On Quasifree States of the Canonical Commutation Relations (I)

By

Huzihiro Araki and Masafumi Shiraiishi

Abstract

A self-dual CCR algebra is defined and arbitrary quasifree state is realized in a Fock type representation of another self-dual CCR algebra of a double size as a preparation for a study of quasi-equivalence of quasifree states.

§ 1. Introduction

A necessary and sufficient condition for the quasi-equivalence of two quasifree representations of the canonical anticommutation relations (CAR) has been derived in [11] for the gauge invariant case and in [3] for the general case. We shall derive an analogous result for the canonical commutation relations (CCR) in this series of papers.

A quasifree state of CCR and Bogoliubov automorphisms have been extensively studied ([5]-[10], [12], [13]). We shall use the formulation developed in [2].

In section 2, we review the formulation in [2]. A self-dual algebra is defined when a linear space $K$, an antilinear involution $\Gamma$ of $K$ and a hermitian form $\gamma$ on $K$ satisfying $\gamma(\Gamma f, \Gamma g)=-\gamma(f, g)^*$ are given. In section 3, we define a quasifree state in terms of a nonnegative hermitian form $S$ on $K$ such that $S(f, g)-S(\Gamma g, \Gamma f)=\gamma(f, g)$. In section 4, the structure of $S$ relative to $(K, \gamma, \Gamma)$ is analyzed.

In section 5, basic properties of a Fock representation are stated and a result in [1] is quoted. A Fock type representation is defined as a generalization of a Fock representation to the case of degenerate $\gamma$ (i.e. Received December 25, 1970.
the case with nontrivial center). In section 6, a quasifree state is realized as the restriction of Fock type state of a CCR algebra for \((\hat{K}_s, \hat{\tau}_s, \hat{F}_s)\) where \(\hat{K}_s\) is about twice as large as \(K\).

An application to the quasi-equivalence of quasifree states will be made in a subsequent paper \([5]\).

§ 2. Basic Notions

Let \(K\) be a complex linear space and \(\tau(f, g)\) be a hermitian form for \(f, g \in K\). Let \(\Gamma\) be an antilinear involution \((\Gamma^2 = 1)\) satisfying \(\tau(\Gamma f, \Gamma g) = -\tau(g, f)\). A self-dual CCR algebra \(\mathfrak{H}(K, \tau, \Gamma)\) over \((K, \tau, \Gamma)\) is the quotient of the complex free * algebra generated by \(B(f), f \in K\), its conjugate \(B(f)^*\), \(f \in K\) and an identity \(1\) over (the two-sided * ideal generated by) the following relations:

1. \(B(f)\) is complex linear in \(f\),
2. \(B(f)^*B(g) - B(g)B(f)^* = \tau(f, g)1\),
3. \(B(\Gamma f)^* = B(f)\).

Any one-to-one linear mapping \(U\) of \(K\) onto \(K_s\) satisfying \(\tau(Uf, Ug) = \tau(f, g)\) and \(\Gamma U = U\Gamma\) preserves the above relations (1)~(3) and hence there exists a unique * automorphism \(\tau(U)\) of \(\mathfrak{H}(K, \tau, \Gamma)\) satisfying \(\tau(U)B(f) = B(Uf)\). \(U\) and \(\tau(U)\) shall be called a Bogoliubov transformation and a Bogoliubov * automorphism.

Any operator \(P\) on \(K\) satisfying

1. \(P^2 = P\),
2. \(\tau(f, Pf) > 0\), if \(Pf \neq 0\),
3. \(\tau(Pf, g) = \tau(f, Pg)\),
4. \(\Gamma Pf = 1 - P\),

is called a basis projection. Such \(P\) is linear.

Let \(L\) be a complex pre-Hilbert space. A CCR algebra \(\mathfrak{A}_{CCR}(L)\) over \(L\) is the quotient of the free * algebra generated by \((a^\dagger, f), (f, a), f \in L\) and an identity by (the two-sided * ideal generated by) the following relations:

1. \((a^\dagger, f)\) is complex linear in \(f\),
2. \((f, a) = (a^\dagger, f)^*\),
(3) \[(f, a), (a^\dagger, g)\] = \[(f, g), L, g, a\].

Let \( P \) be a basis projection. Then the mapping \( \alpha(P) \) from \( \mathfrak{V}(K, \gamma, \Gamma) \) to \( \mathfrak{V}_{\text{CCR}}(PK) \) defined by

\[
(2.1a) \quad \alpha(P, B(f_1) \ldots B(f_n)) = (\alpha(P)B(f_1)) \ldots (\alpha(P)B(f_n)),
\]

\[
(2.1b) \quad \alpha(P)B(f) = (a^\dagger, Pf) + (Pf, a)
\]
is a \(*\) isomorphism of \( \mathfrak{V}(K, \gamma, \Gamma) \) onto \( \mathfrak{V}_{\text{CCR}}(PK) \).

Let \( \mathfrak{V} \) be a \(*\) algebra with an identity. A state \( \varphi \) of \( \mathfrak{V} \) is a complex valued linear functional over \( \mathfrak{V} \) satisfying \( \varphi(1) = 1 \) and \( \varphi(A^*A) \geq 0 \) for all \( A \in \mathfrak{V} \). Associated with every state \( \varphi \), there exists a triplet \( \mathfrak{H}_\varphi, \pi_\varphi, \Omega_\varphi \) of a Hilbert space, a representation of \( \mathfrak{V} \) by densely defined closable operators \( \pi_\varphi(A), A \in \mathfrak{V} \) and a unit vector \( \Omega_\varphi \), cyclic for \( \pi_\varphi(\mathfrak{V}) \), such that \( \varphi(A) = (\pi_\varphi(A)\Omega_\varphi) \), \( \pi_\varphi(A)^* \supset \pi_\varphi(A^*) \) and the domain of \( \pi_\varphi(A) \) is \( \pi_\varphi(\mathfrak{V}) \).

Let \( \text{Re } K \) denote the set of \( f \in K \) such that \( \Gamma f = f \). It is a real linear space. \( f \in \text{Re } K \) if and only if \( B(f)^* = B(f) \).

Let \( \varphi \) be a state of \( \mathfrak{V}(K, \gamma, \Gamma) \) such that \( \pi_\varphi(B(f)) \) is essentially selfadjoint for all \( f \in \text{Re } K \). Let \( W_\varphi(f) = \exp i \pi_\varphi(B(f)) \), \( f \in \text{Re } K \). We shall call such state \( \varphi \) over \( \mathfrak{V}(K, \gamma, \Gamma) \) as a regular state if \( W_\varphi(f) \) satisfies the Weyl-Segal relations:

\[
(2.2) \quad W_\varphi(f)W_\varphi(g) = W_\varphi(f + g)\exp \frac{1}{2} \gamma(g, f).
\]

Let \( \varphi \) be a regular state over \( \mathfrak{V}(K, \gamma, \Gamma) \). Let \( N_\varphi \) be the set of \( f \in K \) with \( \pi_\varphi(B(f)) = 0 \), which is a linear subset of \( K \). Let \( \text{Re } N_\varphi = \text{Re } K \cap \text{Re } N_\varphi \). The collection of distances

\[
(2.3) \quad d_{\mathcal{T}}(f, f') = \sup_{\|T\| \leq 1} \|W_\varphi(tf) - W_\varphi(tf')\|, \quad T \in \mathfrak{H}_{\pi_\varphi},
\]
defines a vector topology on \( \text{Re } K/\text{Re } N_\varphi \), which we shall denote by \( \tau_\varphi \). It also induces a vector topology on \( (\text{Re } K/\text{Re } N_\varphi) + i(\text{Re } K/\text{Re } N_\varphi) = K/\text{N}_\varphi \), which will be denoted again by \( \tau_\varphi \). The topology induced by one distance \( d_{\mathcal{T}} \) for a cyclic \( T \) is mutually equivalent and is equivalent to \( \tau_\varphi \) [4].

(The cyclicity here refers to \( W_\varphi(f), f \in \text{Re } K \).)
§ 3. Quasifree States

Definition 3.1. A state \( \varphi \) on \( \mathfrak{F}(K, \gamma, \Gamma) \) satisfying the following relations is called a quasifree state:

\[
\begin{align*}
(3.1) & \quad \varphi(B(f_1) \cdots B(f_{2n-1})) = 0 \\
(3.2) & \quad \varphi(B(f_1) \cdots B(f_{2n})) = \sum_{s} \prod_{j=1}^{n} \varphi(B(f_{s(j)} B(f_{s(j+n)}))
\end{align*}
\]

where \( n = 1, 2, \ldots \) and the sum is over all permutations \( s \) satisfying \( s(1) < s(2) < \cdots < s(n) \), \( s(j) < s(j+n) \), \( j = 1, \ldots, n \).

Lemma 3.2. For any state over \( \mathfrak{F}(K, \gamma, \Gamma) \), the hermitian form defined by

\[
\begin{align*}
(3.3) & \quad \varphi(B(f)\star B(g)) = S(f, g),
\end{align*}
\]

is positive semidefinite (i.e. \( S(f, f) \geq 0 \)) and satisfies

\[
\begin{align*}
(3.4) & \quad \gamma(g, f) = S(g, f) - S(\Gamma f, \Gamma g).
\end{align*}
\]

Proof. The positivity of \( \varphi \) implies the positive semidefiniteness of \( S \).

\[
\begin{align*}
S(\Gamma f, \Gamma g) = \varphi(B(f)\star B(g)^*) = \varphi(B(g)^* B(f)) - \gamma(g, f)1
= S(g, f) - \gamma(g, f).
\end{align*}
\]
Q. E. D.

Lemma 3.3. The hermitian form

\[
\begin{align*}
(3.5) & \quad (g, f)_S = S(g, f) + S(\Gamma f, \Gamma g)
\end{align*}
\]

is positive semi-definite and satisfies

\[
\begin{align*}
(3.6) & \quad (\Gamma g, \Gamma f)_S = (f, g)_S, \\
(3.7) & \quad |\gamma(g, f)|^2 \leq (f, f)_S S(g, g)_S.
\end{align*}
\]

It is positive definite if \( \gamma \) is non-degenerate.

Proof. From Lemma 3.2,

\[
S(f, f) \geq 0, \quad S(\Gamma f, \Gamma f) \geq 0.
\]

Hence \((g, f)_S\) is positive semidefinite. We also have
By the Schwarz inequality,
\[
|\tau(g, f)| \leq |S(g, f)| + |S(\Gamma f, \Gamma g)|
\leq S(g, g)^{1/2} S(f, f)^{1/2} + S(\Gamma f, \Gamma f)^{1/2} S(\Gamma g, \Gamma g)^{1/2}
\leq (S(g, g) + S(\Gamma g, \Gamma g))^{1/2} (S(f, f) + S(\Gamma f, \Gamma f))^{1/2}
= (g, g)^{1/2} (f, f)^{1/2}.
\]

If \((f, f)_S = 0\), we have \(\tau(f, g) = 0\) for all \(g\). If \(\tau\) is non-degenerate, we have \(f = 0\). Therefore, \((f, g)_S\) is positive definite. Q.E.D.

**Lemma 3.4.** The set \(N_S\) of \(f \in K\) satisfying \((f, f)_S = 0\) is a \(\Gamma\)-invariant subspace of \(K\) such that \(S(f, g) = \tau(f, g) = 0\) for any \(f \in N_S\) and any \(g \in K\). If \(S\) is related to a state \(\phi\) by (3.3), then \(\pi_\phi(B(f)) = 0\) is equivalent to \(f \in N_S\). (\(N_S = N_\phi\) for a regular \(\phi\).)

**Proof.** From the positive semidefiniteness of \((g, f)_S\), it follows that \((g, f)_S = 0\) for any \(g \in K\) whenever \(f \in N_S\). Hence \(N_S\) is a subspace of \(K\). By (3.6), \(N_S\) is \(\Gamma\)-invariant. From (3.7), \(\tau(f, g) = 0\) for any \(g \in K\) whenever \(f \in N_S\). This implies that \(B(f), f \in N_S\) commutes with all \(B(g), g \in K\). In addition, \(0 \leq S(f, f) \leq (f, f)_S = 0\) which implies \(\|\pi_\phi(B(f))Q_\phi\|^2 = S(f, f) = 0\) for \(f \in N_S\). Therefore \(f \in N_S\) implies \(\pi_\phi(B(f)) = 0\). Conversely, \(\pi_\phi(B(f)) = 0\) implies \(S(f, f) = \|\pi_\phi(B(f))Q_\phi\|^2 = 0\), \(S(\Gamma f, \Gamma f) = \|\pi_\phi(B(f))^*Q_\phi\|^2 = 0\), and hence \((f, f)_S = 0\). Q.E.D.

**Lemma 3.5.** For any positive semidefinite hermitian \(S(g, f)\) on \(K \times K\) satisfying (3.4), there exists a unique quasifree state \(\phi_S\) satisfying (3.3). Any quasifree state is regular.

**Proof.** The existence will be seen from Lemma 5.3 and Corollary 6.2. The uniqueness is immediate from (3.1) and (3.2). The regularity will be seen from Corollary 5.6.

**Definition 3.6.** Let \(S, \pi, Q\) denote the Hilbert space, the repre-
sentation and the cyclic unit vector canonically associated with the quasifree state $\varphi_s$ through the relation

$$\varphi_s(A) = (\Omega_s, \pi_s(A) \Omega_s), \quad A \in \mathcal{A}(K, \gamma, \Gamma).$$

If $S$ commutes with a Bogoliubov transformation $U$, then a unitary operator $T_s(U)$ on $\mathcal{Q}_s$ is defined by

$$T_s(U) \pi_s(A) \Omega_s = \pi_s(\tau(U) A) \Omega_s$$

and the continuity. ($S$ is defined in Lemma 4.2.)

§ 4. Structure of $(S, K, \gamma, \Gamma)$

Definition 4.1. $K_s$ denotes the completion of $K/N_s$ with respect to the positive hermitian form induced on $K/N_s$ by $(f, g)_s$. $K/N_s$ is identified with a dense subset of $K_s$. The Hilbert space topology on $K/N_s$ is denoted by $\tau_s$.

Lemma 4.2. (1) There exists an antiunitary involution $\Gamma_s$ on $K_s$ such that $\bar{\Gamma} f = \Gamma_s f$ for all $f \in K$ where $\bar{f} = f + N_s \in K/N_s$.

(2) There exists a bounded operator $\gamma_s$ on $K_s$ such that

$$\gamma(f, g) = (\bar{f}, \gamma_s \bar{g})_s$$

for $f, g \in K$. It satisfies

$$\gamma_s^8 = \gamma_s, \quad \Gamma_s \gamma_s \Gamma_s = -\gamma_s \quad \text{and} \quad \|\gamma_s\|_s \leq 1.$$

(3) There exists a bounded operator $S$ on $K_s$ such that

$$S(f, g) = (\bar{f}, S \bar{g})_s$$

for $f, g \in K$. It satisfies

$$S^* = S, \quad \Gamma_s S \Gamma_s = 1 - S, \quad 0 \leq S \leq 1,$$

and

$$S - \Gamma_s S \Gamma_s = \gamma_s.$$

Proof. Due to the $\Gamma$-invariance of $N_s$ and (3.6), $\bar{\Gamma}_s f = \bar{\Gamma} f$ defines an antilinear isometric operator on $K/N_s$ and hence the closure $\Gamma_s$ of
$\Gamma_s$ is defined on all vectors in $K_s$ and $(\Gamma_s f, \Gamma_s g)_s = (f, g)_s$ for all $f, g \in K_s$. Since $\Gamma^2 = 1$, we have $\Gamma_s^2 = 1$ and hence $\Gamma_s$ is an antiunitary involution on $K_s$.

(3.7) and Lemma 3.4 imply the existence of $\gamma_s$ satisfying (4.1) and $\|\gamma_s\|_s \leq 1$. Since $\gamma(f, g)$ is hermitian, we have $\gamma_s^* = \gamma_s$. Since $\gamma(\Gamma f, \Gamma g) = -\gamma(g, f)$, we have $\Gamma_s \gamma_s \Gamma_s = -\gamma_s$.

From the positivity $S(\Gamma f, \Gamma f) \geq 0$ of $S$, we have $0 \leq S(f, f) \leq \|f\|^2$ for $f \in K$. This together with Lemma 3.4 imply the existence of $S$ satisfying (4.3), $S^* = S$ and $0 \leq S \leq 1$. From (3.5), we have $S + \Gamma_s S \Gamma_s = 1$ and from (3.4), we have (4.5). Q. E. D.

**Definition 4.3.** Let $E_+, E_-$ and $E_0$ be the spectral projection of $\gamma_s$ for $(0, +\infty)$, $(-\infty, 0)$ and $\{0\}$, respectively. Let $K_{\pm} = E_{\pm} K_s$ and $K_0 = E_0 K_s$.

**Lemma 4.4.** $\Gamma_s E_\pm \Gamma_s = E_\pm$, $\Gamma_s E_0 \Gamma_s = E_0$, $\Gamma_s K_{\pm} = K_\pm$ and $\Gamma_s K_0 = K_0$

**Proof.** This follows from $\Gamma_s \gamma_s \Gamma_s = -\gamma_s$. Q. E. D.

§ 5. Fock Representations

**Definition 5.1.** A quasifree state $\varphi_s$ is called a Fock state if the operator $S$ of Lemma 4.2 is a basis projection on $K_s$. $S$ in such a case will be written generally as $P$. The associated representation $\pi_P$ is called a Fock representation.

**Lemma 5.2.** If $P$ is a basis projection of $(K, \gamma, \Gamma)$, then the quasifree state $\varphi_P$ of $\mathfrak{F}(K, \gamma, \Gamma)$ for $P(f, g) = \gamma(f, Pg)$, if it exists, is a Fock state.

**Remark.** In this case $\gamma$ is automatically non-degenerate and $N_p = 0$. $P$ originally given on $K$ is a restriction to $K$ of the operator $P$ on $K_P$ defined by Lemma 4.2 and we have $\gamma(f, Pg) = (f, \gamma_p Pg)_P = (f, Pg)_P$ for $f, g \in K$. Therefore the appearance of two $P$ is probably not confusing.

We shall summarize known properties of a Fock state in the following
Lemma 5.3. Let $P$ be a basis projection for $(K, \gamma, \Gamma)$. A state $\varphi$ of $\mathcal{A}(K, \gamma, \Gamma)$ satisfying

$$(5.1) \quad \varphi(B(f)B(\Gamma f)) = 0, \quad f \in PK,$$

exists, is unique and is a quasifree state $\varphi_P$.

Proof. By splitting $B(f)$ as a sum $B(Pf) + B((1-P)f)$ and bringing $B(Pf)$ to the left of any other $B((1-P)^k f')$ with a help of the commutation relations, any element $A$ in $\mathcal{A}(K, \gamma, \Gamma)$ can be written as $A = \sum \varphi_i B(f_i) + \sum \varphi_j (1-P)K$, $g_j \in PK$. Since (5.1) implies $\varphi(QB(f)) = \varphi(B(g)Q) = 0$ for $f \in (1-P)^k K$, $g \in PK$ and $Q \in \mathcal{A}(K, \gamma, \Gamma)$ by the Schwarz inequality, we have $\varphi(A) = \lambda$. Hence the uniqueness.

The well known Fock state of $\mathcal{A}_{CCR}(PK)$ gives the quasifree state $\varphi_P$ through the identification of $\mathcal{A}_{CCR}(PK)$ with $\mathcal{A}(K, \gamma, \Gamma)$ via $\alpha(P)$. $\varphi_P$ clearly satisfies (5.1). Q. E. D.

Lemma 5.4. Let $f \in \text{Re} K$ and $D_0 = \pi_P[\mathcal{A}(K, \gamma, \Gamma)] \mathcal{O}_P$. $D_0$ is a dense set of entire analytic vectors of $B(f)$. The sum

$$(5.2) \quad \sum_{n=0}^{\infty} n!^{-1} i^n \pi_P(B(f))^n$$

converges on $D_0$. Its closure, denoted by $W_P(f)$, is unitary and satisfies

$$(5.3) \quad W_P(f_1)W_P(f_2) = W_P(f_1 + f_2) \exp(1/2)\gamma(f_2, f_1),$$

$$(5.4) \quad (\mathcal{O}_P, W_P(f)\mathcal{O}_P) = \exp(-1/2)\gamma(f, Pf).$$

$f \mapsto W_P(f)$ is continuous with respect to a norm $\gamma(f, Pf)^{1/2}$ on $\text{Re} K$ and the strong operator topology on $\mathcal{O}_P$.

Proof. Let $(\mathcal{O}_P)_n$ be the subspace of $\mathcal{O}_P$ generated by $\pi_P(B(g_j))\mathcal{O}_P$, $g_j \in PK$. If $\mathcal{V} \in \sum_{n=0}^{N} (\mathcal{O}_P)_n$, then

$$(5.5) \quad ||\pi_P(B(f))\mathcal{V}|| \leq \sqrt{2} (N+1)^{1/2} \gamma(f, (2P-1)f)^{1/2}||\mathcal{V}||.$$

This follows from a well known calculation: Let $\{f_j\}$ be a complete orthonormal basis of $PK$ with $f_0 = Pf$. Then $\Phi = \sum_{n=1}^{N} \pi_P(B(f_j))\mathcal{O}_P(n \leq N)$
is a complete orthonormal basis of $\sum_{n=0}^{N} (\mathcal{S}_{n})$, for which $\pi_P(B(Pf))\Phi$ is also mutually orthogonal and $\|\pi_P(B(Pf))\Phi\| = (k+1)^{1/2} \gamma(f, Pf)^{1/2}\|\Phi\|$ where $k$ is the number of $\nu$ with $j_\nu = 0$. Hence $\|\pi_P(B(Pf))\Psi\| \leq (N+1)^{1/2} \gamma(f, Pf)^{1/2} ||\Psi||$. A similar calculation with $f_0 = I^*(1-P)f$ leads to $\|\pi_P(B(1-P)f)f)f)||\Psi||$. 

From (5.5) we have $\lim_{f_0 \to \infty} \|\pi_P(B(f))^n\Psi||^{1/n} = 0$ for $\Psi \in D_0 = \bigcup N_{n=0}^{N} (\mathcal{S}_{n})$. Hence all such $\Psi$ is an entire analytic vector for $\pi_P(B(f))$, $f \in \text{Re} K$, (5.2) applied on $\Psi$ converges absolutely, the closure $\pi_P(B(f))$ of $\pi_P(B(f))$ is selfadjoint, $W_P(f) = \exp i \pi_P(B(f))$, and $W_P(f)$ is unitary. 

By the commutation relations, we have

$$n!^{-1}B(f_1 + f_2)^n = \sum_{k+l+2m=n} k!^{-1}B(f_1)^k f_1^{-1}B(f_2)^l f_1^{-l}B(f_2)^m f_2^{-m} \gamma(f_2, f_1)^m.$$ 

From the previous result and the Schwarz inequality, $\sum k!^{-1}l!^{-1}(B(f_1)^k, B(f_2)^l)^2$ is absolutely convergent for $\Phi, \Psi \in D_0$ and hence we obtain from (5.6) the equality (5.3) for a matrix element between two vectors $\Phi$ and $\Psi$ in a dense set $D_0$. Hence (5.3) holds.

From (5.3) and (5.4), we have

$$d_P(f_1, f_2)^2 = \|\{W_P(f_1) - W_P(f_2)\} \Omega_P\|^2$$

$$= 2 \{1 - (\exp(-1/4)\|f_2 - f_1\|_p) \cos(i/2) \gamma(f_2, f_1)\}$$

where $||f||_p = \gamma(f, \frac{1}{2}P - 1)f)$, which is $2\gamma(f, Pf)$ for $f \in \text{Re} K$, and $\gamma(f_2, f_1) = \gamma(f_2, f_1) = -\gamma(f_2, f_1)^*$ is pure imaginary for $f_1, f_2 \in \text{Re} K$. Since $\gamma(f_1, f_1) = 0$ for $f_1 \in \text{Re} K$, we have from (3.7)

$$\gamma(f_2, f_1) = \gamma(f_2 - f_1, f_1) \leq \|f_2 - f_1\|_p \|f_1\|_p.$$ 

Hence $f \to W_P(f)\Omega_P$ is continuous. By (5.3) and (3.7), this implies the continuity of $f \to W_P(f)\Omega_P$ for $\Psi = W_P(g)\Omega_P$, $g \in \text{Re} K$. Since $\pi_P(B(g)) = \lim_{r \to 0} (it)^{-1}(W(tg) - 1)$ on $D_0$ for $f \in \text{Re} K$, and since $W_P(g_1)W_P(g_2)\cdots W_P(g_n)$ $\Omega_P$ is a multiple of $W_P(\sum g)\Omega_P$, finite linear combinations of $W_P(g)\Omega_P$, $g \in \text{Re} K$, are dense in $\mathcal{S}_P$. Therefore $f \to W_P(f)\Omega_P$ is continuous. Q. E. D.

**Lemma 5.5.** Let $\text{Re} K_\rho$ be the real Hilbert space obtained by the
completion of $Re K$ with respect to the inner product $(f_1, f_2)_P = \gamma(f_1, (2P-1)f_2)$, $f_1, f_2 \in Re K$. If $f = \lim f_n, f_n \in Re K$, then $W_P(f) = \lim W_P(f_n)$ exists and does not depend on $\{f_n\}$ for a fixed $f$.

Let $H_1$ be a linear subset of $Re K_P$. Denote by $H_1^\perp$ the set of vectors $f \in Re K_P$ such that $(f, \gamma_P g)_P = 0$ for all $g \in H_1$. Let $R_P(H_1)$ be the von Neumann algebra generated by $W_P(f), f \in H_1$. Let $\bar{H}_1$ denote the closure of $H_1$ in $Re K_P$. Then

$(0)$ $R_P(Re K_P)$ is irreducible and $R_P(0)$ is trivial,
$(i)$ $R_P(H_1) = R_P(\bar{H}_1),$
$(ii)$ $R_P(H_1)^\perp = R_P(H_1^\perp),$
$(iii)$ $(R_P(H_1) \cup R_P(H_2))^\perp = R_P(H_1 + H_2),$
$(iv)$ $(R_P(H_1) \cap R_P(H_2))^\perp = R_P(H_1 \cap H_2),$
$(v)$ $\Omega_P$ is cyclic for $R_P(H_1)$ if and only if $\bar{P}(H_1 + iH_1)$ is dense in $PK_P$. ($P$ is the closure of $P$ on $K_P$.)
$(vi)$ $\Omega_P$ is separating for $R_P(H_1)$ if and only if $\bar{P}(H_1^\perp + iH_1^\perp)$ is dense in $PK_P$.
$(vii)$ $R_P(H_1)$ is a factor if and only if $\bar{H}_1 \cap H_1^\perp$ is 0.

Proof. The existence of the unique limit $W_P(f)$ for $f \in Re K_P$ follows from Lemma 5.4. The von Neumann algebra $R_P(H_1)$ is $R(\bar{H}_1/Re K_P)$ in the notation of [1], where $(f_1, f_2)_S$ and $\gamma(f_1, f_2)$ are respectively $(f_1, f_2)$ and $(f_1, \beta f_2)$. (i)~(iv) and (vii) follow from Theorem 1 of [1]. (0) and (v) follow from Lemma 5.1 of [1]. (vi) follows from (v) and (ii).

Q. E. D.

Corollary 5.6. A Fock representation is regular and irreducible.

This is due to Lemmas 5.4 and 5.5.

The Fock representation defined above is applicable only for the case of non-degenerate $\gamma$. We now consider its generalization to the case of degenerate $\gamma$.

Definition 5.7. A quasifree state $\varphi_S$ is called a Fock type state if $N_S = 0$ and the spectrum of the operator $S$ in Lemma 4.2 is contained in $\{0, 1/2, 1\}$. The corresponding representation is called a Fock type
representation.

**Lemma 5.8.** Let $K$, $\gamma$, $\Gamma$ be given. Let $\Pi(f_1, f_2)$ be a positive semidefinite hermitian form on $K$ satisfying (3.4), where $S$ is to be replaced by $\Pi$. Assume that $N_n = 0$ and the spectrum of the operator $\Pi$ defined by Lemma 4.2 is contained in $\{0, 1/2, 1\}$. Let $E_+, E_0$ be defined as in Definition 4.3. Let

\[(5.9)\quad \tilde{K}_n = K_n \oplus E_0 K_n,\]

\[(5.10)\quad \tilde{\Gamma}_n(f \oplus g) = \Gamma_n f \oplus \Gamma_n g,\]

\[(5.11)\quad \tilde{\gamma}_n(f_1 \oplus g_1, f_2 \oplus g_2) = (f_1, \tilde{\gamma}_n f_2)_n + i\{(g_1, f_2)_n - (f_1, g_2)_n\}.\]

Let $\mathfrak{A} = \mathfrak{A}(K, \gamma, \Gamma)$ with the subalgebra $\mathfrak{A}(K \oplus 0, \tilde{\gamma}_n, \tilde{\Gamma}_n)$ of $\mathfrak{A}$. Let

\[(5.12)\quad \tilde{\mathfrak{H}}(f \oplus g) = \{E_+ f + (E_0 f - ig)/2\} \oplus \{(iE_0 f + g)/2\},\]

\[(5.13)\quad \tilde{\mathfrak{H}}(h_1, h_2) = \tilde{\gamma}_n(h_1, \tilde{\mathfrak{H}} h_2).\]

Then $\varphi_{\tilde{\mathfrak{H}}}$ is a Fock state of $\mathfrak{A}$ and its restriction to $\mathfrak{A}$ is the Fock type state $\varphi_{\Pi}$.

**Proof.** $\tilde{\Gamma}_n$ is an antiunitary involution of $\tilde{K}_n$ and $\tilde{\gamma}_n$ is a hermitian form satisfying $\tilde{\gamma}_n(\tilde{\Gamma}_n h_1, \tilde{\Gamma}_n h_2) = -\tilde{\gamma}_n(h_1, h_2)^\ast$. From (5.12), it follows that $\tilde{\Gamma}_n \tilde{\mathfrak{H}} \tilde{\Gamma}_n = 1, \tilde{\mathfrak{H}}^2 = \tilde{\mathfrak{H}}$,

\[(5.14)\quad \tilde{\gamma}_n(f_1 \oplus g_1, \tilde{\mathfrak{H}}(f_2 \oplus g_2))
= (f_1, \Pi f_2)_n + (g_1, \Pi g_2)_n + i\{(g_1, \Pi f_2)_n - (f_1, \Pi g_2)_n\}
= \tilde{\gamma}_n(\tilde{\mathfrak{H}}(f_1 \oplus g_1), f_2 \oplus g_2),\]

and

\[(5.15)\quad \tilde{\gamma}_n(f \oplus g, \tilde{\mathfrak{H}}(f \oplus g)) \geq 0.\]

Therefore $\tilde{\mathfrak{H}}$ is a basis projection and $\varphi_{\tilde{\mathfrak{H}}}$ is a Fock state.

The restriction of $\varphi_{\tilde{\mathfrak{H}}}$ to $\mathfrak{A}$ is $\varphi_{\Pi}$ as is seen from (5.14). Q. E. D.

**Corollary 5.9.** For any $\Pi$ in Lemma 5.8, the Fock type state $\varphi_{\Pi}$
exists. The commutant $\pi(H)(B)$ is abelian and is generated by $\pi(H)(B(f))$, $f \in E_0K_H$.

**Proof.** From Lemmas 5.8 and 5.5 (ii), the following computation suffices: If $f \oplus g \in (K \oplus 0)^+$, then $(f, \gamma(H)(1-E_0)f)_H = (g, E_0f)_H = 0$ for all $f, g \in K$ and hence $f \in E_0K_H$ and $g = 0$. Q. E. D.

§ 6. A Realization of a Quasifree State on a Fock Type Representation

**Lemma 6.1.** (1) Let

$$K'_S = K_S \oplus K_S,$$  

$$\gamma'_S(f_1 \oplus g_1, f_2 \oplus g_2) = (f_1, \gamma S f_2)_S - (g_1, \gamma S g_2)_S,$$  

$$\Gamma'_S = \Gamma'_S \oplus \Gamma'_S.$$

Then $\Gamma'_S$ is an antilinear involution and $\gamma'_S$ is a hermitian form satisfying $\gamma'_S(\Gamma'_S h_1, \Gamma'_S h_2) = -\gamma'_S(h_1, h_2)^*$. If $N_S = N'_S$ and $\gamma_S = \gamma'_S$, then there exists a one-to-one linear map $U$ of $K'_S$ onto $K'_S$ such that $Uh = h$ for $h = (f + N_S) \oplus (g + N_S)$, $f, g \in K$. It satisfies $U\Gamma'_S = \Gamma'_S U$ and $\gamma'_S(h_1, h_2) = \gamma'_S(Uh_1, Uh_2)$.

(2) Let

$$\gamma'_S(f_1 \oplus g_1, f_2 \oplus g_2)_S = (f_1, f_2)_S + (g_1, g_2)_S$$  

$$+ 2(f_1, S^{1/2}(1-S)^{1/2} g_2)_S$$  

$$+ 2(g_1, S^{1/2}(1-S)^{1/2} f_2)_S.$$  

Then it is a $\Gamma'_S$-invariant positive semidefinite form satisfying

$$|\gamma'_S(h_1, h_2)| \leq ||h_1||_S ||h_2||_S.$$  

The kernel $N'_S$ (i.e. the set of $h$ satisfying $||h||_S = 0$) consists of $f \oplus -f$, $f \in E_0K_S$. If $N_S = N'_S$ and $\gamma_S = \gamma'_S$, then $N'_S = UN'_S$.

(3) (6.4), $\gamma'_S$ and $\Gamma'_S$ induce on $K'_S/N'_S$ a positive definite inner product $(\hat{h}_1, \hat{h}_2)_{\hat{S}} = (h_1, h_2)'_S$, a hermitian form $\hat{\gamma}_S(\hat{h}_1, \hat{h}_2) = \gamma'_S(h_1, h_2)$ and an antilinear involution $\hat{\Gamma}_S \hat{h} = (\Gamma'_S h)^*$ satisfying $\hat{\Gamma}_S \hat{h}_1, \hat{\Gamma}_S \hat{h}_2)_{\hat{S}} = (\hat{h}_2, \hat{h}_1)'_{\hat{S}}$ and $\hat{\gamma}_S(\hat{\Gamma}_S \hat{h}_1, \hat{\Gamma}_S \hat{h}_2) = -\hat{\gamma}_S(\hat{h}_2, \hat{h}_1)$ where $\hat{h} = h + N'_S \in K'_S/N'_S$. The closure
of $\hat{\tau}_S$ and $\hat{\tau}'_S$ on the completion $\hat{K}_S$ of $K'_S/N'_S$, denoted by the same letter, satisfy the same properties. $\hat{\tau}'_S$ is antiunitary and there exists an operator $\hat{\tau}'_S$ such that

$$(6.6) \quad \hat{\tau}'_S(h_1, h_2) = (h_1, \hat{\tau}'_S h_2),$$

$$(6.7) \quad \hat{\tau}'_S^* = \hat{\tau}'_S, \quad \hat{\tau}'_S \hat{\tau}'_S = -\hat{\tau}'_S.$$

If $N_S = N'_S$ and $\tau_S = \tau'_S$, then $U$ of (2) induces a one-to-one linear map of $\hat{K}_S$ onto $\hat{K}'_S$ such that $\hat{U}\hat{\tau}'_S = \hat{\tau}'_S \hat{U}$ and $\hat{\tau}_S(\hat{U}h_1, \hat{U}h_2) = \hat{\tau}_S(h_1, h_2)$.

(4) Let

$$(6.8) \quad \Pi_S = (1/2)(1 + \hat{\tau}_S).$$

Then $\hat{\tau}'_S \Pi S \hat{\tau}'_S = 1 - \Pi S$, $\Pi S^* = \Pi S$ and the spectrum of $\Pi S$ is contained in $\{0, 1/2, 1\}$.

(5) For $f \in K$, let $[f] = (\langle f, 0 \rangle + N'_S$ and identify $K'_S/N'_S$ with a dense subset of $\hat{K}_S$. Then

$$(6.9) \quad \hat{\tau}_S([f], [g]) = \tau(f, g).$$

$$(6.10) \quad (\langle f \rangle, \Pi S \langle g \rangle) = S(f, g).$$

(6) If $N_S = N'_S$ and $\tau_S = \tau'_S$, then $\tau_{II_S} = \tau_{II'_S}$ and eigenspaces of $\Pi S$ and $\Pi'_S$ for an eigenvalue $1/2$ are mapped by $\hat{U}$.

**Proof.** (1) The properties of $\Gamma'_S$ and $\hat{\tau}'_S$ are immediate. Since $K_S$ and $K'_S$ is the completion of $K/N_S = K/N'_S$ with respect to $\tau_S = \tau'_S$, there is a natural identification map $U$ which is linear. If $f_j, g_j \in K$ and $h_j = (f_j + N_S) \oplus (g_j + N_S)$, then

$$\hat{\tau}'_S(h_1, h_2) = \tau(f_1, f_2) - \tau(g_1, g_2) = \tau'_S(h_1, h_2),$$

$$\Gamma'_S h_1 = (\Gamma f_1 + N_S) \oplus (\Gamma g_1 + N_S) = \Gamma'_S h_1.$$ 

Since such $f_j$ and $g_j$ are dense in $K_S$, these equalities imply $U \Gamma S = \Gamma'_S U$ and $\gamma_S(h_1, h_2) = \gamma'_S(Uh_1, Uh_2)$.

(2) (6.4) is obviously a $\Gamma'_S$-invariant hermitian form. We have

$$(6.11) \quad (f \oplus g, f \oplus g)'_S = \|S^{1/2}f + (1 - S)^{1/2}g\|^2_S + \|(1 - S)^{1/2}f + S^{1/2}g\|^2_S \geq 0.$$ 

We also have

$$(6.12) \quad \tau'_S(f_1 \oplus g_1, f_2 \oplus g_2) = (S^{1/2}f_1 + (1 - S)^{1/2}g_1, S^{1/2}f_2 + (1 - S)^{1/2}g_2)_S$$
due to \( \gamma_s = 2S - 1 \), which implies
\[
\| \tau_s(f_1 \oplus g_1, f_2 \oplus g_2) \| \leq \| S^{1/2} f_1 + (1 - S)^{1/2} g_1 \|_s \| S^{1/2} f_2 + (1 - S)^{1/2} g_2 \|_s \\
+ \| (1 - S)^{1/2} f_1 + S^{1/2} g_1 \|_s \| (1 - S)^{1/2} f_2 + S^{1/2} g_2 \|_s \\
\leq \| f_1 \oplus g_1 \|_s \| f_2 \oplus g_2 \|_s.
\]

By (6.11), \( \| f \oplus g \|_s = 0 \) is equivalent to \( (2S - 1)f = 0 \) and \( f + g = 0 \). Namely \( N_s \) consists of \( \tau_s = \tau_s' = 0 \) and \( f = g \in E \). \( E_0 \) is the set of \( f \in K_s \) such that \( (f, \tau_s g)_s = 0 \) for all \( g \in K_s \). If \( N_s = N_s' \) and \( \tau_s = \tau_s' \), then there is a natural identification of \( K_s \) with \( K_{s} \) which identifies \( E_0 K_s \) with \( E_0' K_{s} \) due to \( (f, \tau_s g)_s = \tau(f, g) = (f, \tau_s' g)' \) for \( f, g \in K \). (\( E_0 \) and \( E_0' \) are orthogonal projections of \( S \) and \( S' \) for an eigenvalue \( 1/2 \). Since the orthogonality refers to different inner product, \( E_0 \) and \( E_0' \) need not be the same.) This implies \( N_s' = UN_s' \).

(3) Immediate from (1) and (2).

(4) Let \( \hat{\mathcal{K}}_s \), \( \hat{\mathcal{K}}_{s} \), and \( \hat{\mathcal{K}}_s' \) be the subspace of \( \hat{\mathcal{K}}_s \) generated by \( \{ \| f \|_s \} \), \( \{ (1 - S)^{1/2} f \} \), and \( \{ E_0 f \|_s \} \), respectively, where \( f \) runs over \( K_s \). It is easily seen that they are mutually orthogonal and altogether generate \( \hat{\mathcal{K}}_s \). For \( h_s, h_s' \in \hat{\mathcal{K}}_s \), we have \( \hat{\tau}_s(h_s, h_s') = \sigma \delta_{a_s}(h_s, h_s') \) where \( \sigma = +, - \) or 0. Therefore \( \hat{\tau}_s h_s = \sigma h_s \) and the spectrum of \( \Pi_s \) is contained in \( \{ 0, 1/2, 1 \} \).

(5) Immediate from definitions.

(6) From the proof of (4) and the last part of the proof of (2), it follows that \( \hat{\mathcal{K}}_s \) for \( S \) and \( S' \) are mapped by \( U \) if \( N_s = N_s', \pi_s = \pi_{s}' \).

The topology \( \tau_{\pi_s} \) is the strong topology of \( \hat{\mathcal{K}}_s \). Let \( \{ f_a \|_s \} \) be a Cauchy net relative to \( \tau_{\pi_s} \) where \( f_a, g_a \in K_{s} \). \( S^{1/2} f_a + (1 - S)^{1/2} g_a = F_a \) and \( (1 - S)^{1/2} f_a + S^{1/2} g_a = G_a \) are Cauchy in \( K_{s} \). Therefore \( f_a + g_a = \{ S^{1/2} + (1 - S)^{1/2} \}^{-1} (F_a + G_a) \) and \( (2S - 1)(f_a - g_a) = \{ S^{1/2} + (1 - S)^{1/2} \}^{-1} (F_a - G_a) \) are Cauchy. Conversely, if \( f_a + g_a \) and \( (2S - 1)(f_a - g_a) \) are Cauchy in \( K_{s} \), then \( F_a \) and \( G_a \) are Cauchy and hence \( \{ f_a \|_s \} \) is Cauchy in \( \hat{\mathcal{K}}_s \).

If \( N_s = N_s' \) and \( \pi_s = \pi_{s}' \), then the properties of a net \( f_a \) being Cauchy relative to \( \pi_s \) and \( \pi_{s}' \) are the same. Furthermore, \( \gamma_s = 2S - 1 \)
and \((f, \tau_S g)_S = (f, \tau_{S'} g)_{S'}\) imply that \((2S-1)g_a\) is Cauchy relative to \(\tau_S\) if and only if \((2S'-1)g_a\) is Cauchy relative to \(\tau_{S'}\) by the duality.

Combining above two sets of arguments, we see that \((f_a \oplus g_a)^\wedge\) is Cauchy relative to \(\tau_{\pi S}\) if and only if \((f_a \oplus g_a)^\wedge\) is Cauchy relative to \(\tau_{\pi S'}\).

**Q. E. D.**

**Corollary 6.2.** The map \(f \in K \rightarrow \lfloor f \rfloor \in K_S\) induces a \(*\) homomorphism \(\alpha_S\) of \(\mathfrak{A}(K, \tau, \Gamma)\) into \(\mathfrak{A}(\hat{K}_S, \hat{\tau}_S, \hat{\Gamma}_S)\). The restriction of a Fock type state \(\varphi_{\pi S}\) of \(\mathfrak{A}(\hat{K}_S, \hat{\tau}_S, \hat{\Gamma}_S)\) to \(\alpha_S \mathfrak{A}(K, \tau, \Gamma)\) gives a quasifree state \(\varphi_S\) of \(\mathfrak{A}(K, \tau, \Gamma)\) through \(\varphi_{\pi S}(\alpha_S A) = \varphi_S(A)\).

This is immediate from Lemma 6.1.

**Remark 6.3.** It is possible to realize \(\varphi_S\) directly in a Fock representation in the following manner: Define \(K_S^\wedge = K_S \oplus K_S, \Gamma_S^\wedge = \Gamma_S \oplus \Gamma_S,\)

\[
\tau_S^\wedge(f_1 \oplus g_1, f_2 \oplus g_2) = (f_1, \tau_S f_2)_S - (g_1, \tau_S g_2)_S + i\{(g_1, E_0 f_2)_S - (f_1, E_0 g_2)_S\}
\]

and

\[
(6.13) \quad (f_1 \oplus g_1, f_2 \oplus g_2)^\wedge_S = (f_1, f_2)_S + (g_1, g_2)_S + 2(f_1, (1-E_0) S^{1/2}(1-S)^{1/2} g_2)_S + 2(g_1, (1-E_0) S^{1/2}(1-S)^{1/2} f_2)_S.
\]

Then (6.13) is positive definite and

\[
|\tau_S^\wedge(h_1, h_2)| \leq ||h_1||_S^\wedge ||h_2||_S^\wedge.
\]

Let \(K_S^\wedge\) be the completion of \(K_S^\prime\) relative to \(||h||_S^\prime\), \(\tau_S^\wedge\) and \(\Gamma_S^\wedge\) be the closure of \(\tau_S^\prime\) and \(\Gamma_S^\prime\), \(\tau_S^\wedge(h_1, h_2) = (h_1, \tau_S^\prime h_2)_S^\prime\) and \(P_S = (\tau_S^\wedge + 1)/2\). Then \(P_S\) is a basis projection. Let \(\alpha_S^\wedge\) be the \(*\) homomorphism of \(\mathfrak{A}(K, \tau, \Gamma)\) into \(\mathfrak{A}(K_S^\wedge, \tau_S^\wedge, \Gamma_S^\wedge)\) induced by \(f \rightarrow \hat{f} \oplus 0\). Then the restriction of the Fock state \(\varphi_{P_S}\) of \(\mathfrak{A}(K_S^\wedge, \tau_S^\wedge, \Gamma_S^\wedge)\) to \(\alpha_S^\wedge \mathfrak{A}(K, \tau, \Gamma)\) induces the quasifree state \(\varphi_S\) of \(\mathfrak{A}(K, \tau, \Gamma)\).

This method has a defect that a canonical identification map \(U\) can not be defined between \(K_S^\wedge, \tau_S^\wedge, \Gamma_S^\wedge\) and \(K_S^\prime, \tau_S^\prime, \Gamma_S^\prime\) even if \(N_S = N_S^\prime\) and \(\tau_S = \tau_{S'}\), due to the dependence of the operator \(E_0\) on \(S\).
Lemma 6.4. Let $\varphi_\gamma$ be a quasifree state of $\mathcal{H}(K, \gamma, \Gamma)$. The induced topology $\tau_{\varphi_\gamma}$ on $K$ is the same as $\tau_\gamma$ of Definition 4.1.

Proof. Denote $W_{\varphi_\gamma}(f)$ by $W_\gamma(f)$. Since $Q_\gamma$ is cyclic for $\mathcal{H}(K, \gamma, \Gamma)$ and $\pi_\gamma(A)Q_\gamma, A \in \mathcal{H}(K, \gamma, \Gamma)$, is entire for $\pi_\gamma(B(f)) = \lim_{t \to 0} W_\gamma(tf) - 1, f \in \text{Re } K$, $Q_\gamma$ is cyclic for $R_\gamma$.

By [5], it is known that $\tau_\gamma$ is a vector topology and is given by one distance $d_\gamma(f_1, f_2)$ for a cyclic $\mathcal{V}$. Therefore it is enough to show the equivalence of $\|f\|_3 \to 0$ and
\[ d_\gamma(f, 0) = 2\{1 - \exp(-\|f\|_3^2/4)\} \to 0, \]
where (5.7) is used. This equivalence is obvious. Q. E. D.

References

[8] ————, $C^*$-algèbre de relations de commutation, ibid. 139-161.