsion, none of its lattices has $h$-torsion. In particular, one has $\mathcal{M} \cong \mathcal{M}^\text{loc}$ and $\text{gr}_h \mathcal{L} \cong \mathcal{L}_0 = \mathcal{L}/h^1 \mathcal{L}$.

It follows from Lemma 7.3 that lattices locally exist: for a local system of generators $(m_1, \ldots, m_N)$ of $\mathcal{M}$, define $\mathcal{L}$ as the $\mathcal{D}^b_X$-submodule with the same generators.

**Lemma 7.3.** Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of coherent $\mathcal{D}^b_X$-modules. Locally there exist lattices $\mathcal{L}'$, $\mathcal{L}$, $\mathcal{L}''$ of $\mathcal{M}'$, $\mathcal{M}$, $\mathcal{M}''$, respectively, inducing an exact sequence of $\mathcal{D}^b_X$-modules

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0.$$

**Proof.** Let $\mathcal{L}$ be a lattice of $\mathcal{M}$ and let $\mathcal{L}''$ be its image in $\mathcal{M}''$. We set $\mathcal{L} := \mathcal{L} \cap \mathcal{M}'$. These sub-$\mathcal{D}^b_X$-modules give rise to an exact sequence.

Since $\mathcal{L}''$ is of finite type over $\mathcal{D}^b_X$, it is a lattice of $\mathcal{M}''$. Let us show that $\mathcal{L}'$ is a lattice of $\mathcal{M}'$. Being the kernel of a morphism $\mathcal{L} \to \mathcal{L}''$ between coherent $\mathcal{D}^b_X$-modules, $\mathcal{L}'$ is coherent. To show that $\mathcal{L}'$ generates $\mathcal{M}'$, note that any $m' \in \mathcal{M}' \subset \mathcal{M}$ may be written as $m' = h^{-N}m$ for some $N \geq 0$ and $m \in \mathcal{L}$. Hence $m = h^Nm' \in \mathcal{M}' \cap \mathcal{L} = \mathcal{L}'$. \hfill $\square$

For an abelian category $\mathcal{C}$, we denote by $\text{K}(\mathcal{C})$ its Grothendieck group. For an object $M$ of $\mathcal{C}$, we denote by $[M]$ its class in $\text{K}(\mathcal{C})$. We let $\mathcal{K}(\mathcal{D}_X)$ be the sheaf on $X$ associated to the presheaf

$$U \mapsto \text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U)).$$

We define $\mathcal{K}(\mathcal{D}^b_{X,\text{loc}})$ in the same way.

**Lemma 7.4.** Let $\mathcal{L}$ be a coherent $\mathcal{D}^b_X$-module without $h$-torsion. Then, for any $i > 0$, the $\mathcal{D}_X$-module $\mathcal{L}/h^i \mathcal{L}$ is coherent, and we have the equality $[\mathcal{L}/h^i \mathcal{L}] = i \cdot [\text{gr}_h(\mathcal{L})]$ in $\text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X))$.

**Proof.** Since the functor $(\cdot) \otimes_{\text{coh}} \mathbb{C}/h^i \mathbb{C}$ is right exact, $\mathcal{L}/h^i \mathcal{L}$ is a coherent $\mathcal{D}_X$-module. Since $\mathcal{L}$ has no $h$-torsion, multiplication by $h^i$ induces an isomorphism $\mathcal{L}/h^i \mathcal{L} \cong h^i \mathcal{L}/h^{i+1} \mathcal{L}$. We conclude by induction on $i$ with the exact sequence

$$0 \to h^i \mathcal{L}/h^{i+1} \mathcal{L} \to \mathcal{L}/h^{i+1} \mathcal{L} \to \mathcal{L}/h^i \mathcal{L} \to 0.$$ \hfill $\square$

**Lemma 7.5.** For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}^b_{X,\text{loc}})$, $U \subset X$ an open set and $\mathcal{L} \subset \mathcal{M}|_U$ a lattice of $\mathcal{M}|_U$, the class $[\text{gr}_h(\mathcal{L})] \in \text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U))$ only depends on $\mathcal{M}$. This defines a morphism of abelian sheaves $\mathcal{K}(\mathcal{D}^b_{X,\text{loc}}) \to \mathcal{K}(\mathcal{D}_X)$.

**Proof.** (i) We first prove that $[\text{gr}_h(\mathcal{L})]$ only depends on $\mathcal{M}$. We consider another lattice $\mathcal{L}'$ of $\mathcal{M}|_U$. Since $\mathcal{L}$ is a $\mathcal{D}^b_X$-module of finite type, and $\mathcal{L}'$ generates $\mathcal{M}$, there exists $n > 1$ such that $\mathcal{L}' \subset h^{-n} \mathcal{L}'$. Similarly, there exists $m > 1$ with...
$\mathcal{L}' \subset h^{-m}\mathcal{L}$, so that we have the inclusions

$$h^{m+n+2}\mathcal{L} \subset h^{m+n+1}\mathcal{L} \subset h^{m+1}\mathcal{L}' \subset h^m\mathcal{L}' \subset \mathcal{L}.$$ 

Any inclusion $A \subset B \subset C$ yields an identity $[C/A] = [C/B] + [B/A]$ in the Grothendieck group, and we obtain in particular

$$[h^m\mathcal{L}'/h^{m+n+1}\mathcal{L}] = [h^m\mathcal{L}'/h^{m+1}\mathcal{L}'] + [h^{m+1}\mathcal{L}'/h^{m+n+1}\mathcal{L}],$$

$$[\mathcal{L}/h^{m+n+1}\mathcal{L}] = [\mathcal{L}/h^{m+1}\mathcal{L}'] + [h^{m+1}\mathcal{L}'/h^{m+n+1}\mathcal{L}],$$

$$[\mathcal{L}/h^{m+n+2}\mathcal{L}] = [\mathcal{L}/h^{m+1}\mathcal{L}'] + [h^{m+1}\mathcal{L}'/h^{m+n+2}\mathcal{L}].$$

Note that we have isomorphisms of the type $\mathcal{h}^0\mathcal{M}_1/\mathcal{h}^0\mathcal{M}_2 \simeq \mathcal{M}_1/\mathcal{M}_2$ for modules without $h$-torsion. Then Lemma 7.4 and the above equalities give:

$$[\mathcal{L}'/h^{n+1}\mathcal{L}] = [\gr_h(\mathcal{L}')] + [\mathcal{L}'/h^n\mathcal{L}],$$

$$(m + n + 1)[\gr_h(\mathcal{L})] = [\mathcal{L}/h^{m+1}\mathcal{L}'] + [\mathcal{L}'/h^n\mathcal{L}],$$

$$(m + n + 2)[\gr_h(\mathcal{L})] = [\mathcal{L}/h^{m+1}\mathcal{L}'] + [\mathcal{L}'/h^{n+1}\mathcal{L}].$$

A suitable combination of these lines gives $[\gr_h(\mathcal{L})] = [\gr_h(\mathcal{L}')]],$ as desired.

(ii) Now we consider an open subset $V \subset X$ and $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{b,\text{loc}}|v)$. We choose an open covering $\{U_i\}_{i \in I}$ of $V$ such that for each $i \in I$, $\mathcal{M}|_{U_i}$ admits a lattice, say $\mathcal{L}_i$. We have seen that $[\gr_h(\mathcal{L}_i)] \in K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|U_i))$ only depends on $\mathcal{M}$. This implies that

$$[\gr_h(\mathcal{L}_i)]|_{U_{i,j}} = [\gr_h(\mathcal{L}_j)]|_{U_{i,j}} \text{ in } K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|U_{i,j})).$$

Hence the $[\gr_h(\mathcal{L}_i)]]$’s define a section, say $c(\mathcal{M})$, of $\mathcal{K}(\mathcal{D}_X)$ over $V$. By Lemma 7.3, $c(\mathcal{M})$ only depends on the class $[\mathcal{M}]$ in $K(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{b,\text{loc}}|v))$, and $\mathcal{M} \mapsto c(\mathcal{M})$ induces the morphism $\mathcal{K}(\mathcal{D}_X^{b,\text{loc}}) \to \mathcal{K}(\mathcal{D}_X).$ \hfill \Box

By Lemma 7.3, the following definition is correct.

**Definition 7.6.** The characteristic variety of a coherent $\mathcal{D}_X^{b,\text{loc}}$-module $\mathcal{M}$ is defined by

$$\text{char}_{h,\text{loc}}(\mathcal{M}) = \text{char}_h(\mathcal{L}),$$

for $\mathcal{L} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^b)$ a (local) lattice. For $\mathcal{M} \in \mathcal{D}_\text{coh}^b(\mathcal{D}_X^{b,\text{loc}})$, one sets $\text{char}_{h,\text{loc}}(\mathcal{M}) = \bigcup_j \text{char}_{h,\text{loc}}(H^j(\mathcal{M})).$

**Proposition 7.7.** The characteristic variety $\text{char}_{h,\text{loc}}$ is additive both on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{b,\text{loc}})$ and on $\mathcal{D}_\text{coh}^b(\mathcal{D}_X^{b,\text{loc}})$.

**Proof.** This follows from Proposition 3.6 (ii) and Lemma 7.3 \hfill \Box
Consider the functor $$\text{Sol}_h^{\text{loc}} : D^b(\mathcal{D}_X^{\text{h,loc}})^{\text{op}} \to D^b(\mathcal{C}_X^{\text{h,loc}}), \ M \mapsto \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}_X^{\text{h,loc}}}(M, \mathcal{O}_X^{\text{h,loc}}).$$

**Proposition 7.8.** Let $$M \in D^b_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}).$$ Then $$\text{SS}(\text{Sol}_h^{\text{loc}}(M)) \subset \text{char}^{\text{h,loc}}(M).$$

**Proof.** By dévissage, we can assume that $$M \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}).$$ Moreover, since the problem is local, we may assume that $$M$$ admits a lattice $$L.$$ One has the isomorphism $$\text{Sol}_h^{\text{loc}}(M) \simeq \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}_X^{\text{h,loc}}}(L, \mathcal{O}_X^{\text{h,loc}})$$ by extension of scalars. Taking a local resolution of $$L$$ by free $$\mathcal{D}_X^{\text{h}}$$-modules of finite type, we deduce that $$\text{Sol}_h^{\text{loc}}(M) \simeq F^{\text{loc}}$$ for $$F = \text{Sol}_h(L).$$ The statement follows by (7.2) and Corollary 3.14.

One says that $$M$$ is **holonomic** if its characteristic variety is isotropic.

**Proposition 7.9.** The functor $$\text{Sol}_h^{\text{loc}}$$ induces a functor $$\text{Sol}_h^{\text{loc}} : D^b_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}})^{\text{op}} \to D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h,loc}}).$$

**Proof.** By the same arguments and with the same notation as in the proof of Proposition 7.8 we reduce to the case $$\text{Sol}_h^{\text{loc}}(M) \simeq F^{\text{loc}}$$ for $$F = \text{Sol}_h(L)$$ and $$L$$ a lattice of $$M \in \text{Mod}_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}}).$$ Hence $$L$$ is a holonomic $$\mathcal{D}_X^{\text{h}}$$-module, and $$F \in D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h}}).$$

**Remark 7.10.** In general the functor $$\text{Sol}_h^{\text{loc}} : D^b_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}})^{\text{op}} \to D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h,loc}})$$ is not locally essentially surjective. In fact, consider the quasi-commutative diagram of categories

$$
\begin{array}{ccc}
D^b_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}})^{\text{op}} & \xrightarrow{\text{Sol}_h} & D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h}}) \\
(\bullet)^{\text{loc}} \downarrow & & \downarrow (\bullet)^{\text{loc}} \\
D^b_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}})^{\text{op}} & \xrightarrow{\text{Sol}_h^{\text{loc}}} & D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h,loc}})
\end{array}
$$

By the local existence of lattices the left vertical arrow is locally essentially surjective. If $$\text{Sol}_h^{\text{loc}}$$ were also locally essentially surjective, so should be the right vertical arrow. The following example shows that it is not the case.

One can interpret this phenomenon by remarking that $$D^b_{\text{hol}}(\mathcal{D}_X^{\text{h,loc}})$$ is equivalent to the localization of the category $$D^b_{\text{hol}}(\mathcal{D}_X^{\text{h}})$$ with respect to the morphism $$h,$$ in contrast to the category $$D^b_{\text{C-c}}(\mathcal{C}_X^{\text{h,loc}}).$$
Example 7.11. Let $X = \mathbb{C}$, $U = X \setminus \{0\}$ and denote by $j: U \hookrightarrow X$ the embedding. Set $F = \mathrm{R}j_! L$, where $L$ is the local system on $U$ with stalk $\mathbb{C}^{h,\text{loc}}$ and monodromy $h$ around the origin. Since $h$ is not invertible in $\mathbb{C}^{h}$, there is no $F_0 \in D^b_{\mathbb{C}}(\mathbb{C}^{h}_X)$ such that $F \simeq (F_0)^{\text{loc}}$.

§8. Links with deformation quantization

In this last section, we shall briefly explain how the study of deformation quantization algebras on complex symplectic manifolds is related to $D^b_{\mathbb{C}}$. We follow the terminology of [17].

The cotangent bundle $X = T^* X$ to the complex manifold $X$ has the structure of a complex symplectic manifold and is endowed with the $\mathbb{C}^{h}$-algebra $\hat{W}_X$, a non-homogeneous version of the algebra of microdifferential operators. Its subalgebra $\hat{W}_X(0)$ of operators of order at most zero is a deformation quantization algebra. In a system $(x,u)$ of local symplectic coordinates, $\hat{W}_X(0)$ is identified with the star algebra $(\mathcal{O}_{\mathbb{C}}(x), \ast)$ in which the star product is given by the Leibniz product

\begin{equation}
    f \ast g = \sum_{\alpha \in \mathbb{N}^n} \frac{h^{\lvert \alpha \rvert}}{\alpha!} \left( \partial_x^\alpha f \right) \left( \partial_u^\alpha g \right) \quad \text{for } f, g \in \mathcal{O}_X.
\end{equation}

In this section we will set for short $\mathcal{A} := \hat{W}_X(0)$, so that $\mathcal{A}^{\text{loc}} \simeq \hat{W}_X$. Note that $\mathcal{A}$ satisfies Assumption 1.8.

Let us identify $X$ with the zero section of the cotangent bundle $X$. Recall that $X$ is a local model for any smooth Lagrangian submanifold of $X$, and that $\mathcal{O}_X$ is a local model of any simple $\mathcal{A}$-module along $X$. As $\mathcal{O}_X$ has both a $D^b_{\mathbb{C}}$-module and an $\mathcal{A}$-module structure, there are morphisms of $\mathbb{C}^{h}$-algebras

\begin{equation}
    D^b_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}^{h}}(\mathcal{O}_X) \leftarrow \mathcal{A} \mid X.
\end{equation}

Lemma 8.1. The morphisms in (8.2) are injective and induce an embedding $\mathcal{A} \mid X \hookrightarrow D^b_{\mathbb{C}}$.

Proof. Since the problem is local, we may choose a local symplectic coordinate system $(x,u)$ on $X$ such that $X = \{u = 0\}$. Then $\mathcal{A} \mid X$ is identified with $\mathcal{O}_X \mid X$. As the action of $u_i$ on $\mathcal{O}_X$ is given by $h \partial_x$, the morphism $\mathcal{A} \mid X \to \mathrm{End}_{\mathbb{C}^{h}}(\mathcal{O}_X)$ factors through $D^b_{\mathbb{C}}$, and the induced morphism $\mathcal{A} \mid X \hookrightarrow D^b_{\mathbb{C}}$ is described by

\begin{equation}
    \sum_{i \in \mathbb{N}} f_i(x,u) h^i \mapsto \sum_{j \in \mathbb{N}} \left( \sum_{\alpha \in \mathbb{N}^n, \lvert \alpha \rvert \leq j} \partial_x^\alpha f_{j-\lvert \alpha \rvert}(x,0) \partial_x^\alpha \right) h^j,
\end{equation}

which is clearly injective. \qed
Consider the following subsheaves of $\mathcal{D}_X^\hbar$:

$$
\mathcal{D}_X^{\hbar, m} = \prod_{i \geq 0} (F_i + m \mathcal{D}_X) \hbar^i, \quad \mathcal{D}_X^{\hbar, f} = \bigcup_{m \geq 0} \mathcal{D}_X^{\hbar, m}.
$$

Note that $\mathcal{D}_X^{\hbar, 0}$ and $\mathcal{D}_X^{\hbar, f}$ are subalgebras of $\mathcal{D}_X^\hbar$, that $\mathcal{D}_X^{\hbar, 0, \text{loc}} \simeq \mathcal{D}_X^{\hbar, f, \text{loc}}$. By [8.3], the image of $\mathcal{A}|_X$ in $\mathcal{D}_X^\hbar$ is contained in $\mathcal{D}_X^{\hbar, 0}$. (The ring $\mathcal{D}_X^{\hbar, 0}$ should be compared with the ring $\mathcal{R}_X \times \mathbb{C}$ of [16].)

**Remark 8.2.** More precisely, denote by $\mathcal{O}_X^{\hbar}|_X \simeq (\mathcal{O}_X|_X)^\hbar$ the formal completion of $\mathcal{O}_X^\hbar$ along the submanifold $X$. Then the star product in (8.1) extends to this sheaf, and (8.3) induces an isomorphism $(\mathcal{O}_X^{\hbar}|_X, \ast) \simeq \mathcal{D}_X^{\hbar, 0}$.

Summarizing, one has the compatible embeddings of algebras

\[
\begin{array}{cccc}
\mathcal{A}|_X & \hookrightarrow & \mathcal{D}_X^{\hbar, 0, \text{loc}} & \to & \mathcal{D}_X^{\hbar, f, \text{loc}} & \to & \mathcal{D}_X^{\hbar, \text{loc}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{A}|_X & \hookrightarrow & \mathcal{D}_X^{\hbar, 0} & \to & \mathcal{D}_X^{\hbar, f} & \to & \mathcal{D}_X^{\hbar}
\end{array}
\]

One has

\[\text{gr}_h \mathcal{A}|_X \simeq \mathcal{O}_X|_X, \quad \text{gr}_h \mathcal{D}_X^{\hbar, 0} \simeq \mathcal{O}_X^{\hbar}|_X, \quad \text{gr}_h \mathcal{D}_X^{\hbar, f} \simeq \text{gr}_h \mathcal{D}_X^{\hbar} \simeq \mathcal{D}_X.\]

**Proposition 8.3.** (i) The algebra $\mathcal{D}_X^{\hbar, 0}$ is faithfully flat over $\mathcal{A}|_X$.

(ii) The algebra $\mathcal{D}_X^{\hbar, \text{loc}}$ is flat over $\mathcal{A}|_X$.

**Proof.** (i) follows from Theorem 1.12

(ii) follows from (i) and the isomorphism $(\mathcal{D}_X^{\hbar, 0})^{\text{loc}} \simeq \mathcal{D}_X^{\hbar, \text{loc}}$. 

The next examples show that the scalar extension functor

\[\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, 0}) \to \text{Mod}_{\text{coh}}(\mathcal{D}_X)\]

is neither exact nor full.

**Example 8.4.** Let $X = \mathbb{C}^2$ with coordinates $(x, y)$. Then $\hbar \partial_x$ is injective as an endomorphism of $\mathcal{D}_X^{\hbar, 0}/(\hbar \partial_x)$ but it is not injective as an endomorphism of $\mathcal{D}_X^{\hbar}/(\hbar \partial_x)$, since $\partial_x$ belongs to its kernel. This shows that $\mathcal{D}_X^{\hbar}$ is not flat over $\mathcal{D}_X^{\hbar, 0}$.

**Example 8.5.** This example was communicated to us by Masaki Kashiwara. Let $X = \mathbb{C}$ with coordinate $x$, and denote by $(x, u)$ the symplectic coordinates on $X = T^*\mathbb{C}$. Consider the cyclic $\mathcal{A}$-modules

\[\mathcal{M} = \mathcal{A}/(x - u), \quad \mathcal{N} = \mathcal{A}/(x),\]
and their images in \( \text{Mod}(\mathcal{D}_X^b) \)

\[ \mathcal{M}' = \mathcal{D}_X^b/(x - h\partial_x), \quad \mathcal{N}' = \mathcal{D}_X^b/(x). \]

As their supports in \( X \) differ, \( \mathcal{M} \) and \( \mathcal{N} \) are not isomorphic as \( \mathcal{A} \)-modules. On the other hand, in \( \mathcal{D}_X^b \) one has the relation

(8.4) \[ x \cdot e^{h\partial_x^2/2} = e^{h\partial_x^2/2} \cdot (x - h\partial_x), \]

and hence an isomorphism \( \mathcal{M}' \overset{\sim}{\to} \mathcal{N}' \) given by \([P] \mapsto [P \cdot e^{-h\partial_x^2/2}]\). In fact, one checks that

\[ \text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})|_X = 0, \quad \text{Hom}_{\mathcal{D}_X^b}(\mathcal{M}', \mathcal{N}') \simeq \mathbb{C}_X^b. \]

\section{Complements on constructible sheaves}

Let us review some results, well-known to specialists (see, e.g., [18, Proposition 3.10]), but which are usually stated over a field, and we need to work here over the ring \( \mathbb{C}^h \).

Let \( \mathbb{K} \) be a commutative unital Noetherian ring of finite global dimension. Assume that \( \mathbb{K} \) is syzygic, i.e. any finitely generated \( \mathbb{K} \)-module admits a finite projective resolution by finite free modules. (For our purposes we will either have \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{K} = \mathbb{C}^h \)).

Let \( X \) be a real analytic manifold. Denote by \( \text{Mod}_{\mathbb{R}, c}(\mathbb{K}_X) \) the abelian category of \( \mathbb{R} \)-constructible sheaves on \( X \) and by \( \text{Db}_{\mathbb{R}, c}(\mathbb{K}_X) \) the bounded derived category of sheaves of \( \mathbb{K} \)-modules with \( \mathbb{R} \)-constructible cohomology. Under the above assumptions on the base ring, by [9, Propositions 3.4.3, 8.4.9] one has

**Lemma A.1.** The duality functor \( \mathcal{D}_X^c(\mathcal{M}) = \text{RHom}_{\mathbb{K}_X}(\mathcal{M}, \mathbb{K}_X) \) induces an involution of \( \text{Db}_{\mathbb{R}, c}(\mathbb{K}_X) \).

For the next proposition we recall some notation and results of [6, 9]. We consider a simplicial complex \( S = (S, \Delta) \), with set of vertices \( S \) and set of simplices \( \Delta \). We let \( |S| \) be the realization of \( S \). Thus \( |S| \) is the disjoint union of the realizations \( |\sigma| \) of the simplices. For a simplex \( \sigma \in \Delta \), the open set \( U(\sigma) \) is defined in [9, (8.1.3)]. A sheaf \( F \) of \( \mathbb{K} \)-modules on \( |S| \) is said to be weakly \( S \)-constructible if \( F|_{|\sigma|} \) is constant for any \( \sigma \in \Delta \). An object \( F \in \text{Db}(\mathbb{K}_S) \) is said to be weakly \( S \)-constructible if its cohomology sheaves are so. If moreover, all stalks \( F_x \) are perfect complexes, \( F \) is called \( S \)-constructible. By [9, Proposition 8.1.4] we have isomorphisms, for a weakly \( S \)-constructible sheaf \( F \) and for any \( \sigma \in \Delta \) and \( x \in |\sigma| \),

(A.1) \[ \Gamma(U(\sigma); F) \overset{\sim}{\to} \Gamma(|\sigma|; F) \overset{\sim}{\to} F_x, \]

(A.2) \[ H^j(U(\sigma); F) = H^j(|\sigma|; F) = 0 \quad \text{for } j \neq 0. \]
It follows that, for a weakly $S$-constructible $F \in D^b(\mathbb{K}|_S)$, the natural morphisms of complexes of $\mathbb{K}$-modules

(A.3) \[ \Gamma(U(\sigma); F) \to \Gamma(|\sigma|; F) \to F_z \]

are quasi-isomorphisms.

For $U \subset X$ an open subset, we denote by $\mathbb{K}_U := (\mathbb{K}_X)_U$ the extension by 0 of the constant sheaf on $U$.

**Proposition A.2.** Let $F \in D^b_{\mathbb{R},c}(\mathbb{K}_X)$. Then

(i) $F$ is isomorphic to a complex

\[ 0 \to \bigoplus_{i_a \in I_a} \mathbb{K}_{U_{i_a}} \to \cdots \to \bigoplus_{i_b \in I_b} \mathbb{K}_{U_{i_b}} \to 0, \]

where the $\{U_{k; i_b}\}_{k, i_b}$’s are locally finite families of relatively compact subanalytic open subsets of $X$.

(ii) $F$ is isomorphic to a complex

\[ 0 \to \bigoplus_{i_a \in I_a} \Gamma_{V_{i_a}} \mathbb{K}_X \to \cdots \to \bigoplus_{i_b \in I_b} \Gamma_{V_{i_b}} \mathbb{K}_X \to 0, \]

where the $\{V_{k; i_b}\}_{k, i_b}$’s are locally finite families of relatively compact subanalytic open subsets of $X$.

**Proof.** (i) By the triangulation theorem for subanalytic sets (see for example [9, Proposition 8.2.5]) we may assume that $F$ is an $S$-constructible object in $D^b(\mathbb{K}|_S)$ for some simplicial complex $S = (S, \Delta)$. For $i$ an integer, let $\Delta_i \subset \Delta$ be the subset of simplices of dimension $\leq i$ and set $S_i = (S, \Delta_i)$. We denote by $K^b(\mathbb{K})$ (resp. $K^b(\mathbb{K}|_S)$) the category of bounded complexes of $\mathbb{K}$-modules (resp. sheaves of $\mathbb{K}$-modules on $|S|$) with morphisms up to homotopy. We shall prove by induction on $i$ that there exists a morphism $u_i : G_i \to F$ in $K^b(\mathbb{K}|_S)$ such that:

(a) the $G^k_i$ are finite direct sums of $\mathbb{K}_{U(\sigma)}$’s for some $\sigma \in \Delta_i$,

(b) $u_i|_{|S_i|} : G_i|_{|S_i|} \to F|_{|S_i|}$ is a quasi-isomorphism.

The desired result is obtained for $i$ equal to the dimension of $X$.

(i)-(1) For $i = 0$ we consider $F|_{|S_0|} \simeq \bigoplus_{\sigma \in \Delta_0} F_{\sigma}$. The complexes $\Gamma(U(\sigma); F)$, $\sigma \in \Delta_0$, have finite bounded cohomology by the quasi-isomorphisms (A.3). Hence we may choose bounded complexes of finite free $\mathbb{K}$-modules, $R_{0, \sigma}$, and morphisms $u_{0, \sigma} : R_{0, \sigma} \to \Gamma(U(\sigma); F)$ which are quasi-isomorphisms.

We have the natural isomorphism $\Gamma(U(\sigma); F) \simeq a_* \mathcal{H}om_{K^b(\mathbb{K}|_S)}(\mathbb{K}_{U(\sigma)}, F)$ in $K^b(\mathbb{K})$, where $a : |S| \to \text{pt}$ is the projection and $\mathcal{H}om$ is the internal Hom functor.
We deduce the adjunction formula, for $R \in K^b(\mathbb{K})$ and $F \in K^b(\mathbb{K}|\mathcal{S}|)$,

(A.4) \[ \text{Hom}_{K^b(\mathbb{K})}(R, \Gamma(U(\sigma); F)) \simeq \text{Hom}_{K^b(\mathbb{K}|\mathcal{S}|)}(R_U(\sigma), F). \]

Hence the $u_0, \sigma$ induce $u_0: G_0 := \bigoplus_{\sigma \in \Delta_0} (R_{0, \sigma}) \rightarrow F$. By (A.3), $u_0(x)$ is a quasi-isomorphism for all $x \in |\mathcal{S}|$, so that $u_0|_{|\mathcal{S}|}$ also is a quasi-isomorphism, as required.

(i)-(ii) We assume that $u_i$ is built and let $H_i = M(u_i)[-1]$ be the mapping cone of $u_i$, shifted by $-1$. By the distinguished triangle in $K^b(\mathbb{K}|\mathcal{S}|)$

(A.5) \[ H_i \xrightarrow{v_i} G_i \xrightarrow{u_i} F \xrightarrow{+1}, \]

$H_i|_{|\mathcal{S}|}$ is quasi-isomorphic to 0. Hence $\bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (H_i)_{|\mathcal{S}|} \rightarrow H_i|_{|\mathcal{S}|}$ is a quasi-isomorphism. As above we choose quasi-isomorphisms $u_{i+1, \sigma}: R_{i+1, \sigma} \rightarrow \Gamma(U(\sigma); H_i), \sigma \in \Delta_{i+1} \setminus \Delta_i$, where the $R_{i+1, \sigma}$ are bounded complexes of finite free $\mathbb{K}$-modules. By (A.3) again the $u_{i+1, \sigma}$ induce a morphism in $K^b(\mathbb{K}|\mathcal{S}|)$

\[ u'_{i+1}: G'_{i+1} := \bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (R_{i+1, \sigma}) \rightarrow H_i. \]

For $x \in |\mathcal{S}_{i+1}| \setminus |\mathcal{S}_i|$, $(u'_{i+1})_x$ is a quasi-isomorphism by (A.3), and, for $x \in |\mathcal{S}_i|$, this is trivially true. Hence $u'_{i+1}|_{|\mathcal{S}_{i+1}|}$ is a quasi-isomorphism.

Now we let $H_{i+1}$ and $G_{i+1}$ be the mapping cones of $u'_{i+1}$ and $v_i \circ u'_{i+1}$, respectively. We have distinguished triangles in $K^b(\mathbb{K}|\mathcal{S}|)$

(A.6) \[ G'_{i+1} \xrightarrow{u'_{i+1}} H_{i+1} \rightarrow H_{i+1} \xrightarrow{-1}, \quad G'_{i+1} \xrightarrow{v_i \circ u'_{i+1}} G_i \rightarrow G_{i+1} \xrightarrow{-1}. \]

By the construction of the mapping cone, the definition of $G'_{i+1}$ and the induction hypothesis, $G_{i+1}$ satisfies property (a) above. The octahedral axiom applied to triangles (A.5) and (A.6) gives a morphism $u_{i+1}: G_{i+1} \rightarrow F$ and a distinguished triangle $H_{i+1} \rightarrow G_{i+1} \xrightarrow{u_{i+1}} F \xrightarrow{+1}$. By construction $H_{i+1}|_{|\mathcal{S}_{i+1}|}$ is quasi-isomorphic to 0 so that $u_{i+1}$ satisfies property (b) above.

(ii) Set $G = D^b_{\mathcal{K}}(F)$, and represent it by a bounded complex as in (i). Since $U_{k,i,k}$ corresponds to an open subset of the form $U(\sigma)$ in $|\mathcal{S}|$, the sheaves $\mathbb{K}|U_{k,i,k}$ are acyclic for the functor $D^b_{\mathcal{K}}$. Hence $F \simeq D^b_{\mathcal{K}}(G)$ can be represented as claimed.

\begin{lemma} \label{A.3} \textbf{Lemma A.3.} Let $F \rightarrow G \rightarrow 0$ be an exact sequence in $\text{Mod}_{\mathcal{K},c}(\mathbb{K}_X)$. Then for any relatively compact subanalytic open subset $U \subset X$, there exists a finite covering $U = \bigcup_{i \in I} U_i$ by subanalytic open subsets such that, for each $i \in I$, the morphism $F(U_i) \rightarrow G(U_i)$ is surjective. \end{lemma}

\begin{proof} As in the proof of Proposition \ref{A.2} we may assume that $F$, $G$ and $\mathbb{K}_U$ are constructible sheaves on the realization of some finite simplicial complex $(S, \Delta)$.

\end{proof}
For $\sigma \in \Delta$ the morphism $\Gamma(U(\sigma); F) \to \Gamma(U(\sigma); G)$ is surjective, by \ref{A.1}. Since the image of $U$ in $|S|$ is a finite union of $U(\sigma)$’s, this proves the lemma. \hfill $\blacksquare$

§B. Complements on subanalytic sheaves

We review here some well-known results (see \cite[Chapter 7]{11} and \cite{15}) but which are usually stated over a field, and we need to work here over the ring $\mathbb{C}\hbar$.

Let $\mathbb{k}$ be a commutative unital Noetherian ring of finite global dimension (for our purposes we will have either $\mathbb{k} = \mathbb{C}$ or $\mathbb{k} = \mathbb{C}\hbar$). Let $X$ be a real analytic manifold, and consider the natural morphism $\rho: X \to X_{sa}$.

Lemma B.1. The functor $\rho_* : \text{Mod}_{\mathbb{R}-c}(\mathbb{k}X) \to \text{Mod}(\mathbb{k}X_{sa})$ is exact and $\rho^{-1}\rho_*$ is isomorphic to the canonical functor $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X) \to \text{Mod}(\mathbb{k}X_{sa})$.

Proof. Being a direct image functor, $\rho_*$ is left exact. It is right exact thanks to Lemma A.3. The composition $\rho^{-1}\rho_*$ is isomorphic to the identity on $\text{Mod}(\mathbb{k}X_{sa})$ since the open sets of the site $X_{sa}$ give a basis of the topology of $X$. \hfill $\blacksquare$

In the following, we denote by $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X_{sa})$ the image under the functor $\rho_*$ of $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X)$ in $\text{Mod}(\mathbb{k}X_{sa})$. Hence $\rho_*$ induces an equivalence of categories $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X) \simeq \text{Mod}_{\mathbb{R}-c}(\mathbb{k}X_{sa})$. We also denote by $\text{D}^b_{\mathbb{R}-c}(\mathbb{k}X_{sa})$ the full triangulated subcategory of $\text{D}^b(\mathbb{k}X_{sa})$ consisting of objects with cohomology in $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X_{sa})$.

Corollary B.2. The subcategory $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X_{sa})$ of $\text{Mod}(\mathbb{k}X_{sa})$ is thick.

Proof. Since $\rho_*$ is fully faithful and exact, $\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X_{sa})$ is stable under taking kernels and cokernels. It remains to see that, for $F, G \in \text{Mod}_{\mathbb{R}-c}(\mathbb{k}X)$,

\[
\text{Ext}^1_{\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X)}(F, G) \simeq \text{Ext}^1_{\text{Mod}(\mathbb{k}X_{sa})}(\rho_* F, \rho_* G).
\]

By \cite{6} we know that the first $\text{Ext}^1$ may as well be computed in $\text{Mod}(\mathbb{k}X)$. Note that the functors $\rho^{-1}$ and $\mathbb{R}\rho_*$ between $\text{D}^b(\mathbb{k}X)$ and $\text{D}^b(\mathbb{k}X_{sa})$ are adjoint, and moreover $\rho^{-1}\mathbb{R}\rho_* \simeq \text{id}$. Thus, for $F', G' \in \text{D}^b(\mathbb{k}X)$ we have

\[
\text{Hom}_{\text{D}^b(\mathbb{k}X_{sa})}(\mathbb{R}\rho_* F', \mathbb{R}\rho_* G') \simeq \text{Hom}_{\text{D}^b(\mathbb{k}X)}(F', G'),
\]

and this gives the result. \hfill $\blacksquare$

This corollary gives the equivalence $\text{D}^b_{\mathbb{R}-c}(\mathbb{k}X) \simeq \text{D}^b_{\mathbb{R}-c}(\mathbb{k}X_{sa})$, both categories being equivalent to $\text{D}^b(\text{Mod}_{\mathbb{R}-c}(\mathbb{k}X))$. 
References


