Group Actions on Spaces of Rational Functions

By

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Abstract

Let $\text{Hol}_d$ be the space consisting of all holomorphic maps $f : S^2 \to S^2$ of degree $d$. The group $\text{Hol}_1 = \text{PSL}_2(\mathbb{C})$ acts on $\text{Hol}_d$ freely by the post-composition and we shall study the orbit space $X_d = \text{Hol}_1 \setminus \text{Hol}_d$. As an application we shall determine the homotopy types of the universal covering spaces of $\text{Hol}_d$ and $X_d$ explicitly.

§1. Introduction

For each integer $d \geq 0$, let $\text{Hol}_d$ denote the space consisting of all holomorphic maps $f$ of degree $d$ from the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ to itself. The corresponding space of continuous maps $f : S^2 \to S^2$ is denoted by $\text{Map}_d(S^2, S^2)$. Similarly, we denote by $\text{Hol}_d^* \subset \text{Hol}_d$ the subspace consisting of all maps $f \in \text{Hol}_d$ which preserve the base-points, and the corresponding space of continuous maps is also denoted by $\text{Map}_d^*(S^2, S^2) = \Omega_d^2 S^2$. The spaces $\text{Hol}_d$ and $\text{Hol}_d^*$ are of interest both from a classical and modern point of view ([1], [3], [5]), and they can be easily identified with the following spaces of rational functions.

\[
\begin{align*}
\text{Hol}_d &= \{p(z)/q(z) : (i), (ii) are satisfied\} \\
\text{Hol}_d^* &= \{p(z)/q(z) : (i), (iii) are satisfied\},
\end{align*}
\]

where the conditions (i)–(iii) are given by the following:

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(i) \( p(z), q(z) \in \mathbb{C}[z] \) are mutually coprime polynomials.

(ii) \( \max\{\deg(p(z)), \deg(q(z))\} = d \).

(iii) \( p(z), q(z) \in \mathbb{C}[z] \) are monic polynomials of degree \( d \).

In particular, if \( d = 1 \), \( \text{Hol}_1 \) is the group of fractional linear transformations \( \text{PSL}_2(\mathbb{C}) \) and \( \text{Hol}_1^\ast \) may be identified with the affine transformation group of \( \mathbb{C} \). It is an elementary and fundamental fact that \( \text{Hol}_d \) and \( \text{Hol}_d^\ast \) are connected spaces. More generally, the following result is known.

**Theorem 1.1** (S. Epshtein, G. Segal; [6], [17]). Let \( d \geq 1 \) be an integer.

(i) \( \pi_1(\text{Hol}_d^\ast) = \mathbb{Z} \) and \( \pi_1(\text{Hol}_d) = \mathbb{Z}/2d \).

(ii) If \( i_d : \text{Hol}_d \to \text{Map}_d(S^2, S^2) \) and \( \tilde{i}_d : \text{Hol}_d^\ast \to \Omega^2_2 S^2 \) are inclusion maps, then \( i_d \) and \( \tilde{i}_d \) are homotopy equivalences up to dimension \( d \), where the map \( f : X \to Y \) is said to be a homotopy equivalence up to dimension \( N \) if the induced homomorphism \( f_* : \pi_k(X) \to \pi_k(Y) \) is bijective when \( k < N \) and surjective when \( k = N \).

Since the topology of the space \( \text{Hol}_d^\ast \) is now well studied ([3], [7], [9], [10], [17]), we would like to study the topology of \( \text{Hol}_d \). However, because \( \text{Hol}_d \) is non-simply connected, we shall mainly consider the homotopy types of the universal covering of \( \text{Hol}_d \). We denote by \( \tilde{\text{Hol}}_d \) and \( \tilde{\text{Hol}}_d^\ast \) the universal coverings of \( \text{Hol}_d \) and \( \text{Hol}_d^\ast \), respectively. If \( d = 1 \) or 2, the following result is known.

**Theorem 1.2** (M. Guest et al.; [7]).

(i) If \( d = 1 \), \( \tilde{\text{Hol}}_1^\ast \) is contractible and there is a homotopy equivalence \( \tilde{\text{Hol}}_1 \simeq S^3 \).

(ii) If \( d = 2 \), there are homotopy equivalences

\[
\begin{align*}
\tilde{\text{Hol}}_2^\ast & \simeq S^2 \\
\tilde{\text{Hol}}_2 & \simeq S^2 \times S^3.
\end{align*}
\]

We identify \( \text{Hol}_1 = \text{PSL}_2(\mathbb{C}) \) and define the left action of \( \text{Hol}_1 \) on \( \text{Hol}_d \) by post-composition:

\[ A \cdot f(z) = A(f(z)) \quad \text{for} \quad (A, f(z)) \in \text{Hol}_1 \times \text{Hol}_d. \]
We also define the left action of $\text{Hol}^*_d$ on $\text{Hol}_d^*$ by post-composition in a similar way. Let $X_d$ denote the orbit space $X_d = \text{Hol}_1 \backslash \text{Hol}_d$ and let $F_{d,0}$ be the space consisting of all non-singular $(d \times d)$-Toeplitz matrices. We recall the following result.

**Theorem 1.3** (R. J. Milgram; [15]). If $d \geq 1$ is an integer, there is a homeomorphism $X_d \cong F_{d,0}$.

Finite Toeplitz matrices appear in many areas of mathematics ranging from applied mathematics and mathematical physics, through algebraic geometry (e.g. [2], [13]). So it seems interesting to study the topology of the space $F_{d,0}$ and in this paper we shall study the topology of the orbit space $X_d$. Since $X_d$ is not simply connected, we shall also study the universal covering space $\tilde{X}_d$ of $X_d$. The main purpose of this paper is to prove the following two results:

**Theorem 1.4.** Let $d \geq 1$ be an integer and $\tilde{X}_d$ denote the universal covering of $X_d$. Then there is a homotopy equivalence $\tilde{\text{Hol}}_d \simeq S^3 \times \tilde{X}_d$.

**Theorem 1.5.** Let $d \geq 1$ be an integer. Then there is a homotopy equivalence $\tilde{X}_d \simeq \tilde{\text{Hol}}^*_d$.

It follows from the above two results that we also have:

**Corollary 1.1** ([20]). Let $d \geq 1$ be an integer.

(i) There is a homotopy equivalence $\tilde{\text{Hol}}_d \simeq S^3 \times \tilde{\text{Hol}}^*_d$.

(ii) There is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_k(\text{Hol}_d^*) \oplus \pi_k(S^3)$ for any $k \geq 2$.

(iii) In particular, if $d > k \geq 2$, there is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.

**Remark.** The above corollary was first obtained by the analysis of the evaluation fibration in [20]. On the other hand, in this paper, we shall study the $\text{Hol}_1$-action on $\text{Hol}_d$ and prove the above two theorems. Then Corollary 1.1 easily follows from them and so it seems easier and natural to study the topology of $\text{Hol}_d$ from the point of view of group actions.

The plan of this paper is as follows. In Section 2, we shall study the $\text{Hol}_1$ action on $\text{Hol}_d$ and give the proof of Theorem 1.4. In Section 3, we shall prove Theorem 1.5.
§2. Group Actions and Their Orbit Spaces

First recall the following result.

**Lemma 2.1** ([7]).  Let \( d \geq 1 \) be an integer.

(i) The group \( \text{Hol}_1 \) acts on \( \text{Hol}_d \) freely by post-composition. Similarly, the group \( \text{Hol}_1^* \) also acts on \( \text{Hol}_d^* \) freely by post-composition.

(ii) The natural inclusion \( j_d : \text{Hol}_d^* \to \text{Hol}_d \) induces a homeomorphism \( p_d : \text{Hol}_1^* \setminus \text{Hol}_d^* \cong \text{Hol}_1 \setminus \text{Hol}_d = X_d \) such that the diagram

\[
\begin{array}{ccc}
\text{Hol}_1^* & \xrightarrow{s_1^*} & \text{Hol}_d^* \xrightarrow{q_d^*} \text{Hol}_1^* \setminus \text{Hol}_d^* \\
j_1 & \downarrow & \downarrow \ p_d \\
\text{Hol}_1 & \xrightarrow{s_1} & \text{Hol}_d \xrightarrow{q_d} \text{Hol}_1 \setminus \text{Hol}_d = X_d
\end{array}
\]

is commutative, where horizontal sequences \( (\ast)_d^* \) and \( (\ast)_d \) are principal fibration sequences.

**Proof.** This follows from [[7]; (3.1), (3.2)]. \( \square \)

**Lemma 2.2.**  For each integer \( d \geq 1 \), there is a commutative diagram

\[
\begin{array}{ccc}
\pi_1(\text{Hol}_1) & \xrightarrow{(s_1)_*} & \pi_1(\text{Hol}_d) \xrightarrow{(q_d)_*} \pi_1(X_d) \\
\cong & \downarrow & \cong \\
\mathbb{Z}/2 & \xrightarrow{i} & \mathbb{Z}/2d \xrightarrow{\rho} \mathbb{Z}/d
\end{array}
\]

where three vertical maps are isomorphisms, and \( \rho : \mathbb{Z}/2d \to \mathbb{Z}/d \) and \( i : \mathbb{Z}/2 \to \mathbb{Z}/2d \) denote the natural epimorphism and the natural inclusion homomorphism, respectively.

**Proof.** We note ([12], (3.4)) that \( \pi_1(\text{Hol}_1^* \setminus \text{Hol}_d^*) = \mathbb{Z}/d \). Furthermore, since \( p_d : \text{Hol}_1^* \setminus \text{Hol}_d^* \cong X_d \) is a homeomorphism, \( \pi_1(X_d) = \mathbb{Z}/d \). We also remark that \( \pi_1(\text{Hol}_k^*) = \mathbb{Z}/2k \) by Theorem 1.1. Hence, if we consider the homotopy exact sequence of the fibration \( (\ast)_d \), the assertion easily follows. \( \square \)

**Proposition 2.1.**  For each integer \( d \geq 1 \), there is a fibration sequence (up to homotopy),

\[
\begin{array}{c}
\text{Hol}_1 \xrightarrow{s_1} \text{Hol}_d \xrightarrow{q_d} \tilde{X}_d.
\end{array}
\]
Proof. We remark that $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$ and $\pi_1(X_d) = \mathbb{Z}/d$. Let $\tilde{\iota}_d : \text{Hol}_d \to K(\mathbb{Z}/2d, 1) = B(\mathbb{Z}/2d)$ and $\iota'_d : X_d \to K(\mathbb{Z}/d, 1) = B(\mathbb{Z}/d)$ denote the maps which represent the generators of $[\text{Hol}_d, K(\mathbb{Z}/2d, 1)] \cong H^1(\text{Hol}_d, \mathbb{Z}/2d) = \mathbb{Z}/2d$ and $[X_d, K(\mathbb{Z}/d, 1)] \cong H^1(X_d, \mathbb{Z}/d) = \mathbb{Z}/d$, respectively.

Consider the universal coverings $\pi_d : \tilde{\text{Hol}}_d \to \text{Hol}_d$ and $\pi'_d : \tilde{X}_d \to X_d$.

Then it follows from Lemma 2.2 that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
\tilde{\text{Hol}}_d & \xrightarrow{\tilde{q}_d} & \tilde{X}_d \\
\downarrow{\pi_d} & & \downarrow{\pi'_d} \\
\text{Hol}_d & \xrightarrow{q_d} & X_d \\
\downarrow{i_1} & & \downarrow{i'_d} \\
B(\mathbb{Z}/2) & \xrightarrow{B_i} & B(\mathbb{Z}/2d) & \xrightarrow{B_p} & B(\mathbb{Z}/d)
\end{array}
$$

where all vertical and horizontal sequences are fibration sequences.

Because $\text{Hol}_1 \simeq SO_3$ and $\tilde{i}_1$ induces an isomorphism on $\pi_i$, the homotopy fibre of $\tilde{i}_1$ is $S^3$ (up to homotopy). Hence it follows from [4, (2.1)] that we obtain the homotopy commutative diagram

$$
(2.1.2)
\begin{array}{ccc}
S^3 & \xrightarrow{s_1} & \tilde{\text{Hol}}_d & \xrightarrow{\tilde{q}_d} & \tilde{X}_d \\
\downarrow{q} & & \downarrow{\pi_d} & & \downarrow{\pi'_d} \\
\text{Hol}_1 & \xrightarrow{s_1} & \text{Hol}_d & \xrightarrow{q_d} & X_d \\
\downarrow{i_1} & & \downarrow{i'_d} & & \downarrow{i'_d} \\
B(\mathbb{Z}/2) & \xrightarrow{B_i} & B(\mathbb{Z}/2d) & \xrightarrow{B_p} & B(\mathbb{Z}/d)
\end{array}
$$

where all vertical and horizontal sequences are fibration sequences and $q : S^3 \to SO_3 \simeq \text{Hol}_1$ denotes the universal covering projection. So we obtain the desired fibration sequence (2.1.1).

Definition 2.1. Let $ev_d : \text{Hol}_d \to S^2$ denote the evaluation map defined by $ev_d(f) = f(\infty)$ for $f \in \text{Hol}_d$, where we identify $S^2 = \mathbb{C} \cup \infty$.

There is a fibration sequence $\text{Hol}_d^* \xrightarrow{3d} \text{Hol}_d \xrightarrow{ev_d} S^2$.

Lemma 2.3. The following two diagrams are homotopy commutative:

(i) $\begin{array}{ccc}
\text{Hol}_1 & \xrightarrow{s_1} & \text{Hol}_d \\
\downarrow{ev_1} & & \downarrow{ev_d} \\
S^2 & \xrightarrow{s} & S^2
\end{array}$

(ii) $\begin{array}{ccc}
\text{Hol}_d & \xrightarrow{ev_d} & S^2 \\
\downarrow{i_1} & & \downarrow{s} \\
B(\mathbb{Z}/2d) & \xrightarrow{Bp'} & BS^1 = K(\mathbb{Z}, 2)
\end{array}$
where $\rho' : \mathbb{Z}/2d \to S^1$ and $i : S^2 \to BS^1 = K(\mathbb{Z}, 2)$ denote the natural inclusion homomorphism and the map which represents the generator of $\pi_2(K(\mathbb{Z}, 2)) = \mathbb{Z}$, respectively.

Proof. (i) Without loss of generalities, we may suppose that the map $s_1 : \text{Hol}_1 \to \text{Hol}_d$ is given by $s_1((az + b)/(cz + d)) = (az^d + b)/(cz^d + d)$. Then it is easy to see that the diagram (i) is commutative.

(ii) It suffices to show that two induced homomorphisms

$$
\begin{align*}
\alpha_1 : \mathbb{Z} = H^2(K(\mathbb{Z}, 2), \mathbb{Z}) &\cong H^2(S^2, \mathbb{Z}) \\ \alpha_2 : \mathbb{Z} = H^2(K(\mathbb{Z}, 2), \mathbb{Z}) &\cong H^2(\text{Hol}_d, \mathbb{Z})
\end{align*}
$$

coincide. Remark that $i^*$ is an isomorphism and that $(B\rho')^*$ can be identified with the natural projection homomorphism $\pi' : \mathbb{Z} \to \mathbb{Z}/2d$. Next, consider the Serre spectral sequence of the evaluation fibration: $\text{Hol}_d^* \xrightarrow{j_d} \text{Hol}_d \xrightarrow{ev_d} S^2$,

$$E_2^{p,q} = H^p(S^2, \mathbb{Z}) \otimes H^q(\text{Hol}_d^*, \mathbb{Z}) \Rightarrow H^{p+q}(\text{Hol}_d, \mathbb{Z}).$$

Since $H^1(\text{Hol}_1, \mathbb{Z}) = \mathbb{Z}$, $H^2(\text{Hol}_d^*, \mathbb{Z})$ is a torsion group and $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$, the differential $d_2 : \mathbb{Z} = E_2^{0,1} \to E_2^{2,0} = \mathbb{Z}$ is identified with the 2$d$-times multiplication. Hence $E_2^{0,0} = E_2^{2,0} = H^2(\text{Hol}_d, \mathbb{Z}) = \mathbb{Z}/2d$ and we obtain that $ev_1^* = \pi' : \mathbb{Z} = H^2(S^2, \mathbb{Z}) \to H^2(\text{Hol}_d, \mathbb{Z}) = \mathbb{Z}/2d$. Therefore, $\alpha_1 = \pi' : \mathbb{Z} \to \mathbb{Z}/2d$. Similarly, if we compute the Serre spectral sequence of the fibration sequence $\text{Hol}_d \xrightarrow{\pi_d} \text{Hol}_d \xrightarrow{i_d} B(\mathbb{Z}/2d)$, we can easily see that $i_d^*$ is an isomorphism. Hence $\alpha_2$ can be also identified with $\pi'$. So $\alpha_1 = \alpha_2$ and the diagram (ii) is homotopy commutative.

**Theorem 2.1 (Theorem 1.4).** For each integer $d \geq 1$, there is a homotopy equivalence $\phi_d : \text{Hol}_d \xrightarrow{\cong} S^3 \times \hat{X}_d$.

Proof. We remark that there is a fibration sequence $S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{i} BS^1$, where $\eta_2 \in \pi_3(S^2) = \mathbb{Z}$; $\eta_2$ denotes the Hopf map. Then it follows from Lemma 2.3 that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
\text{Hol}_d \xrightarrow{\pi_d} \text{Hol}_d & \xrightarrow{i_d} & B(\mathbb{Z}/2d) \\
\downarrow{ev_d} & & \downarrow{B\rho'} \\
S^3 & \xrightarrow{\eta_2} & S^2 \xrightarrow{i} BS^1
\end{array}
$$

where two horizontal sequences are fibration sequences.
Since \( \iota \circ ev_d \circ \pi_d \) is null-homotopic, there is a map \( \theta_d : \overline{\text{Hol}}_d \to S^3 \) such that \( \eta_2 \circ \theta_d = ev_d \circ \pi_d \) (up to homotopy).

Then by using the diagram \((2.1.2)\) and Lemma 2.3, we have

\[
\eta_2 \circ \theta_d \circ \tilde{s}_1 = ev_d \circ \pi_d \circ \tilde{s}_1 = ev_d \circ s_1 \circ q = ev_1 \circ q.
\]

On the other hand, because \( \text{Hol}_1^* \simeq S^1 \) and \( q : S^3 \to SO_3 \simeq \text{Hol}_1 \) is a universal covering projection, \((ev_1)_* : \mathbb{Z} \cdot q = \pi_3(\text{Hol}_1) \xrightarrow{\cong} \pi_3(S^2) = \mathbb{Z} \cdot \eta_2 = \text{an isomorphism} \). Hence, \( ev_1 \circ q = \pm \eta_2 \) and we also obtain

\[
(2.1.3) \quad \eta_2 \circ \theta_d \circ \tilde{s}_1 = \pm \eta_2 = \pm \eta_2 \circ \iota_3,
\]

where we denote by \( \iota_n \in \pi_n(S^n) = \mathbb{Z} \cdot \iota_n \) the homotopy class of identity map of \( S^n \).

Now we recall the isomorphism \((\eta_2)_* : \pi_3(S^3) = \mathbb{Z} \cdot \iota_3 \xrightarrow{\cong} \pi_3(S^2) = \mathbb{Z} \cdot \eta_2 \).

Then it follows from \((2.1.3)\) that we have the equality

\[
(2.1.4) \quad \theta_d \circ \tilde{s}_1 = \pm \iota_3.
\]

Consider the fibration sequence \((2.1.1)\):

\[
\text{Hol}_1 \xrightarrow{s_i^*} \overline{\text{Hol}}_d \xrightarrow{\tilde{q}_d} \tilde{X}_d.
\]

Define the map \( \phi_d : \overline{\text{Hol}}_d \to S^3 \times \tilde{X}_d \) by \( \phi_d = (\theta_d, \tilde{q}_d) \). Then using \((2.1.4)\) and the homotopy exact sequence induced from \((2.1.1)\), we can easily see that \((\phi_d)_* : \pi_k(\overline{\text{Hol}}_d) \xrightarrow{\cong} \pi_k(S^3 \times \tilde{X}_d) \) is an isomorphism for any \( k \geq 0 \). Hence, \( \phi_d \) is a homotopy equivalence.

\[\Box\]

§3. The Universal Covering Space \( \tilde{X}_d \)

Since \( p_d : \text{Hol}_1^* \setminus \text{Hol}_d^* \xrightarrow{\cong} \text{Hol}_1 \setminus \text{Hol}_d = X_d \) is a homeomorphism, without loss of generalities we may assume \( X_d = \text{Hol}_1^* \setminus \text{Hol}_d^* \) and that there is a fibration sequence

\[
\text{Hol}_1^* \xrightarrow{s_i^*} \text{Hol}_d^* \xrightarrow{\tilde{q}_d^*} X_d.
\]

**Lemma 3.1.** For each integer \( d \geq 1 \), there is a commutative diagram

\[
\begin{array}{ccc}
\pi_1(\text{Hol}_1^*) & \xrightarrow{\cong} & \pi_1(\text{Hol}_d^*) & \xrightarrow{\cong} & \pi_1(X_d) \\
\approx & \downarrow & \approx & \downarrow & \approx \\
\mathbb{Z} & \xrightarrow{\mu_d} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/d
\end{array}
\]

where \( \pi : \mathbb{Z} \to \mathbb{Z}/d \) denotes the natural epimorphism and \( \mu_d : \mathbb{Z} \to \mathbb{Z} \) denotes the multiplication map given by \( \mu_d(x) = dx \) for \( x \in \mathbb{Z} \).
Proof. Since the proof is analogous to that of Lemma 2.2, we omit the detail.

Theorem 3.1 (Theorem 1.5). For each integer \( d \geq 1 \), there is a homotopy equivalence \( \tilde{q}_d : \tilde{\text{Hol}}_d^* \cong \tilde{X}_d \).

Proof. It follows from Lemma 3.1, [4], (2.1) and the analogous method given in the proof of Proposition 2.1 that we can easily obtain the homotopy commutative diagram

\[
\begin{array}{cccc}
F & \rightarrow & \tilde{\text{Hol}}_d^* & \rightarrow & \tilde{X}_d \\
\downarrow & \nearrow & \nearrow & \nearrow & \downarrow \\
\text{Hol}_1^* & \rightarrow & \text{Hol}_d^* & \rightarrow & X_d \\
\downarrow & \nearrow & \nearrow & \nearrow & \downarrow \\
\tilde{\iota}_1 & \rightarrow & \tilde{\iota}_d & \rightarrow & \iota_d \\
B\mathbb{Z} & \rightarrow & B\mathbb{Z} & \rightarrow & B(\mathbb{Z}/d)
\end{array}
\]

where all horizontal and vertical sequences are fibrations.

Since \( \tilde{\iota}_1 \) is a homotopy equivalence ([7]), the homotopy fibre \( F \) of \( \tilde{q}_d \) is contractible. Hence the map \( \tilde{q}_d : \tilde{\text{Hol}}_d^* \cong \tilde{X}_d \) is a homotopy equivalence.

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References


