Weyl Groups Associated with
Affine Reflection Systems of Type $A_1$
(Coxeter Type Defining Relations)

by

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Abstract

We offer a presentation for the Weyl group of an affine reflection system $R$ of type $A_1$ as well as a presentation for the so called hyperbolic Weyl group associated with an affine reflection system of type $A_1$. Applying these presentations to extended affine Weyl groups, and using a description of the center of the hyperbolic Weyl group, we also give a new finite presentation for an extended affine Weyl group of type $A_1$. Our presentation for the (hyperbolic) Weyl group of an affine reflection system of type $A_1$ is the first nontrivial presentation given in this generality, and can be considered as a model for other types.

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§0. Introduction

Weyl groups, as reflection groups, always give a geometric meaning to underlying structures such as root systems, Lie algebras and Lie groups. Thus to get a “good” perspective of these structures, one needs to have a better understanding of their Weyl groups. The present work is dedicated to the study of some new presentations for (hyperbolic) Weyl groups associated with affine reflection systems of type $A_1$.

Affine reflection systems are the most general known extensions of finite and affine root systems introduced by E. Neher and O. Loos in [LN2]. They include


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locally finite root systems, toroidal root systems, extended affine root systems, locally extended affine root systems, and root systems extended by an abelian group. In 1985, K. Saito [S] introduced the notion of an extended affine root system as a generalization of finite and affine root systems. For a systematic study of such root systems the reader is referred to [AP]. Another generalization of finite root systems are locally finite root systems; for a complete account see [LN1]. Root systems extended by an abelian group and locally extended affine root systems, introduced in [Y2] and [Y1], are two other generalizations which include extended affine root systems and locally finite root systems, respectively.

In [AYY], the authors introduce an equivalent definition for an affine reflection system (see Definition 1.1) which we will use here. In the finite and affine cases, the corresponding Weyl groups are essentially known. In particular, it is known that they are Coxeter groups and that their actions implement a specific geometric and combinatorial structure on their underlying root systems. In the extended affine case, however, it is known that if the nullity is greater than one, then the corresponding hyperbolic Weyl groups, called extended affine Weyl groups, are not Coxeter groups (see [H3, Theorem 3.6]). Here we record some advances made on presentations of (hyperbolic) Weyl groups of certain subclasses of affine reflection systems. As the starting point in this direction, we can name the works of [Kr], [A2] and [A1] which consider certain presentations for toroidal Weyl groups and extended affine Weyl groups. In [ST], the authors give a generalized Coxeter presentation for extended affine Weyl groups of nullity 2. In [AS4], [AS3], [AS2], [H2] and [H3], the authors offer some new presentations for extended affine Weyl groups; they achieve this by a group-theoretical analysis of the structure and in particular of the center of extended affine Weyl groups. For a comprehensive account of the structure of extended affine Weyl groups, the reader is referred to [MS], [A2], [AS1], [H1] and [H2].

In the study of groups associated with affine reflection systems and other extensions of finite and affine root systems, type $A_1$ has always played a special role and is usually considered as a model for other types. A philosophical justification for this is that any (tame) affine reflection system can be considered as the union of a family of affine reflection systems of type $A_1$.

In this work, we study two groups associated with an affine reflection system $R$. The first one, defined in [LN2], is called the Weyl group of $R$ which we denote by $W$, and the other, denoted by $\tilde{W}$, is defined when the ground abelian group is torsion-free. We call $\tilde{W}$ the hyperbolic Weyl group of $R$ (see Definition 1.5; compare with [H2, Definition 3.1]). One should note that the notion of a hy-
perbolic Weyl group is a generalization of the definition of an extended affine Weyl group.

Let $R$ be an affine reflection system of type $A_1$ in an abelian group $A$. In Section 1, we give preliminary definitions as well as record some results and facts on affine reflection systems. In Sections 2 and 3, we offer two presentations for $W$ and $\tilde{W}$, respectively (see Theorems 2.6 and 3.5). A quick look at these presentations shows that both $W$ and $\tilde{W}$ have soluble word problems. Also, using the description of the center of $\tilde{W}$ given in Proposition 3.6, one perceives that $\tilde{W}$ is a central extension of $W$. In Theorem 3.9, we prove under a very particular set of conditions that the existence of a presentation for $W$ is equivalent to the existence of a presentation for $\tilde{W}$ in the case when the given reflectable base is elliptic-like. In Section 4, we assume that $R$ is an extended affine root system of type $A_1$, i.e., $R$ is an affine reflection system in a free abelian group $A$ of rank $\nu + 1$. Then we offer two finite presentations, one for $W$ in Theorem 4.3 and the other for the so called baby extended affine Weyl group $\tilde{W}$ of type $A_1$ in Proposition 4.5. The latter presentation has $\nu + 1$ generators and $\nu(\nu + 1)/2 + \nu + 1$ relations, where the relations consist of $\nu(\nu + 1)/2$ central elements and $\nu + 1$ involutions. The paper concludes with an appendix, Section 5, in which we provide a geometric approach to the proof of Theorem 4.3.

Some of our results are suggested by running a computer program designed specifically for calculating certain relations among elements of an extended affine Weyl group of type $A_1$. This program consists of several algorithms, written in Visual Basic .Net. The interested reader can find the program and its source code at http://sourceforge.net/projects/central-exp/files/.

§1. Affine reflection systems and their Weyl groups

In this section, we recall the definition and some properties of affine reflection systems, introduced by E. Neher and O. Loos in [LN2]. Here we follow an equivalent definition given in [AYY]. Let $A$ be an abelian group. By a symmetric form on $A$, we mean a symmetric bi-homomorphism $(\cdot, \cdot) : A \times A \to \mathbb{Q}$. The radical of the form is the subgroup $A^0 = \{\alpha \in A \mid (\alpha, A) = 0\}$ of $A$. Also, let $A^\times = A \setminus A^0$, $\bar{A} = A/A^0$ and $\bar{\cdot} : A \to \bar{A}$ be the canonical map. The form $(\cdot, \cdot)$ is called positive definite (resp. positive semidefinite) if $(\alpha, \alpha) > 0$ (resp. $(\alpha, \alpha) \geq 0$) for all $\alpha \in A \setminus \{0\}$. If $(\cdot, \cdot)$ is positive semidefinite, then it is easy to see that

$$A^0 = \{\alpha \in A \mid (\alpha, \alpha) = 0\}.$$ 

Assume from now on that $(\cdot, \cdot)$ is positive semidefinite on $A$. For a subset $B$ of $A$,
let $B^\times = B \setminus A^0$ and $B^0 = B \cap A^0$. For $\alpha, \beta \in A$, if $(\alpha, \alpha) \neq 0$ set $(\beta, \alpha^\vee) := 2(\beta, \alpha)/(\alpha, \alpha)$, and if $(\alpha, \alpha) = 0$ set $(\beta, \alpha^\vee) := 0$. A subset $X$ of $A$ is called connected if it cannot be written as a disjoint union of two nonempty orthogonal subsets. The form $(\cdot, \cdot)$ induces a unique form on $\bar{A}$ by

$$(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta) \quad \text{for } \alpha, \beta \in A.$$ 

This form is positive definite on $\bar{A}$. Thus, $\bar{A}$ is a torsion-free group. For a subset $S$ of $A$, we denote by $\langle S \rangle$ the subgroup generated by $S$.

Here is the definition of an affine reflection system given in [AYY].

**Definition 1.1.** Let $A$ be an abelian group equipped with a nontrivial symmetric positive semidefinite form $(\cdot, \cdot)$. Let $R$ be a subset of $A$. The triple $(A, (\cdot, \cdot), R)$, or just $R$ if there is no confusion, is called an affine reflection system if it satisfies the following three axioms:

1. (R1) $R = -R$,
2. (R2) $\langle R \rangle = A$,
3. (R3) for $\alpha \in R^\times$ and $\beta \in R$, there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$(\beta + \mathbb{Z} \alpha) \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\} \quad \text{and} \quad d - u = (\beta, \alpha^\vee).$$

The affine reflection system $R$ is called irreducible if

1. (R4) $R^\times$ is connected.

Moreover, $R$ is called tame if

1. (R5) $R^0 \subseteq R^\times - R^\times$ (elements of $R^0$ are nonisolated).

Finally $R$ is called reduced if

1. (R6) $\alpha \in R^\times \Rightarrow 2\alpha \notin R^\times$.

Elements of $R^\times$ (resp. $R^0$) are called nonisotropic roots (resp. isotropic roots). An affine reflection system $(A, (\cdot, \cdot), R)$ is called a locally finite root system if $A^0 = \{0\}$.

Here, we recall some results about the structure of affine reflection systems from [AYY]. The image of $R$ under $\bar{\cdot}$ is denoted by $\bar{R}$. Since the form is nontrivial, $A \neq A^0$. It then follows from the axioms that $0 \notin R$.

**Proposition 1.2** ([AYY, Corollary 1.9]). If $(A, (\cdot, \cdot), R)$ is an affine reflection system, then $(\bar{R}, (\cdot, \cdot), \bar{A})$ is a locally finite root system. In particular, if $R$ is irreducible, the induced form on $\bar{V} := \mathbb{Q} \otimes_{\mathbb{Z}} \bar{A}$ is positive definite.
The type of $R$ is defined to be the type of $\hat{R}$. Since this work is devoted to the study of Weyl groups associated with affine reflection systems of type $A_1$, for the rest of this work we assume that $(A,(\cdot,\cdot), R)$ is a tame irreducible affine reflection system of type $A_1$. By [AYY, Theorem 1.13], $\hat{R}$ contains a finite root system $\hat{\mathcal{R}} = \{0, \pm \epsilon\}$ and a subset $S \subseteq R^0$ such that

\begin{equation}
R = (S + S) \cup (\hat{\mathcal{R}} + S),
\end{equation}

where $S$ is a pointed reflection subspace of $A^0$, in the sense that

\begin{equation}
0 \in S, \quad S + 2S \subseteq S, \quad \langle S \rangle = A^0.
\end{equation}

In fact by [AYY, Theorem 1.13], any tame irreducible affine reflection system of type $A_1$ arises this way.

**Remark 1.3.** By [AYY, Remark 1.16], the definition of a tame irreducible affine reflection system of type $A_1$ is equivalent to the definition of an irreducible root system extended by an abelian group, of type $A_1$, defined by Y. Yoshii [Y2]. If $A$ is a free abelian group of finite rank and $R$ is a tame irreducible affine reflection system, we may identify $R$ with the subset $1 \otimes R$ of $R \otimes_{\mathbb{Q}} A$. Then $R$ is isomorphic to an extended affine root system in the sense of [AP].

Let $\text{Aut}(A)$ be the group of automorphisms of $A$. For $\alpha \in A$, one defines $w_\alpha \in \text{Aut}(A)$ by

\[ w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha. \]

We call $w_\alpha$ the reflection based on $\alpha$, since it sends $\alpha$ to $-\alpha$ and fixes pointwise the subgroup $\{ \beta \in A \mid (\beta, \alpha) = 0 \}$. Note that if $\alpha \in A^0$, then according to our convention, $(\beta, \alpha^\vee) = 0$ for all $\beta$ and so $w_\alpha = \text{id}_A$. For a subset $S$ of $R$, the subgroup of $\text{Aut}(A)$ generated by $w_\alpha$, $\alpha \in S$, is denoted by $W_S$.

Now we can define the Weyl group of the affine reflection system $R$.

**Definition 1.4.** $W := W_R$ is called the Weyl group of $R$.

In a similar way, one defines $w_\alpha$, the reflection based on $\bar{\alpha}$, for $\bar{\alpha} \in \hat{A}$, and $\hat{W}$, the Weyl group of $\hat{R}$, which is a subgroup of $\text{Aut}(\hat{A})$ generated by $w_\alpha$, $\bar{\alpha} \in \hat{R}$.

Next we associate with $R$ another reflection group. To do so, until the end of this section we assume that $A$ is a torsion-free abelian group. To justify this assumption, we recall that the root system of any invariant affine reflection algebra over a field of characteristic zero is an affine reflection system whose $\mathbb{Z}$-span is a torsion-free abelian group. Indeed, affine reflection systems corresponding to invariant affine reflection algebras are contained in the dual space of the so called toral subalgebras. Thus the root system is a subset of a torsion-free group.
According to [AYY, Remark 1.6(ii)], one can transfer the form on $A$ to the vector space $V := \mathbb{Q} \otimes \mathbb{Z} A$. Then the radical of the form can be identified with $V^0 := \mathbb{Q} \otimes A^0$ and one may conclude that the form on $V$ is a positive semidefinite symmetric bilinear form. Then $A$ and $R$ can be identified with subsets $1 \otimes A$ and $1 \otimes R$ of $V$, respectively. Set $\hat{A} = \mathbb{Z} e$ and $\hat{V} := \mathbb{Q} \otimes \hat{A}$; then we have $V = \hat{V} \oplus V^0$.

Since $(S) = A^0$, it follows that $S \equiv 1 \otimes S$ spans $V^0$. So $S$ contains a basis $\mathfrak{B}^0 = \{\sigma_j | j \in J\}$ of $V^0$. Therefore, if $\mathfrak{B} = \{e\}$, then $\mathfrak{B} = \mathfrak{B} \cup \mathfrak{B}^0$ forms a basis for $V$. For $j \in J$, define $\lambda_j$ to be the element of the dual space $(V^0)^*$ of $V^0$ given by $\lambda_j(\sigma_i) = \delta_{ij}$. We call $(\mathfrak{B}^0)^* := \{\lambda_j | j \in J\}$ the dual basis of $\mathfrak{B}^0$. Let $(V^0)^\dagger$ be the subspace of $(V^0)^*$ spanned by the dual basis $(\mathfrak{B}^0)^*$, that is,

$$(V^0)^\dagger := \sum_{j \in J} \mathbb{Q}\lambda_j.$$

Note that $(V^0)^\dagger$, as a subspace of $(V^0)^*$, can be identified with the restricted dual space $\sum_{j \in J}(\mathbb{Q}\sigma_j)^*$ of $V^0$ with respect to the basis $\{\sigma_j | j \in J\}$. We now set

$$\tilde{V} := V \oplus (V^0)^\dagger = \hat{V} \oplus V^0 \oplus (V^0)^\dagger.$$

We then extend the form $(\cdot, \cdot)$ on $V$ to a nondegenerate form on $\tilde{V}$, denoted again by $(\cdot, \cdot)$, as follows:

- $(\tilde{V}, (V^0)^\dagger) := ((V^0)^\dagger, (V^0)^\dagger) := \{0\}$,
- $(\sigma, \lambda) := \lambda(\sigma)$ for $\sigma \in V^0$ and $\lambda \in (V^0)^\dagger$.

We call $\tilde{V}$ the hyperbolic extension of $V$ with respect to the basis $\{\sigma_j | j \in J\}$. For each $\alpha \in V$, we define $w_\alpha \in \text{Aut}(\tilde{V})$ by

$$w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha,$$

where $(\beta, \alpha^\vee) := 2(\beta, \alpha)/(\alpha, \alpha)$ if $(\alpha, \alpha) \neq 0$, and $(\beta, \alpha^\vee) := 0$ if $(\alpha, \alpha) = 0$. Clearly, $w_\alpha$ is a reflection with respect to the vector space $\tilde{V}$. For a subset $S$ of $V$, we define $\tilde{W}_S$ to be the subgroup of $\text{Aut}(\tilde{V})$ generated by $w_\alpha$, $\alpha \in S$.

**Definition 1.5.** We call $\tilde{W} := \tilde{W}_R$ the hyperbolic Weyl group of $R$.

We note that for $w \in \tilde{W}$, $\alpha \in V$ and $\beta, \gamma \in \tilde{V}$, we have

$$(1.3) \quad w^2_\alpha = 1, \quad (w\beta, w\gamma) = (\beta, \gamma), \quad \text{and} \quad w w_\alpha w^{-1} = w_{w(\alpha)}.$$

Since $A$ as a torsion-free abelian group is embedded in $V$, the restriction of elements of $\tilde{W}$ to $V$ induces an epimorphism $\varphi : \tilde{W} \to W$. 


§2. Alternating presentation

Let $A$ be an arbitrary abelian group and $(A, (\cdot, \cdot), R)$ be a fixed tame irreducible affine reflection system of type $A_1$. As we have seen in (1.1),

$$R = (S + S) \cup (\hat{R} + S),$$

where $S$ is a subset of $R^0$ satisfying (1.2), and $\hat{R} = \{0, \pm \epsilon\}$. We may assume that

(2.1) \((\epsilon, \epsilon) = 2\).

Note that $A = \hat{A} \oplus A^0, \langle \hat{R} \rangle = \hat{A}$ and $\langle R^0 \rangle = \langle S \rangle = A^0$. Let $p : A \to A^0$ be the projection onto $A^0$.

For each $\alpha \in A$, we have

$$\alpha = \text{sgn}(\alpha) \epsilon + p(\alpha),$$

where $\text{sgn} : A \to \mathbb{Z}$ is a group epimorphism. Clearly, each $\alpha \in A$ is uniquely determined by $\text{sgn}(\alpha)$ and $p(\alpha)$. Then for $\alpha \in R^\times$ and $\beta \in R$, we have

(2.2) \((\beta, \alpha^\vee) = (\beta, \alpha) = 2 \text{sgn}(\beta) \text{sgn}(\alpha).$$

**Lemma 2.1.** Let $w := \omega_{\alpha_t} \cdots \omega_{\alpha_1} \in W$. Then for $\beta \in R^\times$, we have

$$\text{sgn}(w\beta) = (-1)^t \text{sgn}(\beta) \quad \text{and} \quad p(w\beta) = p(\beta - 2(-1)^t \text{sgn}(\beta) \sum_{i=1}^{t} (-1)^i \text{sgn}(\alpha_i) \alpha_i).$$

**Proof.** First, let $t = 1$; then $w\beta = \omega_{\alpha}(\beta) = \beta - (\beta, \alpha)\alpha$, so by (2.2) we have

(2.3) \(\text{sgn}(w\beta) = \text{sgn}(\beta) - 2 \text{sgn}(\alpha) \text{sgn}(\beta) \text{sgn}(\alpha) = - \text{sgn}(\beta) = (-1)^1 \text{sgn}(\beta),$$

and

(2.4) \(p(w\beta) = p(\beta - (\beta, \alpha)\alpha) = p(\beta - 2 \text{sgn}(\alpha) \text{sgn}(\beta)\alpha).$$

So the result holds for $t = 1$. Now, assume that the statement holds for $t \leq k$. Now if $w' = \omega_{\alpha_k} \cdots \omega_{\alpha_1}$, then for $t = k + 1$, we have $w\beta = w'\beta - (w'\beta, \alpha_1)\alpha_1$, so

\[
\begin{align*}
\text{sgn}(w\beta) &= \text{sgn}(w'\beta) - 2 \text{sgn}(w'\beta) \text{sgn}(\alpha_1) \text{sgn}(\alpha_1) \\
&= - \text{sgn}(w'\beta) = (-1)^k \text{sgn}(\beta) = (-1)^{k+1} \text{sgn}(\beta).
\end{align*}
\]
Also
\[
p(w/\beta) = p(w'/\beta) - 2 \text{sgn}(w'/\beta) \text{sgn}(\alpha_1)p(\alpha_1)
\]
\[
= p(\beta - 2(-1)^k \text{sgn}(\beta) \sum_{i=2}^{k+1} (-1)^{i-1} \text{sgn}(\alpha_i)\alpha_i) - 2(-1)^k \text{sgn}(\beta) \text{sgn}(\alpha_1)p(\alpha_1)
\]
\[
= p\left(\beta - 2(-1)^{k+1} \text{sgn}(\beta) \sum_{i=1}^{k+1} (-1)^i \text{sgn}(\alpha_i)\alpha_i\right).
\]

**Proposition 2.2.** For \(\alpha_1, \ldots, \alpha_n \in \mathbb{R}^\times\), we have
\[w := w_{\alpha_1} \cdots w_{\alpha_n} = 1\]
in \(W\) if and only if \(n\) is even and
\[
\sum_{i=1}^{n} (-1)^i \text{sgn}(\alpha_i)p(\alpha_i) = 0.
\]
In particular, if \(n\) is odd, then \(w^2 = 1\).

**Proof.** To prove the first assertion, since \(\mathbb{R}^\times\) generates \(A\), it is enough to show that, for each \(\beta \in \mathbb{R}^\times\), \(w\beta = \beta\) if and only if \(n\) is even and \(\sum_{i=1}^{n} (-1)^i \text{sgn}(\alpha_i)p(\alpha_i) = 0\). However, since the maps \(p\) and \(\text{sgn}\) determine \(w\beta\) uniquely, the result immediately follows from Lemma 2.1.

Now assume that \(n\) is odd; then for \(w^2 = w_{\alpha_1} \cdots w_{\alpha_n} w_{\alpha_1} \cdots w_{\alpha_n}\), we have
\[
\sum_{i=1}^{n} (-1)^i \text{sgn}(\alpha_i)p(\alpha_i) + \sum_{i=n+1}^{2n} (-1)^i \text{sgn}(\alpha_{i-n})p(\alpha_{i-n})
\]
\[
= \sum_{i=1}^{n} [(-1)^i + (-1)^{n+i}] \text{sgn}(\alpha_i)p(\alpha_i) = 0,
\]
where the last equality holds since \(n\) is odd. So \(w^2 = 1\) by the first assertion. \(\square\)

Inspired by Proposition 2.2, we make the following definition.

**Definition 2.3.** Let \(P\) be a subset of \(\mathbb{R}^\times\). We call a \(k\)-tuple \((\alpha_1, \ldots, \alpha_k)\) of roots in \(P\) an alternating \(k\)-tuple in \(P\) if \(k\) is even and \(\sum_{j=1}^{k} (-1)^j \text{sgn}(\alpha_j)p(\alpha_j) = 0\). We denote by \(\text{Alt}(P)\) the set of all alternating \(k\)-tuples in \(P\) for all \(k\). By Proposition 2.2, if \((\alpha_1, \ldots, \alpha_k) \in \text{Alt}(P)\), then \(w_{\alpha_1} \cdots w_{\alpha_k} = 1\) in \(W\).

**Corollary 2.4.** For \(\alpha_1, \ldots, \alpha_t \in \mathbb{R}^\times\) we have, as elements of \(W\),
\[w_{\alpha_1} \cdots w_{\alpha_t} = w_{\alpha_1} \cdots w_{\alpha_{t-1}} w_{\alpha_t} w_{\alpha_{t+1}} w_{\alpha_{t+2}} \cdots w_{\alpha_t}\]
whenever \(1 \leq i \leq t - 2\).
Proof. Clearly equality holds if and only if \( w_{\alpha_i}w_{\alpha_{i+1}}w_{\alpha_{i+2}} = w_{\alpha_{i+2}}w_{\alpha_{i+1}}w_{\alpha_i} \), and this in turn holds if and only if \( (w_{\alpha_i}w_{\alpha_{i+1}}w_{\alpha_{i+2}})^2 = 1 \), which holds by Proposition 2.2.

For our next result, we recall a definition from [AYY, Definition 1.19(ii)]. First we recall that for a subset \( S \) of \( A \), \( W_S \) is by definition the subgroup of \( W \) generated by all reflections \( w_\alpha, \alpha \in S \).

**Definition 2.5.** Let \( P \subseteq R^\times \).

(i) The set \( P \) is called **reflectable** if \( W_P = R^\times \).

(ii) The set \( P \) is called a **reflectable base** if \( P \) is a reflectable set and no proper subset of \( P \) is reflectable.

Obviously, if \( P \) is a reflectable set, then \( W = W_P \). In [AYY], reflectable sets and reflectable bases are characterized for tame irreducible affine reflection systems of reduced types. As shown in [A2], one can find finite reflectable sets and bases for many interesting affine reflection systems, such as extended affine root systems.

Here is our main result of this section which provides a presentation for the Weyl group of an affine reflection system of type \( A_1 \).

**Theorem 2.6.** Let \( (A, (\cdot, \cdot), R) \) be an affine reflection system of type \( A_1 \) and let \( \Pi \subseteq R^\times \) be a reflectable set for \( R \). Then \( W \) is isomorphic to the group \( G \) defined by

- **generators:** \( x_\alpha, \alpha \in \Pi \),
- **relations:** \( x_{\alpha_1} \cdots x_{\alpha_k}, (\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi) \).

Proof. By definition of a reflectable set we have \( W = \{ w_\alpha \mid \alpha \in \Pi \} \). Let \( x_{\alpha_1} \cdots x_{\alpha_k} \) be a defining relation in \( G \). Then by definition, \( k \) is even, \( \sum_{j=1}^{k} (-1)^j \text{sgn}(\alpha_j)p(\alpha_j) = 0 \) and \( \alpha_i \in \Pi \) for \( 1 \leq i \leq k \). By Proposition 2.2, we have \( w_{\alpha_1} \cdots w_{\alpha_k} = 1 \).

So, the assignment \( x_\alpha \mapsto w_\alpha, \alpha \in \Pi \), induces an epimorphism \( \phi : G \to W \). Let \( x = x_{\alpha_1} \cdots x_{\alpha_k} \in \text{Ker} \phi, \alpha_i \in \Pi \). Then we have

\[
1 = \phi(x) = w_{\alpha_1} \cdots w_{\alpha_k}.
\]

By Proposition 2.2, \( k \) is even and \( p(\sum_{j=1}^{k} (-1)^j \text{sgn}(\alpha_j)\alpha_j) = 0 \). Therefore, we have \( (\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi) \) and so \( \phi \) is an isomorphism. \( \square \)

**Remark 2.7.** Let \( \Pi' \) be obtained from \( \Pi \) by changing the signs of all elements of a subset of \( \Pi \). Then it is clear that \( \Pi \) is reflectable if and only if \( \Pi' \) is reflectable. So without loss of generality we may assume that \( \Pi \subseteq \epsilon + S \). Now if we set \( \hat{\Pi} = p(\Pi) \), then the above presentation can be written in the form:

- **generators:** \( x_\tau, \tau \in \hat{\Pi} \),
- **relations:** \( x_{\tau_1} \cdots x_{\tau_k} \), \( k \) is even and \( \sum_{j=1}^{k} (-1)^j \tau_j = 0 \).
§3. Hyper-alternating presentation

Throughout this section, we assume that \( A \) is a torsion-free abelian group. Let \((A, (\cdot, \cdot), R)\) be an affine reflection system of type \( A_1 \). As in Section 1, let \( \mathcal{V} = \mathbb{Q} \otimes \mathbb{Z} A \) and consider the basis \( \mathfrak{B} = \{ \varepsilon \} \cup \mathfrak{B}^0 \) of \( \mathcal{V} \) where \( \mathfrak{B}^0 = \{ \sigma_j \mid j \in J \} \) is a basis of \( \mathcal{V}^0 \). Let \( (\mathfrak{B}^0)^* = \{ \lambda_j \mid j \in J \} \) be the dual basis of \( \mathfrak{B}^0 \) defined in Section 1 and consider the corresponding hyperbolic extension \( \tilde{\mathcal{V}} = \mathcal{V} \oplus (\mathcal{V}^0)^\dagger \) of \( \mathcal{V} \). As we have already seen, since \( A \) is torsion-free, we can identify \( R \) with \( 1 \otimes \mathbb{Z} A \) as a subset of \( \mathcal{V} \subseteq \tilde{\mathcal{V}} \). This is done in order to study the hyperbolic Weyl group \( \tilde{W} \) of \( R \) (see Definition 1.5). We fix a reflectable set \( \Pi = \{ \alpha_i \mid i \in I \} \subseteq R^\times \)

where \( I \) is an index set. Recall that \( p : A \to A^0 \) is the projection with respect to the decomposition \( A = \hat{A} \oplus A^0 \) of \( A \). Since \( A \) is identified with the subgroup \( 1 \otimes \mathbb{Z} A \) of \( \mathcal{V} \), one can consider \( p \) as the restriction of the projection \( \mathcal{V} \to \mathcal{V}^0 \) with respect to the decomposition \( \mathcal{V} = \hat{\mathcal{V}} \oplus \mathcal{V}^0 \). For \( \alpha \in R \), let \( p_j(\alpha) \in \mathbb{Q} \) be the \( j \)-th coordinate of \( p(\alpha) \) with respect to the basis \( \mathfrak{B}^0 \) of \( \mathcal{V}^0 \), that is,

\[
0 = \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p(\alpha) = \sum_{j \in J} \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i)\sigma_j,
\]

So for \( s \in J \), we have

\[
0 = \lambda_s \left( \sum_{j \in J} \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i)\sigma_j \right) = \sum_{j \in J} \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i)\lambda_s(\sigma_j) = \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_s(\alpha_i).
\]

Summarizing the above discussion, we have

(3.1) \((\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi) \iff \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_s(\alpha_i) = 0 \) for all \( s \in J \).

For our next result, we note that \( \lambda_j - w(\lambda_j) \in \mathcal{V} \) for all \( j \in J \) and \( w \in \tilde{W} \).

**Lemma 3.1.** Let \( w = w_{\alpha_1} \cdots w_{\alpha_k} \in \tilde{W} \), \( \alpha_i \in \Pi \). Then, for \( j \in J \), we have

\[
\text{sgn}(\lambda_j - w(\lambda_j)) = \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i)
\]
and
\[ p(\lambda_j - w(\lambda_j)) = \sum_{s=1}^{k} p_j(\alpha_s)p(\alpha_s) + 2 \sum_{s=2}^{k} (-1)^s p_j(\alpha_s) \text{sgn}(\alpha_s) \sum_{r=1}^{s-1} (-1)^r \text{sgn}(\alpha_r)p(\alpha_r). \]

Proof. We have \( w_{\alpha_k}(\lambda_j) = \lambda_j - (\lambda_j, \alpha_k)\alpha_k \) so

\[ p(w_{\alpha_k}(\lambda_j) - \lambda_j) = (\lambda_j, \alpha_k)p(\alpha_k) = \lambda_j(p(\alpha_k))p(\alpha_k) = p_j(\alpha_k)p(\alpha_k). \]

Now if \( w' = w_{\alpha_1} \cdots w_{\alpha_{k-1}} \), then

\[ w(\lambda_j) = w'(\lambda_j - p_j(\alpha_k)\alpha_k) = w'(\lambda_j) - p_j(\alpha_k)w'(\alpha_k). \]

Then by the induction hypothesis and Lemma 2.1, we have

\[ p(\lambda_j - w(\lambda_j)) = p(\lambda_j - w'(\lambda_j)) + p_j(\alpha_k)p(w'(\alpha_k)) \]

\[ = \sum_{s=1}^{k-1} p_j(\alpha_s)p(\alpha_s) + 2 \sum_{s=2}^{k-1} (-1)^s \text{sgn}(\alpha_s) \sum_{r=1}^{s-1} (-1)^r \text{sgn}(\alpha_r)p(\alpha_r) \]

\[ + p_j(\alpha_k) \left( p(\alpha_k) - 2(-1)^{k-1} \text{sgn}(\alpha_k) \sum_{r=1}^{k-1} (-1)^r \text{sgn}(\alpha_r)p(\alpha_r) \right) \]

\[ = \sum_{s=1}^{k} p_j(\alpha_s)p(\alpha_s) + 2 \sum_{s=2}^{k} (-1)^s \text{sgn}(\alpha_s) \sum_{r=1}^{s-1} (-1)^r \text{sgn}(\alpha_r)p(\alpha_r). \]

Also

\[ \text{sgn}(\lambda_j - w(\lambda_j)) = \text{sgn}(w'(\lambda_j)) - p_j(\alpha_k) \text{sgn}(w'(\alpha_k)) \]

\[ = \sum_{i=1}^{k-1} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i) - (-1)^{k-1}p_j(\alpha_k) \text{sgn}(\alpha_k) \]

\[ = \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i). \]

The following is now clear from Lemma 3.1.

**Proposition 3.2.** Let \( w = w_{\alpha_1} \cdots w_{\alpha_k} \in \tilde{W}, \alpha_i \in \mathbb{R}^a \). Then \( w = 1 \) in \( \tilde{W} \) if and only if \( k \) is even and for all \( j \in J \),

\[ \sum_{i=1}^{k} (-1)^i \text{sgn}(\alpha_i)p_j(\alpha_i) = 0 \]

and

\[ \sum_{s=1}^{k} p_j(\alpha_s)p(\alpha_s) + 2 \sum_{s=2}^{k} (-1)^s p_j(\alpha_s) \text{sgn}(\alpha_s) \sum_{r=1}^{s-1} (-1)^r \text{sgn}(\alpha_r)p(\alpha_r) = 0. \]
Proof. Let \( w \) be as in the statement. From the definition of \( \tilde{V} \), it is clear that \( w = 1 \) in \( \tilde{W} \) if and only if \( w = 1 \) in \( W \) and \( w(\lambda_j) = \lambda_j \) for all \( j \in J \). Now from Proposition 2.2, we have \( w = 1 \) in \( W \) if and only if \( k \) is even and (3.2) is satisfied. Also by Lemma 3.1, \( w(\lambda_j) = \lambda_j \) if and only if (3.2) and (3.3) hold.

Remark 3.3. Since \( w_\alpha = w_{-\alpha} \), we may assume in Proposition 3.2 that \( \text{sgn}(\alpha_i) = 1 \) for all \( i \). Then the statement can be written in the form: \( w = 1 \) in \( \tilde{W} \) if and only if \( k \) is even and for all \( j \in J \),

\[
\sum_{i=1}^{k} (-1)^{i} p_j(\alpha_i) = 0
\]

and

\[
\sum_{s=1}^{k} p_j(\alpha_s)p(\alpha_s) + \sum_{s=2}^{k} (-1)^{s} p_j(\alpha_s) \sum_{r=1}^{s-1} (-1)^{r} p(\alpha_r) = 0.
\]

Inspired by Proposition 3.1, we make the following definition.

Definition 3.4. Let \( P \) be a subset of \( R^\times \). We call \((\alpha_1, \ldots, \alpha_k)\), \( \alpha_i \in P \), a hyper-alternating \( k \)-tuple in \( P \) if \( k \) is even and (3.2) and (3.3) hold for all \( j \in J \). We denote the set of all hyper-alternating \( k \)-tuples in \( P \) by \( \tilde{\text{Alt}}(P) \).

Using an argument similar to the proof of Theorem 2.6, we summarize the results of this section in the following theorem.

Theorem 3.5. Let \( \Pi \subseteq R^\times \) be a reflectable set for \( R \). Then \( \tilde{W} \) is isomorphic to the group \( G \) defined by

- generators: \( y_\alpha, \alpha \in \Pi \),
- relations: \( y_{\alpha_1} \cdots y_{\alpha_k}, (\alpha_1, \ldots, \alpha_k) \in \tilde{\text{Alt}}(\Pi) \).

We note that if \( w = w_{\alpha_1} \cdots w_{\alpha_k} \) with \( \alpha_i \)'s in \( R^\times \), then depending on the context we may consider \( w \) as an element of either \( W \) or \( \tilde{W} \).

Proposition 3.6. Let \( R^0 \neq \{0\} \) and \( w := w_{\alpha_1} \cdots w_{\alpha_k} \) with \( \alpha_i \)'s in \( R^\times \). The following statements are equivalent:

(i) \((\alpha_1, \ldots, \alpha_k) \in \text{Alt}(R^\times)\),
(ii) \( w = 1 \) in \( W \) (or equivalently \( w|_V = \text{id}_V \)),
(iii) if \( w = w_{\beta_1} \cdots w_{\beta_m} \) with \( \beta_i \)'s in \( R^\times \), then \((\beta_1, \ldots, \beta_m) \in \text{Alt}(R^\times)\),
(iv) \( w \in Z(\tilde{W}) \).
Proof. The implications (i)⇔(ii)⇔(iii) follow at once from Proposition 2.2 or Theorem 2.6.

(ii)⇒(iv). Assume \( w = 1 \) in \( W \). Therefore \( w(\alpha) = \alpha \) for all \( \alpha \in V \) and so for any \( \alpha \in R^x \) we have \( ww_\alpha w^{-1} = w_{w(\alpha)} = w_\alpha \). Thus \( w \in Z(\tilde{W}) \).

(iv)⇒(ii). Let \( z \in Z(\tilde{W}) \). For each \( \alpha \in R^x \), we have \( zw_\alpha z^{-1} = w_\alpha \). By (1.3),
\[ w_{z(\alpha)} = w_\alpha. \]
The definition of a reflection and the fact that \( R \) is a reduced root system yield
\[ z(\alpha) = \pm \alpha. \]
It is enough to show that \( z \) acts as the identity on \( \epsilon + S \), which is a spanning set of \( V \). We use the fact that each \( w \in \tilde{W} \) acts as the identity map on \( V^0 \). Since \( R^0 \neq \{0\} \), there exists \( 0 \neq \sigma \in S \). Then if \( z(\epsilon + \sigma) = -\epsilon - \sigma \), we have
\[ z(\epsilon) + \sigma = -\epsilon - \sigma, \]
which is a contradiction in both cases \( z(\epsilon) = \pm \epsilon \). Thus
\[ z(\epsilon + \sigma) = \epsilon + \sigma. \]
Then
\[ z(\epsilon) = z(\epsilon + \sigma - \sigma) = \epsilon + \sigma - \sigma = \epsilon. \]

Corollary 3.7. The group \( \tilde{W} \) is a central extension of \( W \) by \( Z(\tilde{W}) \), that is,
\[ 1 \to Z(\tilde{W}) \xrightarrow{i} \tilde{W} \xrightarrow{\varphi} W \to 1 \]
is a short exact sequence, where \( \varphi(w) = w|_V \) for \( w \in \tilde{W} \).

Proof. As mentioned at the end of Section 1, the map \( \varphi \) is an epimorphism. Now, by Proposition 3.6, we have \( \text{Ker} \varphi = Z(\tilde{W}) \).

Convention 3.8. Suppose that \( H \) is a group and \( \{h_\alpha \mid \alpha \in P\} \) is a fixed subset of \( H \). For a \( k \)-tuple \( f = (\alpha_1, \ldots, \alpha_k) \) in \( P \), \( k \) a positive integer, we set
\[ f_H := h_{\alpha_1} \cdots h_{\alpha_k} \in H. \]
Now let $\Pi$ be a reflectable set for $R$, and \( \{z_l\}_{l \in L} \) be a fixed set of generators for $Z(\tilde{W})$. Then, using Proposition 3.6 together with the fact that $\{w_\alpha \mid \alpha \in \Pi\}$ generates $\tilde{W}$, each $z_\ell$, $\ell \in L$, can be written in the form

$$z_\ell = f_{\tilde{W}}$$

for some $f = (\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi)$ (see Convention 3.8).

As usual, $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$ of elements $x, y$ of a group.

**Theorem 3.9.** Let $\Pi$ be a reflectable set for $R$ and assume that $Z(\tilde{W})$ is a free abelian group. Fix a free basis $\{z_l\}_{l \in L}$ for $Z(\tilde{W})$, and for each $\ell \in L$ let $f_\ell = (\alpha_1, \ldots, \alpha_k)$ be a fixed element of Alt(\Pi) such that $z_\ell = f_{\tilde{W}}$. Then the following statements are equivalent:

(i) The assignment $w_\alpha \mapsto x_\alpha$, $\alpha \in \Pi$, induces an isomorphism $\psi$ from the Weyl group $W$ of $R$ onto the group $G$ defined by

- generators: $x_\alpha$, $\alpha \in \Pi$,
- relations: $x_\alpha^2$, $f_\ell^G$, $\alpha \in \Pi$, $\ell \in L$.

(ii) The assignment $w_\alpha \mapsto \tilde{x}_\alpha$, $\alpha \in \Pi$, induces an isomorphism $\theta$ from the hyperbolic Weyl group $\tilde{W}$ of $R$ onto the group $\tilde{G}$ defined by

- generators: $\tilde{x}_\alpha$, $\alpha \in \Pi$,
- relations: $\tilde{x}_\alpha^2$, $[\tilde{x}_\alpha, f_\ell^G]$, $\alpha \in \Pi$, $\ell \in L$.

**Proof.** (i)$\Rightarrow$(ii). Consider the assignment

$$\theta : \{\tilde{x}_\alpha\}_{\alpha \in \Pi} \to \{w_\alpha\}_{\alpha \in \Pi}, \quad \tilde{x}_\alpha \mapsto w_\alpha.$$ 

Since any defining relation in $\tilde{G}$ corresponds to a relation in $\tilde{W}$, the map $\theta$ can be extended to an epimorphism from $\tilde{G}$ onto $\tilde{W}$. We proceed with the proof by showing that $\theta$ is injective.

Consider the subgroup $\tilde{Z} := \langle f_\ell^G \mid \ell \in L \rangle$ of $\tilde{G}$. We show that $\tilde{Z} \cong Z(\tilde{W})$ and $\tilde{G}/\tilde{Z} \cong W$. From the defining relations of $\tilde{G}$, it is clear that $\tilde{Z}$ is contained in $Z(\tilde{G})$. Since

$$\theta(f_\ell^G) = f_{\tilde{W}} = z_\ell$$

and $\{z_\ell \mid \ell \in L\}$ is basis of $Z(\tilde{W})$, it follows easily that

$$\theta_1 := \theta|_{\tilde{Z}} : \tilde{Z} \to Z(\tilde{W})$$

is an isomorphism.

Next, from Section 1, recall that $\varphi : \tilde{W} \to W$ is the epimorphism which is defined by $\varphi(w) = w|_\psi$, with $\text{Ker}\varphi = Z(\tilde{W})$, and consider the epimorphism $\varphi \circ \theta :$
$\tilde{G} \rightarrow W$. We have $\tilde{Z} \subseteq \text{Ker}(\varphi \circ \theta)$. So, there is an epimorphism $\theta_2 : \tilde{G}/\tilde{Z} \rightarrow W$ such that $\theta_2(\tilde{x}Z) = \varphi \circ \theta(\tilde{x})$ for $\tilde{x} \in \tilde{G}$. In particular, for $\alpha \in \Pi$, we have $\theta_2(\tilde{x}_\alpha Z) = w_\alpha$.

Let $\pi : \tilde{G} \rightarrow \tilde{G}/\tilde{Z}$ be the canonical map and $i : \tilde{Z} \rightarrow \tilde{G}$ be the inclusion. Then we have the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\tilde{W}) & \xrightarrow{i} & \tilde{W} & \xrightarrow{\varphi} & W & \longrightarrow & 1 \\
& & \downarrow{\theta_1} & & \downarrow{\theta} & & \downarrow{\theta_2} & & \\
1 & \longrightarrow & \tilde{Z} & \xrightarrow{i} & \tilde{G} & \xrightarrow{\pi} & \tilde{G}/\tilde{Z} & \longrightarrow & 1
\end{array}
$$

Since $\theta_1$ is injective, if we show that $\theta_2$ is injective, it will follow that $\theta$ is injective, and thus $\tilde{W} \cong \tilde{G}$.

To show that $\theta_2$ is injective we show that $\theta_2$ is invertible. For $\alpha \in \Pi$, we have $(\tilde{x}_\alpha \tilde{Z})^2 = \tilde{Z}$ and, for $l \in L$, we have $f^l \tilde{Z} = \tilde{Z}$. Thus any defining relation in $G$ corresponds to a relation in $\tilde{G}/\tilde{Z}$. So, there is an epimorphism $\kappa : \tilde{G} \rightarrow \tilde{G}/\tilde{Z}$, where $\kappa(x_\alpha) = \tilde{x}_\alpha \tilde{Z}$ for $\alpha \in \Pi$. Ultimately, we have an epimorphism $\kappa \circ \psi : W \rightarrow \tilde{G}/\tilde{Z}$ such that $\kappa \circ \psi(w_\alpha) = \tilde{x}_\alpha \tilde{Z}$. Clearly $\kappa \circ \psi$ is the inverse of $\theta_2$.

(ii) $\Rightarrow$ (i). From Theorem 2.6, we know that the assignment $w_\alpha \mapsto y_\alpha$, $\alpha \in \Pi$, induces an isomorphism from $W$ onto the group $G'$ defined by

- generators: $y_\alpha$, $\alpha \in \Pi$,
- relations: $y_{\alpha_1} \cdots y_{\alpha_k}, \ (\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi)$.

It is clear that every relation in $\tilde{G}$ is a relation in $G$. Thus we can extend the natural one-to-one correspondence $\rho : \{x_\alpha\}_{\alpha \in \Pi} \rightarrow \{x_\alpha\}_{\alpha \in \Pi}$ to an epimorphism $\rho : \tilde{G} \rightarrow G$, where $\rho(\tilde{x}_\alpha) = x_\alpha$. Thus $\rho \circ \theta$ is an epimorphism from $W$ onto $G$. We will show that $G \cong G'$.

Let $\phi : \{x_\alpha\}_{\alpha \in \Pi} \rightarrow \{y_\alpha\}_{\alpha \in \Pi}$ be the natural one-to-one correspondence between the set of generators of $G$ and the set of generators of $G'$. Since, for $\alpha \in \Pi$, we have $(\alpha, \alpha) \in \text{Alt}(\Pi)$ and $f^l \in \text{Alt}(\Pi)$ for $l \in L$, each relation in $G$ corresponds naturally to a defining relation in $G'$. Thus $\phi$ can be extended to a group epimorphism $\psi : G \rightarrow G'$, where $\psi(x_\alpha) = y_\alpha$ for $\alpha \in \Pi$.

Now, let $y_{\alpha_1} \cdots y_{\alpha_k}$ be a relation in $G'$. Thus $(\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi)$. By Proposition 3.6, we have $w := w_{\alpha_1} \cdots w_{\alpha_k} \in Z(\tilde{W})$. So there are $l_1, \ldots, l_t \in L$ and $n_1, \ldots, n_t \in \mathbb{Z}$ such that

$$
w = (z_{l_1})^{n_1} \cdots (z_{l_t})^{n_t} = (f^1_{W})^{n_1} \cdots (f^t_{W})^{n_t}.
$$

Thus

$$
x_{\alpha_1} \cdots x_{\alpha_k} = \rho \circ \theta(w) = \rho \circ \theta((f^1_{W})^{n_1} \cdots (f^t_{W})^{n_t}) = (f^1_{G})^{n_1} \cdots (f^t_{G})^{n_t} = 1.
$$
So, any defining relation in \( G' \) corresponds to a relation in \( G \). Thus \( \phi^{-1} \) can be extended to a group epimorphism \( \eta : G' \to G \). Since \( \psi \) is an extension of \( \phi \) and \( \eta \) is an extensions of \( \phi^{-1} \), they are inverse of each other. Thus \( G \cong G' \cong W \).

**Remark 3.10.** (i) The implication (i)\( \Rightarrow \) (ii) of Theorem 3.9 is in fact a consequence of a more general result concerning presented groups (see [J, Theorem 10.2]). However, to be applicable to our situation, this general fact would need quite a few adjustments. For this reason, we preferred to give a direct proof for this special case.

(ii) According to [A2, Lemma 3.18(i) and Corollary 3.29], if \( R \) is an extended affine root system in the sense of [AP, Definition II.2.1], then \( Z(\tilde{W}) \) is a free abelian group of finite rank. So Theorem 3.9 is applicable to extended affine Weyl groups of type \( A_1 \). In the next section we will show that when \( R \) is an extended affine root system, \( W \) is isomorphic to the group \( G \) defined in Theorem 3.9 for a special reflectable set \( \Pi \) and a particular set of \( f' \)'s.

§4. Application to extended affine Weyl groups

Let \((A, \langle \cdot, \cdot \rangle, R)\) be an affine reflection system. In this section, we assume that \( A^0 \) is a free abelian group of rank \( \nu \). Then \( 1 \otimes R \) as a subset of \( V := \mathbb{R} \otimes \mathbb{Q} A \) turns out to be an extended affine root system of type \( A_1 \) in the sense of [AP, Definition II.2.1]. So the hyperbolic Weyl group \( \tilde{W} \) is just an extended affine Weyl group in the sense of [A2, Definition 2.14]. Now, similar to what we have seen in Section 3, \( R \) can be identified with \( 1 \otimes R \) and we have \( V = \tilde{V} \oplus V^0 \), where \( \tilde{V} = \text{span}_\mathbb{R} \tilde{A} \) and \( V^0 = \text{span}_\mathbb{R} A^0 \). Set

\[
\Lambda := A^0.
\]

Then

\[
R = (S + S) \cup (\hat{R} + S),
\]

where, in this case, the pointed reflection space \( S \) is a semilattice in \( \Lambda \) in the sense of [AP, Definition II.1.2], that is, a subset of \( \Lambda \) satisfying

\[
0 \in S, \quad S \pm 2S \subseteq S, \quad \langle S \rangle = \Lambda.
\]

By [AP, Remark II.1.6], we have

\[
S = \bigcup_{i=0}^{m} (\tau_i + 2\Lambda),
\]

where \( \tau_i \)'s represent distinct cosets of \( 2\Lambda \) in \( \Lambda \) for \( 1 \leq i \leq m \), and \( \tau_0 = 0 \). By [AP, Proposition II.1.11], \( \Lambda \) has a \( \mathbb{Z} \)-basis consisting of elements of \( S \). So we may
assume that
\[ \{\sigma_1 := \tau_1, \ldots, \sigma_\nu := \tau_\nu\} \]
is a \( \mathbb{Z} \)-basis of \( \Lambda \). It follows from [AYY, Theorem 3.1] that
\begin{equation}
(4.1) \quad \Pi := \{\epsilon, \epsilon + \tau_1, \ldots, \epsilon + \tau_m\}
\end{equation}
is a reflectable base for \( R \). Considering these facts, there are two extreme cases for \( S \) which we would like to treat separately. The first case is when \( m = \nu \). We call the corresponding root system the baby extended affine root system and we denote it by \( R_b \). Another extreme case is when \( S \) is a lattice, namely \( S = \Lambda \). The corresponding root system is called the toroidal root system, which we denote it by \( R_t \). With our conventions, for any extended affine root system \( R \) of type \( A_1 \) in \( A \), we have \( R_b \subseteq R \subseteq R_t \). This justifies special treatment of \( R_b \) and \( R_t \).

The following proposition is essential for obtaining a new presentation for \( W \).

**Proposition 4.1.** We have
\begin{enumerate}[(i)]
\item \( w_{\alpha + \sigma + \delta} = w_{\alpha + \sigma} w_{\alpha + \delta} \) for \( \alpha \in R_t^\times \) and \( \sigma, \delta \in \Lambda \).
\item \( w_{\alpha + k\sigma} w_{\alpha} = (w_{\alpha + \sigma} w_{\alpha})^k \) for \( k \in \mathbb{Z} \), \( \alpha \in R_t^\times \) and \( \sigma \in \Lambda \).
\item \( W = W_b \) for any extended affine root system \( R \) in \( A \).
\end{enumerate}

**Proof.** (i)–(ii) The tuples
\[(\alpha + \sigma + \delta, \alpha + \sigma, \alpha + \sigma, \alpha + \delta),\]
\[(\alpha, \alpha + k\sigma, \alpha, \alpha + \sigma, \ldots, \alpha + \sigma, \alpha) \quad \text{for } k > 0,\]
\[(\alpha, \alpha + k\sigma, \alpha, \alpha + \sigma, \ldots, \alpha, \alpha + \sigma) \quad \text{for } k < 0,\]
are elements of \( \text{Alt}(R_t^\times) \). Also (ii) is nothing but \( w_\alpha^2 = 1 \) for \( k = 0 \). Thus by Proposition 2.2 and Definition 2.3, (i) and (ii) hold.

(iii) It is enough to show that \( w_\alpha \in W_b \) for each \( \alpha \in \pm\epsilon + \Lambda \). Since \( w_\alpha = w_{-\alpha} \), we assume that \( \text{sgn}(\alpha) = 1 \). Let \( \alpha = \epsilon + \sigma \) and
\[ \sigma = \sum_{i=1}^\nu k_i \sigma_i, \]
where \( k_i \in \mathbb{Z} \). Using (i), we have
\[ w_\alpha = w_{\epsilon + k_1 \sigma_1} w_{\epsilon} \cdots w_{\epsilon + k_{\nu-1} \sigma_{\nu-1}} w_{\epsilon} w_{\epsilon + k_\nu \sigma_\nu}. \]
Now for each \( i \), from (ii) we have
\[ w_{\epsilon + k_i \sigma_i} w_{\epsilon} = (w_{\epsilon + \sigma_i} w_{\epsilon})^{k_i}. \]
This way we obtain an expression of $w_\alpha$ with respect to the reflections based on elements of $\Pi_0 = \{\epsilon, \epsilon + \sigma_1, \ldots, \epsilon + \sigma_\nu\}$. Since $\Pi_0$ is a subset of $R_b$, we have $w_\alpha \in W_b$.

Here we offer a finite presentation for $W$ which is essential for the rest of this section. Let $\Pi_0 := \{\alpha_0 := \epsilon, \alpha_1 := \epsilon + \sigma_1, \ldots, \alpha_\nu := \epsilon + \sigma_\nu\}$. First we analyze the elements of $\text{Alt}(\Pi_0)$ which we need for our presentation.

**Remark 4.2.** Let $f = (\alpha_{j_1}, \ldots, \alpha_{j_k})$ be an element of $\text{Alt}(\Pi_0)$. The following can be easily checked from the definition of $\text{Alt}(\Pi_0)$.

(i) From Corollary 2.4, for any $1 \leq s \leq k$, we have

$$(\alpha_{j_1}, \ldots, \alpha_{j_{s-1}}, \alpha_{j_s}, \alpha_{j_{s+1}}, \alpha_{j_{s+2}}, \ldots, \alpha_{j_k}) \in \text{Alt}(\Pi_0).$$

(ii) Note that $\Pi_0$ is a linearly independent set. So, if $n_i$ is the number of $\alpha_i$ in $f$ for $0 \leq i \leq \nu$, that is,

$$n_i = |\{s \mid 1 \leq s \leq k, \ j_s = i\}|,$$

then $n_i$ is even. Also, for each $1 \leq r \leq k$ there is an odd integer $p$ such that $j_r = j_{r+p}$.

(iii) If $k = 2$ then we have $f = (\alpha_i, \alpha_i)$ for some $0 \leq i \leq \nu$. If $k = 4$ then either $f = (\alpha_i, \alpha_i, \alpha_j, \alpha_j)$ or $f = (\alpha_i, \alpha_j, \alpha_i, \alpha_i)$, for some $0 \leq i, j \leq \nu$. The only possible alternating 6-tuple $f = (\alpha_{j_1}, \ldots, \alpha_{j_6})$ which does not contain an alternating 4-tuple of the form $(\alpha_{j_s}, \ldots, \alpha_{j_{s+3}})$ has to be of the form $f = (\alpha_i, \alpha_j, \alpha_m, \alpha_i, \alpha_j, \alpha_m)$ for some $0 \leq i, j, m \leq \nu$.

**Theorem 4.3.** Let $R$ be an extended affine root system of type $A_1$ and nullity $\nu$. Then the Weyl group $W$ of $R$ is isomorphic to the group $G$ defined by

- **generators:** $x_i, 0 \leq i \leq \nu$,
- **relations:** $x_k^2$, $(x_0 x_i x_j)^2$, $0 \leq k \leq \nu$, $1 \leq i < j \leq \nu$.

**Proof.** By Proposition 4.1, we have $W = W_b$. Since $\Pi_0$ is a reflectable base for $R_b$, Theorem 2.6 implies that $W_b$ is isomorphic to the group $G'$ defined by

- **generators:** $y_\alpha, \alpha \in \Pi_0$,
- **relations:** $y_{\alpha_1} \cdots y_{\alpha_k}, (\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi_0)$.

Let $\phi : \{x_i\}_{i=0}^\nu \to \{y_{\alpha_i}\}_{i=0}^\nu$ be the natural one-to-one correspondence between the set of generators of $G$ and the set of generators of $G'$. Since, for $\alpha \in \Pi_0$, we have $(\alpha, \alpha) \in \text{Alt}(\Pi_0)$, and $(\alpha_0, \alpha_1, \alpha_j, \alpha_i, \alpha_1, \alpha_i) \in \text{Alt}(\Pi_0)$ for $0 \leq i < j \leq \nu$, each relation in $G$ corresponds naturally to a defining relation in $G'$. Thus $\phi$ can be
extended to a group epimorphism \( \psi : G \to G' \), where \( \psi(x_i) = y_{\alpha_i} \) for \( 0 \leq i \leq \nu \).

We will show that \( \psi \) is an isomorphism.

Let \( y_{\alpha_1} \cdots y_{\alpha_k} \) be a defining relation in \( G' \). Then \((\alpha_{j_1}, \ldots, \alpha_{j_k}) \in \text{Alt}(\Pi_0)\).

We now prove that \( x := x_{j_1} \cdots x_{j_k} \) is a relation in \( G \). Notice that \( k \) is even. We argue by induction on \( m \), where \( k = 2m \).

From Remark 4.2(iii), for \( k = 2, 4 \), we see that \( x \) is one of the expressions \( x_1^2, x_1^2 x_2 \) or \( x_1 x_2^2 x_1 \), which are clearly relations in \( G \).

Next we examine the case \( k = 6 \). It follows easily from the defining relations of \( G \) that

\[
(x_r x_s x_t)^2 = 1 \quad \text{and} \quad x_r x_s x_t = x_t x_r x_s, \quad \text{if at least one of } r, s \text{ or } t \text{ is 0.}
\]

If \((\alpha_{j_1}, \ldots, \alpha_{j_k})\) contains an alternating 4-tuple \( f' = (\alpha_{j_r}, \ldots, \alpha_{j_{r+2}}) \) and \( y \) is the element in \( G \) corresponding to \( f' \), then \( x \) has to have one of the forms \( x_r^2 y, x_r y x_s \) or \( y x_s^2 \), and so by the case \( k = 4 \), \( x \) is a relation in \( G \). Thus by Remark 4.2, we may assume that the alternating 6-tuple under consideration is \((\alpha_r, \alpha_s, \alpha_t, \alpha_r, \alpha_s, \alpha_t)\), where none of \( r, s \) and \( t \) is 0. By (4.2), we have

\[
x = (x_r x_s x_t)^2 = x_r x_s x_t x_r x_s x_t = x_r x_s x_t x_r x_t x_r x_s x_t = x_0 x_s x_t x_r x_s x_t = x_0 x_s x_t x_0 x_s x_t = 1.
\]

Thus \( x \) is a relation in \( G \) when \( k = 6 \), and

\[
(4.3) \quad x_r x_s x_t = x_t x_r x_s \quad \text{for all } 0 \leq r, s, t \leq \nu.
\]

Now, let \( m > 3 \) (or \( k > 6 \)) and assume every expression in \( G \) corresponding to an alternating \( 2n \)-tuple is a relation in \( G \), for all \( n < m \). First assume that for some \( 1 \leq r \leq k - 1, j_r = j_{r+1} \). Since \( x_{j_r}^2 = 1 \), we have

\[
x = x_{j_1} \cdots x_{j_k} = x_{j_1} \cdots x_{j_{r-1}} x_{j_{r+2}} \cdots x_{j_k}
\]

and \((\alpha_{j_1}, \ldots, \alpha_{j_{r-1}}, \alpha_{j_{r+2}}, \ldots, \alpha_{j_k})\) is an alternating \((k-2)\)-tuple. Thus \( x \) is a relation in \( G \), by the induction hypothesis. So, we may assume that \( j_r \neq j_{r+1} \) for all \( 1 \leq r \leq k - 1 \). From Remark 4.2(ii), the number of \( \alpha_{j_1} \) appearing in \((\alpha_{j_1}, \ldots, \alpha_{j_k})\) is even and there is an even integer \( 2 \leq s \leq k \) such that \( j_1 = j_s \). Since by (4.3), \( x_{j_1} x_{j_2} x_{j_3} = x_{j_1} x_{j_2} x_{j_3} \), we have

\[
x = x_{j_1} \cdots x_{j_k} = x_{j_s} x_{j_2} x_{j_1} x_{j_4} \cdots x_{j_k}.
\]

By repeating this process we can move \( x_{j_1} \) next to \( x_{j_2} \), namely

\[
x = x_{j_1} \cdots x_{j_k} = x_{j_2} x_{j_1} x_{j_3} x_{j_k} \cdots x_{j_{s-1}} x_{j_{s-2}} x_{j_1} x_{j_4} \cdots x_{j_k}.
\]
By Remark 4.2(i), \((\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}, \alpha_{j_4}, \ldots, \alpha_{j_{k-1}}, \alpha_{j_k})\) is an alternating \(k\)-tuple. Thus \((\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}, \ldots, \alpha_{j_{k-1}}, \alpha_{j_k})\) is an alternating \((k-2)\)-tuple and
\[
x = x_{j_1} \cdots x_{j_k} = x_{j_1}x_{j_2}x_{j_3}x_{j_4} \cdots x_{j_{k-1}}x_{j_k}.
\]
By the induction hypothesis, the right hand side is a relation in \(G\). Thus every relation in \(G'\) corresponds to a relation in \(G\). So, \(\phi^{-1}\) can be extended to a group epimorphism \(\eta : G' \rightarrow G\) satisfying \(\psi \circ \eta = 1\) and \(\eta \circ \psi = 1\). Thus \(G \cong G' \cong W\).

A slight modification of generators and relations in the statement of Theorem 4.3 provides a new presentation for \(W\) which is more useful for our purposes. We do this in the following proposition. Even though the proof is elementary and straightforward, we provide the details for the convenience of the reader.

**Proposition 4.4.** Let \(B\) be a subset of \(\{(i, j) \mid 0 \leq i < j \leq \nu\}\). Then \(W\) is isomorphic to the group \(G\) defined by

- **generators:** \(x_k, x_{(i,j)}, 0 \leq k \leq \nu, (i, j) \in B\),
- **relations:** \(x_k^2, x_{(i,j)}^2, 0 \leq k \leq \nu, (i, j) \in B; x_{(i,j)}x_{0}\), \((i, j) \in B; (x_0x)\)^2, \((i, j) \notin B\).

**Proof.** If \(B = \emptyset\) this is Theorem 4.3. Suppose that \(B \neq \emptyset\). By Theorem 4.3, \(W\) is isomorphic to the group \(G'\) defined by

- **generators:** \(y_i, 0 \leq i \leq \nu\),
- **relations:** \(y_k^2, (y_0y_{ij})^2, 0 \leq k \leq \nu, 1 \leq i < j \leq \nu\).

By the proofs of Theorems 2.6 and 4.3, this isomorphism is in fact induced by the assignment \(w_{i+1} \mapsto y_i\) for \(0 \leq i \leq \nu\). We show that \(G \cong G'\). For \((i, j) \in B\), we set \(y_{(i,j)} := y_iy_j \in G'\). Let \(Y = \{y_k\}_{k=0}^\nu \cup \{y_{(i,j)}\}_{(i,j) \in B}\) and define \(\phi : Y \rightarrow \{x_k\}_{k=0}^\nu \cup \{x_{(i,j)}\}_{(i,j) \in B}\) by \(\phi(y_k) = x_k\) for \(0 \leq k \leq \nu\), and \(\phi(y_{(i,j)}) = x_{(i,j)}\) for \((i, j) \in B\). It is obvious that \(Y\) is a set of generators for \(G'\). In \(G\), we have \(x_k^2 = 1\) for \(1 \leq k \leq \nu\). Since \((x_0x)^2 = 1\), \((i, j) \notin B\), we have \((x_0x)^2 = x_j^2 = 1\). Thus
\[
(x_0x)^2 = (x_jx_0)^{-2} = 1
\]
for \((i, j) \notin B\). Also, \((x_0x)^2 = x_{(i,j)}^2 = 1\) for \((i, j) \in B\). Thus
\[
(x_0x)^2 = x_j(x_0x)^2x_j = x_j^2 = 1
\]
for \((i, j) \in B\). Thus \(\phi\) can be extended to an epimorphism \(\psi : G' \rightarrow G\).

On the other hand, in \(G'\), we have \(y_{(i,j)}y_iy_jy_j = y_jy_0y_iy_0y_j = 1\) and \(y_{(i,j)}^2 = 1\) for \((i, j) \in B\),
\[
(y_iy_jy_j)^2 = y_j(y_iy_j)^2y_j = y_j(y_iy_j)^{-2}y_j = y_j^2 = 1
\]
where \( i \supp( W ) \) is a function. Let \( \alpha \in \mathbb{R}^\times \), then for all \( 0 \leq k \leq \nu \), so any defining relation in \( G \) corresponds to a relation in \( G' \). Thus \( \phi^{-1} \) can be extended to a group homomorphism \( \eta : G \rightarrow G' \). Since \( \psi \) is an extension of \( \phi \) and \( \eta \) is an extension of \( \phi^{-1} \), they are inverse to each other and \( G \cong G' \cong W \). This completes the proof. Note that under this isomorphism the generator \( x_k \) of \( G \) maps to the element \( w_{c+\sigma_k} \) of \( W \) for \( 0 \leq k \leq \nu \), and the generator \( x_{i,j} \) of \( G \) maps to the element \( w_{c+\alpha_{ij}} w_{i,j} w_{c+\sigma_i} \) of \( W \) for \( (i,j) \in B \).

Recall that each \( \alpha \in \mathbb{R}^\times \) can be uniquely written in the form \( \pm \epsilon + \sum_{i=1}^\nu s_i \sigma_i \) mod \( 2\Lambda \), where \( s_i \in \{0,1\} \) for all \( i \). Let \( \Pi = \{ \alpha_0, \ldots, \alpha_m \} \) be as in (4.1). Set \( \supp(\alpha) = \{ i \mid s_i \neq 0 \} \), and

\[
\supp(\Pi) = \{ \supp(\alpha) \mid \alpha \in \Pi \}.
\]

Since \( \Pi \) is a reflectable base, \( \tau_i \)'s represent distinct cosets of \( 2\Lambda \) in \( \Lambda \), so \( \tau_i = \tau_j \) if and only if \( \supp(\alpha_i) = \supp(\alpha_j) \). Here we consider all sets \( \supp(\alpha) \) as ordered sets, namely if \( \supp(\alpha) = \{ i_1, \ldots, i_t \} \), then \( i_1 < \cdots < i_t \). We call the reflectable base \( \Pi \) elliptic-like if \( |\supp(\alpha)| \in \{ 0, 1, 2 \} \) for all \( \alpha \in \Pi \), or equivalently \( |\supp(\alpha_k)| = 2 \) for \( \nu + 1 \leq k \leq m \). Since \( |\supp(\alpha)| \leq \nu \) for \( \alpha \in \mathbb{R}^\times \), all extended affine root systems of nullity \( \leq 2 \) are elliptic-like. Finally we set

\[
B_{\Pi} = \{ (i,j) \mid \{ i,j \} \in \supp(\Pi), 1 \leq i < j \leq \nu \}.
\]

For \( (i,j) \in B_{\Pi} \), we denote by \( \alpha_{ij} \) the unique element in \( \Pi \) with \( \supp(\alpha) = \{ i,j \} \).

**Proposition 4.5.** Let \( R \) be an extended affine root system in \( A \) and \( \Pi \) be the reflectable base for \( R \) as in (4.1). Assume that \( \Pi \) is elliptic-like. Then \( \tilde{W} \) is isomorphic to the group \( \tilde{G} \) defined by

- **generators:** \( \tilde{x}_k \), \( 0 \leq k \leq m \),
- **relations:** \( \tilde{x}_i \), \( \{ \tilde{x}_i, \tilde{x}_j \} \) if \( \{ i,j \} = \supp(\alpha) \); \( \tilde{x}_k, (\tilde{x}_i \tilde{x}_j \tilde{x}_k)^2 \) if \( \{ i,j \} \not\subset \supp(\Pi) \), \( 0 \leq k \leq m \), \( 1 \leq i < j \leq \nu \), \( \nu + 1 \leq s \leq m \).

**Proof.** We proceed with the proof in the following steps.

**Step 1.** We show that \( Z(\tilde{W}) \) is a free abelian group with basis \( \{ z_{ij} \mid 1 \leq i < j \leq \nu \} \), where \( z_{ij} \)'s are defined, as follows. If \( \{ i,j \} \in \supp(\Pi) \) set \( z_{ij} := w_{c+\sigma_i} w_{c+\sigma_j} \), where \( i,j \) is the unique integer satisfying \( \supp(\alpha_{ij}) = \{ i,j \} \). If \( \{ i,j \} \not\subset \supp(\Pi) \) set \( z_{ij} := (w_{c+\sigma_i} w_{c+\sigma_j})^2 \). From [A2, Lemma 3.18(i)] and Corollary 3.29), we know that the center \( Z(\tilde{W}) \) of \( \tilde{W} \) is a free abelian group of rank \( \nu(\nu - 1)/2 \). For \( 1 \leq i < j \leq \nu \) define \( c_{ij} \in \text{GL}(\tilde{V}) \) by

\[
c_{ij}(v) = v \quad \text{and} \quad c_{ij} \lambda_k = \lambda_k - \delta_{kj} \sigma_i + \delta_{ki} \sigma_j \quad (v \in \tilde{V}, 1 \leq k \leq \nu).
\]
By [AS4, Proposition 2.2(vi)],
\[
\{c_{ij} \mid (i,j) \in B_H\} \cup \{c_{ij}^2 \mid (i,j) \notin B_H\}
\]
is a free basis for the group $Z(W)$. We are done if we show that $z_{ij} = c_{ij}$ for $(i,j) \in B_H$ and $z_{ij} = c_{ij}^2$ for $(i,j) \notin B_H$. Now for $(i,j) \in B_H$, we have $\epsilon + \sigma_i + \sigma_j \in \mathbb{R}_x$, and so $z_{ij} \in W$. Moreover, by Lemma 3.1 (or a simple verification),
\[
z_{ij|\nu} = \text{id}_\nu \quad \text{and} \quad z_{ij} \lambda_k = \lambda_k - \delta_{k} \sigma_i + \delta_{k} \sigma_j = c_{ij} \lambda_k
\]
for $1 \leq k \leq \nu$. Thus $z_{ij} = c_{ij}$. For $(i,j) \notin B_H$, we have
\[
z_{ij|\nu} = \text{id}_\nu
\]
and
\[
z_{ij} \lambda_k = \lambda_k - 2 \delta_{k} \sigma_i + 2 \delta_{k} \sigma_j = c_{ij}^2 \lambda_k.
\]
Thus $z_{ij} = c_{ij}^2$.

**Step 2.** We show that the assignment $w_{\alpha_k} \mapsto x_{\alpha_k}$, $0 \leq k \leq m$, induces an isomorphism from the Weyl group $W$ onto the group $G$ defined by

- generators: $x_{\alpha_k}$, $0 \leq k \leq m$,
- relations: $x_{\alpha_k}^2$, $0 \leq k \leq m$, $f_{ij}^\nu$, $1 \leq i < j \leq \nu$, where $f_{ij}^\nu := (\alpha_i, \alpha_i, \alpha_0, \alpha_j)$ for $(i,j) \in B_H$, and $f_{ij}^\nu := (\alpha_i, \alpha_0, \alpha_j, \alpha_i, \alpha_0, \alpha_j)$ if $(i,j) \notin B_H$.

(Recall that for $(i,j) \in B_H$, $\alpha_{ij}$ is the unique element in $\Pi$ with $\text{supp}(\alpha_{ij}) = \{i,j\}$.) By Proposition 4.4, the assignment $w_{\alpha_k} \mapsto x_k$, $0 \leq k \leq m$, induces an isomorphism from $W$ onto the group $G'$ defined by

- generators: $x_k$, $x_{(i,j)}$, $0 \leq k \leq \nu$, $(i,j) \in B_H$,
- relations: $x_k^2$, $x_{(i,j)}^2$, $0 \leq k \leq \nu$, $(i,j) \in B_H$, $x_{(i,j)}x_{(i,j)}$, $(i,j) \in B_H$, $(x_{(i,j)}x_{(i,j)})^2$, $(i,j) \notin B_H$.

Using the correspondence $x_{\alpha_i} \leftrightarrow x_i$ for $0 \leq i \leq \nu$ and $x_{\alpha_k} \leftrightarrow x_{(i,j)}$ for $\nu + 1 \leq k \leq m$ with $\text{supp}(\alpha_k) = \{i,j\}$, the defining generators and relations of the groups $G$ and $G'$ coincide and so we identify them.

**Step 3.** By Step 1, Step 2 and Theorem 3.9, $\widetilde{W}$ is isomorphic to the group $\widetilde{G}$ defined by

- generators: $\tilde{x}_{\alpha_k}$, $0 \leq k \leq m$,
- relations: $\tilde{x}_{\alpha_k}^2$, $0 \leq k \leq m$, $[\tilde{x}_{\alpha_k}, f_{ij}^\nu]$, $1 \leq i < j \leq \nu$, where $f_{ij}^\nu = (\alpha_i, \alpha_i, \alpha_0, \alpha_j)$ if $\{i,j\} \in \text{supp}(\Pi)$, and $f_{ij}^\nu = (\alpha_i, \alpha_0, \alpha_j, \alpha_i, \alpha_0, \alpha_j)$ if $\{i,j\} \notin \text{supp}(\Pi)$. 
Now using the correspondence $x_{\alpha_i} \leftrightarrow x_i$ for $0 \leq i \leq \nu$ and $x_{\alpha_{ij}} \leftrightarrow x_s$ for $\nu + 1 \leq s \leq m$ with $\text{supp}(\alpha_s) = \{i, j\}$, we see that $\tilde{W}$ is isomorphic to the group defined by

- generators: $\tilde{x}_k$, $0 \leq k \leq m$,
- relations: $\tilde{x}_k^2$, $[\tilde{x}_k, \tilde{x}_s\tilde{x}_j]$ if $\{i, j\} = \text{supp}(\alpha_s)$; $[\tilde{x}_k, (\tilde{x}_s\tilde{x}_j)^2]$ if $\{i, j\} \not\in \text{supp}(\Pi)$, $0 \leq k \leq m$, $1 \leq i < j \leq \nu$, $\nu + 1 \leq s \leq m$.

**Remark 4.6.**

(i) We recall that if $\nu = 0$ ($\nu = 1$), then $R$ is a finite (affine) root system of type $A_1$. Therefore Proposition 4.4, together with Proposition 4.5, reproduces the following known presentations for $W$ and $\tilde{W}$:

$$W \cong (x_0 | x_0^2) \cong \tilde{W} \cong \mathbb{Z}_2 \quad (\nu = 0),$$

$$W \cong (x_0, x_1 | x_0^2, x_1^2) \cong \tilde{W} \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \quad (\nu = 1).$$

(ii) Proposition 4.5 provides a finite presentation for the baby and the toroidal extended affine Weyl groups of type $A_1$ of nullity 2. In this case, these are the only possible extended affine Weyl groups. Considering the nature of the relations one might consider this presentation as a generalized Coxeter presentation. We encourage the interested reader to compare our defining set of generators and relations with those given in [ST] for types $A_1^{(1,1)}$ and $A_1^{(1,1)*}$.

§5. Appendix: A geometric approach

In this section, we provide a geometric approach to the proof of Theorem 4.3. For $w \in W$ suppose that $w_{\alpha_1} \cdots w_{\alpha_k}$ is an expression of $w$ with respect to $R^\vee$. We define

$$\varepsilon(w) := (-1)^k \quad \text{and} \quad T(w) := \sum_{i=1}^{k} (-1)^{k-i} \text{sgn}(\alpha_i)p(\alpha_i).$$

From Lemma 2.1, it follows that the maps $\varepsilon : W \to \{-1, 1\}$ and $T : W \to \Lambda$ are well-defined, that is, their definitions are independent of the choice of expressions for an element of $W$. Furthermore, $\varepsilon$ is a group homomorphism and

$$T(w_1w_2) = \varepsilon(w_2)T(w_1) + T(w_2) \quad (w_1, w_2 \in W).$$

One can easily see that for $\alpha \in R$,

$$w(\alpha) = \varepsilon(w) \text{sgn}(\alpha)\varepsilon + p(\alpha) - 2\text{sgn}(\alpha)T(w),$$

and so $w$ is uniquely determined by $\varepsilon(w)$ and $T(w)$.
For $\sigma = \sum_{i=1}^{\nu} k_i \sigma_i \in \Lambda$ and $\eta \in \{ \pm 1 \}$, consider the $\nu$-simplex

$$B_{\sigma, \eta} := \left\{ \sum_{i=1}^{\nu} (2k_i + \eta t_i) \sigma_i \mid t_i \geq 0, 0 \leq t_1 + \cdots + t_\nu \leq 1 \right\}.$$ 

Let $\mathcal{B} = \{ B_{\sigma, \eta} \mid \sigma \in \Lambda, \eta \in \{ \pm 1 \} \}$. Now from (5.2) and the fact that $\varepsilon$ is a group homomorphism, it follows that $W$ acts on $\mathcal{B}$ by

$$(5.4) \quad w \cdot B_{\sigma, \eta} = B_{\sigma + \eta T(w), \varepsilon(w)\eta}.$$ 

If $w \cdot B_{\sigma, \eta} = B_{\sigma, \eta}$, then $T(w) = 0$ and $\varepsilon(w) = 1$, so $w = 1$. This shows that $W$ acts on $\mathcal{B}$ freely, in particular the action is faithful. Moreover, (5.4) shows that $\mathcal{B} = W \cdot B_{0, 1}$, so the action is transitive.

**Remark 5.1.** In (i)–(iii) below, we explain our motivation for the action of $W$ on $\mathcal{B}$ defined in (5.4) (a similar idea is given in [Ka, §6.6]).

(i) We show how the action of $W$ on $R$ can be transferred to the action (5.4) on $\mathcal{B}$. From (5.3), it is easy to see that the action of $W$ on $\pm \varepsilon + 2\Lambda$ is faithful. Using the map $\eta \varepsilon + 2\sigma \mapsto B_{\sigma, \eta}, \eta \in \{ \pm 1 \}, \sigma \in \Lambda$, we can identify the set $\pm \varepsilon + 2\Lambda$ with $\mathcal{B}$. We use this identification to transfer the action of $W$ to an action, denoted by $\bullet$, on $\mathcal{B}$. In fact, since for $w \in W$ we have

$$w(\eta \varepsilon + 2\sigma) = \varepsilon(w)\eta \varepsilon + 2(\sigma - \eta T(w)),$$

the corresponding action on $\mathcal{B}$ reads

$$w \bullet B_{\sigma, \eta} = B_{\sigma - \eta T(w), \varepsilon(w)\eta}.$$ 

The action of $W$ on $\pm \varepsilon + 2\Lambda$, and so on $\mathcal{B}$, is faithful and transitive. The two actions $\bullet$ and $\cdot$ on $\mathcal{B}$ are related as follows:

$$w_{\bullet} w_{\bullet} w_{\bullet} \bullet B_{\sigma, \eta} = w_{\bullet} w_{\bullet} \bullet B_{\sigma, -\eta} = w_{\bullet} \bullet B_{\sigma + \eta T(w), -\varepsilon(w)\eta} = B_{\sigma + \eta T(w), \varepsilon(w)\eta} = w \cdot B_{\sigma, \eta}.$$ 

(ii) Since we have identified $\varepsilon + 2\sigma$ and $-\varepsilon + 2\sigma$ with $B_{\sigma, 1}$ and $B_{\sigma, -1}$, respectively, one can interpret this identification as a polarization of elements of $2\Lambda$ through elements of $R^\times$, i.e., we consider $B_{\sigma, 1}$ and $B_{\sigma, -1}$ as positive and negative poles respectively for each element $2\sigma \in 2\Lambda$. In this way, the action of $W$ on $\mathcal{B}$ can be interpreted on $2\Lambda$ as a translation together with a polarization.

(iii) In contrast to the action $\bullet$, the action $\cdot$ on $\mathcal{B}$ given in (5.4) has this “nice” property that for any $\alpha \in \Pi_0$, $w_\alpha$ takes any simplex in $\mathcal{B}$ to another simplex in $\mathcal{B}$, topologically connected to it. This has been our main reason for choosing the action $\cdot$ instead of $\bullet$. This completes our remark.
For $\alpha_1, \ldots, \alpha_k \in \mathbb{R}^k$, we define the path corresponding to $w_{\alpha_1} \cdots w_{\alpha_k}$ in $\mathcal{B}$ starting at $B_{\sigma, \eta}$ to be the $(k + 1)$-tuple

$$(B_{\sigma, \eta}, w_{\alpha_k} \cdot B_{\sigma, \eta}, \ldots, w_{\alpha_1} \cdot w_{\alpha_k} \cdot B_{\sigma, \eta}).$$

We denote this path by $P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k})$, and we say that this is a path of length $k$. If $\sigma = 0$ and $\eta = 1$, we simply denote this path by $P(w_{\alpha_1} \cdots w_{\alpha_k})$. We call a path a loop based at $B_{\sigma, \eta}$ if its starting and ending points are $B_{\sigma, \eta}$. We call the single tuple $(B_{\sigma, \eta})$ the trivial loop based at $B_{\sigma, \eta}$ and denote it by $P_{B_{\sigma, \eta}}(1)$. Since $\mathcal{W}$ acts freely on $\mathcal{B}$, the path $P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k})$ is a loop if and only if $w_{\alpha_1} \cdots w_{\alpha_k} = 1$. Thus, by Proposition 2.2, the path $P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k})$ is a loop if and only if $(\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\mathbb{R}^k)$. Finally, note that $\mathcal{W}$ acts on the set of paths in $\mathcal{B}$ corresponding to $w_{\alpha_1} \cdots w_{\alpha_k} \in \mathcal{W}$ by

$$w \cdot P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k}) := P_{w \cdot B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k}) \quad (w \in \mathcal{W}).$$

Let $\mathcal{P}$ be the set of all paths corresponding to $w^2 \sigma_{i+1}$ for $0 \leq i \leq \nu$, and $(w^2 \sigma_{i+1} w^2 \sigma_{i+1})^2$ for $1 \leq i < j \leq \nu$. Any path in $\mathcal{P}$ is a loop of length either 2 or 6.

**Definition 5.2.** (i) Let $P = (B_{\sigma_1, \eta_1}, \ldots, B_{\sigma_n, \eta_n})$ be a path in $\mathcal{B}$. For $1 \leq i < j \leq n$, we call $(B_{\sigma_i, \eta_i}, \ldots, B_{\sigma_j, \eta_j})$ a subpath of $P$. The trivial loops $(B_{\sigma_1, \eta_1})$ for $1 \leq i \leq n$ are called trivial subloops.

(ii) Let $P_1 = (B_{\sigma_1, \eta_1}, \ldots, B_{\sigma_n, \eta_n})$ and $P_2 = (B_{\sigma_n, \eta_1}, \ldots, B_{\sigma_n, \eta_n})$ be two paths in $\mathcal{B}$, where the ending point of $P_1$ is the same as the starting point of $P_2$. We define

$$P_1 \cdot P_2 := (B_{\sigma_1, \eta_1}, \ldots, B_{\sigma_n, \eta_n}, \ldots, B_{\sigma_n, \eta_n}).$$

(iii) Let $P_1$ and $P_2$ be two paths in $\mathcal{B}$. A move of $P_1$ is obtained either by replacing a trivial subloop based at $B_{\sigma, \eta}$ with a loop based at $B_{\sigma, \eta}$ from $\mathcal{P}$, or by replacing a subloop based at $B_{\sigma, \eta}$ which is an element of $\mathcal{P}$ with the trivial loop $(B_{\sigma, \eta})$. We say that $P_1$ can be moved to $P_2$, and we write $P_1 \rightarrow P_2$, if $P_2$ is obtained from $P_1$ by a finite number of moves.

It is easy to see that if $P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k}) \rightarrow P_{B_{\sigma, \eta}}(w_{\beta_1} \cdots w_{\beta_n})$ then

$$w \cdot P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k}) \rightarrow w \cdot P_{B_{\sigma, \eta}}(w_{\beta_1} \cdots w_{\beta_n}),$$

and

$$P_{B_{\sigma, \eta}}(w_{\alpha_1} \cdots w_{\alpha_k}) \cdot P' \rightarrow P_{B_{\sigma, \eta}}(w_{\beta_1} \cdots w_{\beta_n}) \cdot P',$$

for any $w \in \mathcal{W}$ and any path $P'$ in $\mathcal{B}$ for which the product on the left of (5.6) is defined.

Recall that $\mathcal{W} = \mathcal{W}_0 = \langle w_\alpha \mid \alpha \in \Pi_0 = \{\epsilon, \epsilon + \sigma_1, \ldots, \epsilon + \sigma_\nu\} \rangle$. 


Theorem 5.3. Let $R$ be an extended affine root system of type $A_1$ and nullity $\nu$. Then the Weyl group $W$ of $R$ is isomorphic to the group $G$ defined by

- generators: $x_i$, $0 \leq i \leq \nu$,
- relations: $x_i^2$, $(x_0x_ix_j)^2$, $0 \leq k \leq \nu$, $1 \leq i < j \leq \nu$.

Proof. By Theorem 2.6, $W$ has the presentation with

- generators: $w_\alpha$, $\alpha \in \Pi_0$,
- relations: $w_{\alpha_1} \cdots w_{\alpha_k}$, $(\alpha_1, \ldots, \alpha_k) \in \text{Alt}(\Pi_0)$.

This means that every relation in $W$ corresponds to a loop in $B$. For $0 \leq i \leq \nu$, let us denote $w_{i+\sigma}$ by $x_i$. Then the theorem is proved if we show that, for $\sigma \in \Lambda$ and $\eta \in \{\pm 1\}$, any loop based at $B_{\sigma, \eta}$ corresponding to $x_{i_1} \cdots x_{i_k}$ can be moved to the trivial loop $(B_{0,0})$. Now, using (5.5) and the fact that the action of $W$ on $B$ is transitive, it is enough to show that any loop based at $B_{0,1}$ can be moved to the trivial loop $(B_{0,1})$. We show this by induction on the length $2m$ of a loop based at $(B_{0,1})$.

Unless otherwise mentioned, all loops are considered based at $B_{0,1}$. First, we show that the assertion holds for $m = 1, 2, 3$. Let $m = 1$. From Remark 4.2(iii), we know that any loop of length 2 corresponds to $x_i^2$ for some $0 \leq i \leq \nu$. Thus any loop of length 2 is an element of $P$ and so by definition can be moved to the trivial loop. Next, let $m = 2$. By Remark 4.2(iii), a loop of length 4 is of the form either $P(x_i^2x_j^2)$ or $P(x_ix_jx_i)$, for some $0 \leq i, j \leq \nu$. Each of these loops can be moved to the trivial loop as follows:

$$P(x_i^2x_j^2) = P(x_j^2) \cdot P(x_i^2) \to P(x_j^2) \to P(1)$$

and

$$P(x_ix_jx_i) = P(x_i) \cdot P_{x_i, B_{0,1}}(x_j^2) \cdot P_{x_i, B_{0,1}}(x_i) \to P(x_i^2) \to P(1).$$

Let $m = 3$ and consider a loop corresponding to $x = x_{j_1} \cdots x_{j_k}$. Then $(\alpha_{j_1}, \ldots, \alpha_{j_k}) \in \text{Alt}(\Pi_0)$. If $(\alpha_{j_1}, \ldots, \alpha_{j_k})$ contains an alternating 4-tuple $f' = (\alpha_{j_1}, \ldots, \alpha_{j_{k+3}})$ and $y$ is the element in $W$ corresponding to $f'$, then $x$ has to have one of the forms $x_y^2y$, $x_jy_jx_j$ or $yx_j^2$, so by the cases $m = 1, 2$, the loop corresponding to $x$ can be moved to the trivial loop. So we may assume that $(\alpha_{j_1}, \ldots, \alpha_{j_k})$ contains no alternating 4-tuple $f'$ as above. By our assumption we know that the loop corresponding to $(x_0x_ix_j)^2$ for $1 \leq i < j \leq \nu$ is an element of $P$, so by definition it can be moved to the trivial loop. Now, consider $(x_jx_0x_j)^2$, the inverse of $(x_0x_ix_j)^2$. Since $P((x_0x_ix_j)^2) \to P(1)$, using (5.6) we have

$$P(x_0x_ix_jx_0x_i) \to P(x_j^2) \cdot P(x_0x_ix_jx_0x_i)$$

$$= P(x_j) \cdot P_{x_j, B_{0,1}}((x_0x_ix_j)^2) \to P(x_j).$$
By repeating this process, we obtain $P((x_j x_i x_0)^2) \rightarrow P(1)$. Using similar arguments, we conclude that

(5.7) \[ \text{if } 0 \in \{r, s, t\} \text{ then } P((x_r x_s x_t)^2) \rightarrow P(1). \]

Also if $0 \in \{r, s, t\}$, we have

(5.8) \[ P(x_r x_s x_t) \rightarrow P((x_r x_s x_t)^2) \cdot P(x_r x_s x_t) \]

\[ = P(x_1 x_s x_r) \cdot P_{x_0 x_r x_s, x_0} (x_1 x_s x_r) \rightarrow P(x_1 x_s x_r). \]

Now to finish the case $m = 3$, consider the element $x = (x_r x_s x_t)^2$, where none of $r, s$ and $t$ is zero. We have

\[ P(x) \rightarrow P(x) \cdot P(x_0^2) \]

\[ = P(x_1 x_s x_r) \cdot P_{x_0 x_r x_s, x_0} (x_1 x_s x_r) \cdot P_{x_0 x_r x_s, x_0} (x_1 x_s x_r) \cdot P_{x_0 x_r x_s, x_0} (x_1 x_s x_r) \]

\[ \rightarrow P(x_1 x_s x_r) \cdot P_{x_0 x_r x_s, x_0} (x_1 x_s x_r) = P((x_r x_s x_t)^2) \rightarrow P(1). \]

Thus any loop of length 6 corresponding to expressions with respect to $\Pi_0$ can be moved to the trivial loop. Also using the same argument as in (5.8), we get

(5.9) \[ P(x_r x_s x_t) \rightarrow P(x_r x_s x_r) \quad (0 \leq r, s, t \leq \nu). \]

Now, we assume that $m > 3$ and that any loop of length smaller than $2m$ can be moved to a trivial loop. Let $P(x)$ be a loop of length $2m$, where $x = x_i \cdots x_{2m}$.

First assume that for some $1 \leq r \leq 2m - 1$, $j_r = j_r + 1$. Since $P_{x_r, x_r}^2 (x_r) \rightarrow P_{x_r, x_r} (1)$, we have

\[ P(x) \rightarrow P(x_{j_1} \cdots x_{j_{r-1}} x_{j_r+2} \cdots x_{2m}). \]

Now since $(\alpha_{j_1}, \ldots, \alpha_{j_r-1}, \alpha_{j_r+2}, \ldots, \alpha_{2m})$ is an alternating $(2m - 2)$-tuple, the path $P(x_{\alpha_{j_1}} \cdots x_{\alpha_{j_r-1}} x_{\alpha_{j_r+2}} \cdots x_{\alpha_{2m}})$ is a loop of length $2m - 2$ in $B$ and so by the induction hypothesis it can be moved to the trivial loop. So, we may assume that $j_r \neq j_{r+1}$ for all $1 \leq r \leq 2m - 1$. From Remark 4.2(ii), the root $\alpha_{j_1}$ appears in $(\alpha_{j_1}, \ldots, \alpha_{2m})$ an even number of times and there is an even integer $2 \leq s \leq 2m$ such that $j_1 = j_s$. From (5.9), we have

\[ P(x) \rightarrow P(x_{j_1} x_{j_2} x_{j_3} x_{j_4} \cdots x_{2m}). \]

By repeating this process, we can move $x_{j_1}$ next to $x_{j_s}$, so that

\[ P(x) \rightarrow P(x_{j_3} x_{j_2} x_{j_1} x_{j_4} \cdots x_{j_{s-1}} x_{j_{s-2}} x_{j_{j_1}} x_{j_s} \cdots x_{2m}). \]
By Remark 4.2(i), the $2m$-tuple
\[
(\alpha_{j_3}, \alpha_{j_2}, \alpha_{j_1}, \alpha_{j_4}, \ldots, \alpha_{j_{m-1}}, \alpha_{j_m}, \alpha_{j_{m+1}}, \ldots, \alpha_{j_{2m}})
\]
is alternating. Thus $(\alpha_{j_3}, \alpha_{j_2}, \alpha_{j_1}, \alpha_{j_4}, \ldots, \alpha_{j_{m-1}}, \alpha_{j_m}, \alpha_{j_{m+1}}, \ldots, \alpha_{j_{2m}}$) is an alternating $(2m - 2)$-tuple and
\[
P(x) \rightarrow P(x_{j_3}x_{j_2}x_{j_1}x_{j_4} \cdots x_{j_{m-1}}x_{j_m}x_{j_{m+1}} \cdots x_{j_{2m}}).
\]
By the induction hypothesis, the right hand side, which is a loop in $\mathcal{B}$ of length $2m - 2$, can be moved to the trivial path. Thus every loop in $\mathcal{B}$ can be moved to the trivial loop.

We conclude this section with the following example which gives a geometric illustration of the method we used in the proof of Theorem 5.3. We use the same notation as in Sections 4 and 5.

Example 5.4. Let $\mathcal{Y}^3 = \mathbb{R}\sigma_1 \oplus \mathbb{R}\sigma_2$ and
\[
w = w_{\epsilon+\sigma_2}w_{\epsilon+\sigma_1}w_{\epsilon+\sigma_2}w_{\epsilon+\sigma_1}w_{\epsilon+\sigma_2}w_{\epsilon+\sigma_1}w_{\epsilon+\sigma_2}w_{\epsilon+\sigma_1}w_{\epsilon}.
\]
Since $(\sigma_2, 0, \sigma_2, \sigma_1, 0, \sigma_1, 0, \sigma_2, \sigma_1, \sigma_2, \sigma_1, 0)$ is an alternating 12-tuple, $w$ is a relation in $\mathcal{W}$. Now, we use our approach to illustrate geometrically how the path $P(w)$ can be moved to the trivial path $P(1)$. With the notation of Theorem 5.3, we write $w = x_2x_0x_2x_1x_0x_1x_2x_2x_1x_2x_0$, where $x_i = w_{\epsilon+\sigma_i}$ for $0 \leq i \leq \nu$. We have
\[
P(w) = (B_{0,1}, B_{0,-1}, B_{-\sigma_1,1}, B_{\sigma_2-\sigma_1,1}, B_{\sigma_2-\sigma_1,-1}, B_{\sigma_2-2\sigma_1,1}, B_{\sigma_2-2\sigma_1,-1}, B_{\sigma_2-2\sigma_1,1}, B_{\sigma_2-2\sigma_1,-1}, B_{\sigma_2,1}, B_{\sigma_2,-1}, B_{0,1}).
\]
In the first move, we have
\[
P(w) \rightarrow P(x_2x_0x_2x_1x_0x_1x_2x_2x_1x_2x_0) = P(x_2x_1x_2x_1x_0x_2x_1x_2x_1x_2x_0).
\]
Let $w_1 := x_2x_0x_2x_1x_0x_2x_1x_2x_1x_0 = x_2x_1x_2x_1x_0x_2x_1x_2x_1x_0$. Then
\[
P(w_1) = (B_{0,1}, B_{0,-1}, B_{-\sigma_1,1}, B_{\sigma_2-\sigma_1,-1}, B_{\sigma_2-2\sigma_1,1}, B_{\sigma_2-2\sigma_1,-1}, B_{\sigma_2-2\sigma_1,1}, B_{\sigma_2-2\sigma_1,-1}, B_{\sigma_2,1}, B_{\sigma_2,-1}, B_{0,1}).
\]
As one can see, the first move replaces the path $(B_7, B_8, B_9, B_{10}, B_{11})$ in Figure 5.1 by the path $(B_7, B_8, B_9)$ in Figure 5.2.

In the second move, we have
\[
P(w_1) \rightarrow P(x_2x_1x_0x_1x_2x_2x_1x_2x_1x_0) = P((x_2x_1x_0)^2).
\]
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Figure 5.1. The complex $P(w)$ in $V^0$

Figure 5.2. The complex $P(w_1)$ in $V^0$

Figure 5.3. The complex $P(w_2)$ in $V^0$
Let \( w_2 := (x_2 x_1 x_0)^2 \). Then

\[
P(w_2) = (B_{0,1}, B_{0,-1}, B_{-\sigma_1,1}, B_{\sigma_2-\sigma_1,1}, B_{\sigma_2,-1}, B_{0,1}).
\]

The second move replaces the path \((B_3, B_4, B_5, B_6, B_7, B_8)\) in Figure 5.2 with the path \((B_3, B_4)\) in Figure 5.3. Since \( w_2 \) belongs to \( \mathcal{P} \), from Definition 5.2(iii), \( P(w_2) \) moves to \( P(1) \).

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References


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