On some fields of meromorphic functions on fibers

By

Takashi Okano*

§1. Introduction

1.1. In this paper we consider the extension problem of meromorphic functions on fibers of complex analytic fiber spaces to neighborhoods of the fibers.

Let \( X \rightarrow Y \) be a complex analytic fiber space, where \( X \) and \( Y \) are normal and connected complex spaces and \( \pi \) is a proper holomorphic mapping of \( X \) onto \( Y \) with irreducible fibers. We denote by \( K_t \) the meromorphic function field of a fiber \( X_t := \pi^{-1}(t) \), and by \( K'_t \) the subfield of \( K_t \) consisting of all elements of \( K_t \) which can be extended to some neighborhoods of \( X_t \). By [6] or [9], the field \( K_t \) is isomorphic to a finite algebraic extension of a rational function field.

We discuss here the following problem.

Let \( f_1, \ldots, f_l \) be meromorphic functions on \( X \) and \( g \) be a meromorphic function on a fiber \( X_t \) which is dependent on \( f_1, \ldots, f_l \), where \( f_i, (i = 1, \ldots, l) \) is the analytic restriction of \( f_i \) to \( X_t \). Then, can we extend the function \( g \) to a meromorphic function on some neighborhood of \( X_t \)?

We can answer this problem as follows.

(1) The complement of the set \( \{ t \in Y \mid \text{any meromorphic function on } X, \text{which is dependent on } f_1, \ldots, f_l, \text{can be extended to some neighborhoods of } X_t \} \) is nowhere dense in \( Y \).

The proof of this theorem is essentially due to the Stein factorization of a proper holomorphic mapping. This notion (or the
notion of complex base is useful to research dependency of holomorphic or meromorphic mappings (for example, see [5], [6], [8], [9]).

Using (I) we obtain:

(II) The set \( \{ t \in Y | K_t \text{ is not algebraically closed in } K \} \) is nowhere dense in \( Y \).

Furthermore, by a similar method to the proof of (I) we have:

(III) If the transcendence degree of \( K_t \) over the complex number field \( \mathbb{C} \) is equal to the (complex) dimension of the fiber \( X_h \), then \( K_t = K_r \).

1.2. In this paper, we assume all complex spaces to be reduced, and we denote the complex projective space of dimension \( m \) by \( P_m \), and the Osgood space of dimension \( l \) by \( P_l \).

We recall here the concepts of rank and of degeneracy of mappings.

Let \( \sigma: M \to N \) be a holomorphic mapping of an irreducible complex space \( M \) to a complex space \( N \). We define the local rank of \( \sigma \) at a point \( x \) of \( M \) by \( \dim_x M - \dim_x \sigma^{-1}(\sigma(x)) \) and denote it by \( r_x(\sigma) \). Further we define the rank of \( \sigma \) by \( \sup_{x \in M} r_x(\sigma) \) and denote it by \( r(\sigma) \).

Now, if \( r_x(\sigma) = r(\sigma) \) for a point \( x \) of \( M \), we call this point \( x \) a point of degeneracy of \( \sigma \). By R. Remmert [8], the set of all points of degeneracy is an analytic subset, and any holomorphic mapping without points of degeneracy (we say such a mapping is non-degenerated or is of constant rank) to a normal complex space whose dimension is equal to the rank of the mapping is an open mapping.

§2. Some remarks on fiber spaces and meromorphic mappings

2.1. Let \( X \) and \( Y \) be complex spaces and \( \{ X_i \} \) be the set of irreducible components of \( X \).

Now let \( f \) be a correspondence between \( X \) and \( Y \). We denote the graph of \( f \) by \( G \) and the natural projections of \( G \) to \( X \) and \( Y \) by \( \tilde{f} \) and \( \tilde{f} \) respectively. Conforming to [9], we call the correspondence \( f \) to be a meromorphic mapping of \( X \) to \( Y \) if the following condi-
tions are satisfied;

(a) there is a dense open set of $X$ on which $f$ defines a holomorphic mapping to $Y$,

(b) the graph $G$ is an analytic subset of $X \times Y$, and $f^{-1}(X_i)$ is an irreducible component of $G$ for each $X_i$,

(c) the projection $\tilde{f}$ is proper.

Let $f$ be a meromorphic mapping of $X$ to $Y$. We call a point $x$ of $X$ a singular point of $f$ if $f$ is not holomorphic at $x$, and call $f$ to be proper (resp. surjective) if $\tilde{f}$ is proper (resp. surjective). Further we define the rank of $f$ by $r(\tilde{f})$ and denote it by $r(f)$. Moreover we say that a meromorphic mapping $f$ of $X$ to $Y$ is bimeromorphic if the correspondence $f$ defines a meromorphic mapping of $Y$ to $X$.

Next, we recall some fundamental properties of meromorphic mappings.

(i) The set of all singular points of a meromorphic mapping is an analytic subset.

(ii) A meromorphic mapping of a certain complex space $X$ to the complex projective space $P^1$ which maps $X$ not constantly to $\infty$ is nothing but a meromorphic function in the usual sense.

(iii) Let $X$, $Y$ and $Z$ be complex spaces and $f$ and $g$ be meromorphic mappings of $X$ to $Y$ and of $Y$ to $Z$ respectively. We define naturally a correspondence between $X$ and $Z$ such that a point $x$ of $X$ corresponds to the subset $g(f(x))$ of $Z$. If there is a dense open set $U$ of $X$ on which the above correspondence between $X$ and $Z$ is single-valued, then we can define naturally one meromorphic mapping $h$ of $X$ to $Z$ such that $h(x) = g(f(x))$ for $x \in U$. We denote it by $g \circ f$. In particular, if $X$ is a subspace of $Y$ and $f$ is the inclusion map, we denote $g \circ f$ by $g\|X$.

(iv) Let $X, Y_1, \ldots, Y_l$ be complex spaces and $f_i$ be a meromorphic mapping of $X$ to $Y_i$ ($i=1, \ldots, l$). Then we can naturally define one meromorphic mapping of $X$ to the product space $Y_1 \times \cdots \times Y_l$. We denote it by $f_1 \times \cdots \times f_l$.

(v) Let $X$ and $Y$ be irreducible complex spaces of the same
dimension and $f$ be a proper and surjective meromorphic mapping of $X$ to $Y$. Then there is a thin analytic subset $N$ of $Y$ such that $f$ is holomorphic on $X - \tilde{f}(\tilde{f}^{-1}(N))$ and the map $f \| (X - \tilde{f}(\tilde{f}^{-1}(N)))$ is a proper holomorphic covering map of $X - \tilde{f}(\tilde{f}^{-1}(N))$ to $Y - N$. We call such a meromorphic mapping to be a \textit{meromorphic covering}.

Next, we recall the notion of \textit{dependency} of meromorphic mappings. Let $X$, $Y$ and $Z$ be complex spaces and $f$ and $g$ be meromorphic mappings of $X$ to $Y$ and of $X$ to $Z$ respectively. Then we say that $g$ depends on $f$ if $r(f \times g) = r(f)$. Further let $f_1, \ldots, f_i$ be meromorphic functions on $X$. Then we say that the system \{$f_1, \ldots, f_i$\} is independent if $r(f_1 \times \cdots \times f_i) = 1$.

\section{2.2} Let $X$ and $Y$ be complex spaces and $\pi$ be a proper holomorphic mapping of $X$ to $Y$. We denote the set of all connected components of all fibers of the map $\pi$ by $X'$. By [1] we can define on the set $X'$ a topology and a complex structure which have the following properties:

\begin{itemize}
  \item[(a)] the natural maps $\pi_1: X \rightarrow X'$ and $\pi_2: X' \rightarrow Y$ are holomorphic.
  \item[(b)] an arbitrary map $h$ of $X'$ to a complex space $Z$ such that $h \circ \pi_1$ is holomorphic is holomorphic.
\end{itemize}

We call this sequence $X' \xrightarrow{\pi_1} X' \xrightarrow{\pi_2} Y$ the \textit{Stein factorization} of $\pi$.

\textbf{Proposition 1.} Let $X$ be a compact irreducible complex space, and $f_1, \ldots, f_i$ be meromorphic functions on $X$. We put $F = f_1 \times \cdots \times f_i$, $G =$ the graph of $F$. Let $\tilde{G} \xrightarrow{\alpha} G$ be the normalization of $G$ and $\tilde{G} \xrightarrow{\beta} H \xrightarrow{\lambda} P^i$ be the Stein factorization of the proper holomorphic mapping $\tilde{F} \circ \mu$, where $\tilde{F}$ is the natural projection of $G$ to $P^i$.

Then, for any meromorphic function $g$ on $X$ dependent on $F$, there is a meromorphic function $g'$ on $H$ such that $g = g' \circ h_1 \circ \mu^{-1} \circ \tilde{F}^{-1}$.

\textbf{Proof.} Since $X$ and $\tilde{G}$ are bimeromorphically equivalent, we may assume that $X$ is normal and connected and $F$ is holomorphic on $X$. Under these assumptions we may identify the sequence $\tilde{G} \xrightarrow{\beta_1} H \xrightarrow{\lambda} P^i$ with the Stein factorization $X \xrightarrow{\beta_1} X' \xrightarrow{\lambda} P^i$ of the proper holomorphic mapping $F$.  

\textit{Takashi Okano}
Let $S(g)$ be the singular set of $g$. Since $X$ is compact and $g$ depends on $F$, there is a polynomial $P_r(X_0, \ldots, X_t)X^r + \cdots + P_0(X_0, \ldots, X_t)$, where $s > 0$, such that $P_r(f_1, \ldots, f_t)g^r + \cdots + P_0(f_1, \ldots, f_t)$ $= 0$ on $X$ and $P_r(f_1, \ldots, f_t) \equiv 0$ on $X$ (see [9], p. 864). Now we take a point $z = (z_0, \ldots, z_t)$ of $F(X)$ such that $z_i \neq \infty$ for all $i$ and $P_r(z_0, \ldots, z_t) \neq 0$. Let $x$ be a point of $F^{-1}(z)$. Then $g(x)$ is a finite set in $P^1$ since $P_r(z_0, \ldots, z_t) \neq 0$. This fact and the normality of $X$ yield the holomorphy of $g$ at $x$ (see [9], Prop. 3.1.3). Hence $F(S(g)) \neq F(X)$.

Since $F(X)$ is an irreducible complex space, $F(S(g))$ is a thin analytic subset of $F(X)$ and so $F^{-1}(F(S(g)))$ is a thin analytic set of $X$. We put $X_b = X - F^{-1}(F(E) \cup F(S(g)))$, where $E$ is the set of degeneracy of $F$. ($F^{-1}(F(E))$ is thin in $X$.) We denote the Stein factorization of the proper holomorphic mapping $F|X_b$ by $X_b \to X'_b \to F(X_b)$. Then we may consider that $X'_b = h_b(X_b) \subset X'$. For a point $x$ of $X_b$, $r_x(F) = r_x(F \times g)$ because $r_x(F) \leq r_x(F \times g) \leq r(F \times g) = r(F) = r_x(F)$. Hence $g$ is constant along each connected component of $F^{-1}(z)$ for any $z$ of $F(X_b)$ (see [8], p. 300). Therefore we obtain a holomorphic function $g'_b$ on $X'_b$ such that $g|X_b = g'_b \circ h_b$.

Put $G(g) =$ the graph of $g$, and $G' =$ the Image of $G(g)$ by the map $h_b \times 1$ of $X \times P^1$ to $X' \times P^1$. Then $G'$ gives a morphomorphic function $g'$ on $X'$ such that $g'|X_b = g'_b$.

**Remark.** Proposition 1 can be generalized as follows:

Let $\pi: X \to Y$ be a proper holomorphic mapping, where $X$ is irreducible, and $f_1, \ldots, f_t$ be meromorphic functions on $X$. We put $\sigma = f_1 \times \cdots \times f_t \times \pi$, $G =$ the graph of $\sigma$, and $\tilde{\sigma}, \tilde{\sigma} = \sigma$ the natural projections of $G$ to $P^t \times Y$ and to $X$. Let $\tilde{G} \to G$ be the normalization of $G$ and $\tilde{G} \to H \to P^t \times Y$ be the Stein factorization of $\tilde{\sigma} \circ \mu$.

Then, for any meromorphic function $g$ on $X$ dependent on $\sigma$, there is a meromorphic function $g'$ on $H$ such that $g = g' \circ h_\sigma \circ \mu^{-1} \circ \tilde{\sigma}^{-1}$.

**Proposition 2.** Let $V$ be an irreducible analytic subspace of $P_m$. Then any element of the field $K(V)$ of all meromorphic functions on $V$ is the restriction of a rational function of $P_m$. 

---

*On some fields of meromorphic functions on fibers* 57
Furthermore let \( \{f_1, \ldots, f_i\} \) be a transcendence base of \( K(V) \) over the complex number field \( \mathbb{C} \). Then the degree of \( K(V) \) over the field \( C(f_1, \ldots, f_i) \) is equal to the number of sheet of the meromorphic covering map \( F : V \to P' \), where \( F=f_1 \times \cdots \times f_i \).

**Proof.** Let \( K_s(V) \) be the subfield of \( K(V) \) consisting of all elements of \( K(V) \) which can be extended to a rational function of \( P_m \). Then the transcendence degrees of \( K(V) \) and \( K_s(V) \) over \( \mathbb{C} \) are equal to the dimension of \( V \). Let \( \{f_1, \ldots, f_i\} \) be a transcendence base of \( K_s(V) \) over \( \mathbb{C} \) and \( F \) be the meromorphic mapping \( f_1 \times \cdots \times f_i \) of \( V \) to \( P' \). Then \( F \) is a meromorphic covering map, because \( \dim V = l = \dim P' \) and the system \( \{f_1, \ldots, f_i\} \) is independent. So there is an analytic subset \( N \) of \( P' \) such that \( F \) is holomorphic on \( V - F^{-1}(N) \) and \( F|[V-V^{-1}(N)] \) is a proper unramified holomorphic covering map to \( P' - N \). We put \( b=\) the number of sheet of \( F|[V-V^{-1}(N)] \), \( d=[K(V): C(f_1, \ldots, f_i)] \) and \( d'=[K_s(V): C(f_1, \ldots, f_i)] \). Then clearly \( b \geq d \geq d' \), because any element \( f \) of \( K(V) \) satisfies; \( f^i + H_{i-1}f^{i-1} + \cdots + H_0 = 0 \), where \( H_i (i=0,1,\ldots,b-1) \) is a suitable rational function of \( P' \) which is considered as an element of \( C(f_1, \ldots, f_i) \).

On the other hand, we can find an element \( g \) of \( K_s(V) \) whose degree over \( C(f_1, \ldots, f_i) \) is not smaller than \( b \). In fact, fix a point \( p \) of \( P' - N \), and put \( F^{-1}(p) = \{p_1, \ldots, p_i\} \). Then we can easily find two linear forms \( w_i=a_0z_0+\cdots+a_nz_n, w_i=b_0z_0+\cdots+b_nz_n \), for a system of homogeneous coordinate \( \{z_1, \ldots, z_n\} \) of \( P_m \), such that \( w_i(p_i) \neq 0 \) for all \( i \), and \( \frac{w_i(p_i)}{w_i(p_i)} = \frac{w_i(p_i)}{w_i(p_i)} (\text{for } i \neq j) \). Now we put \( \alpha = \frac{w_2}{w_1} \) and \( \alpha = \frac{w_2}{w_1} \). Then it can be easily proved that the degree of \( \alpha \) over \( C(f_1, \ldots, f_i) \) is not smaller than \( b \).

**Theorem.** (H. Grauert and R. Remmert, [2], [4]). Let \( X \) and \( Y \) be complex spaces and \( \sigma \) be a proper holomorphic mapping of \( X \) to the product space \( \mathbb{P}_m \times Y \) with discrete fibers. Let \( U \) be a relatively compact Stein open set of \( Y \). We put \( X_U = \sigma^{-1}(\mathbb{P}_m \times U) \).

Then, there is a natural number \( N \) and a biholomorphic mapping \( \omega \) of \( X_U \) to an analytic subspace of the product space...
On some fields of meromorphic functions on fibers

$P_n \times U \times P_n$ such that $\sigma| X_o = p \circ \omega$, where $p$ is the natural projection of $P_n \times Y \times P_n$ to $P_n \times Y$.

**Proposition 3.** Let $\pi : X \to Y$ be a proper holomorphic mapping of a normal complex space $X$ onto a complex space $Y$. Then the set \{ $t \in Y$ | the space $\pi^{-1}(t)$ is not locally irreducible \} is nowhere dense in $Y$.

The proof of this proposition is essentially due to W. Thimm [11]. We prove this in the next section.

§3. Proof of Proposition 3

To prove our proposition we use local descriptions of the normal complex space $X$. Therefore we start by setting the following notations. We put;

$$T = \{(t_1, \ldots, t_n) \in \mathbb{C}^n | \sum |t_i| < \varepsilon, \quad i = 1, \ldots, n\},$$

$$Z_m = \{z_1, \ldots, z_m \in \mathbb{C}^n | \sum |z_j| < \zeta, \quad j = 1, \ldots, m\},$$

$$D_m = T \times Z_m,$$

$p = \text{the natural projection of } D_m \text{ to } T,$

$$Z_m,t = p^{-1}(t), \text{ where } t \text{ is a point of } T.$$

Now let $A$ be an analytic set of $D_m$ and $T_o$ be the set \{ $t \in T$ | $Z_m,t \cap A = Z_m \}$ . We consider the following condition ($*$) for a point $x$ of $D_m$ with respect to $A$:

$(*)$ The point $p(x)$ does not belong to $T_c$ and there is a fundamental system of neighborhoods $\{ U_i \}$ of the point $x$ which satisfies the following condition (C):

(C) for a curve $C$ in $U_i$ such that $C \cap A = \phi$ and $p(C(0)) = p(C(1)) = p(x)$, there is a deformation of the curve $C$ to a curve in $U_i \cap Z_m,x$, through the space $U_i - A$, with the end points $C(0)$ and $C(1)$ fixed.

**Lemma 3.1.** Let $M$ be a connected normal complex space and $r$ be a proper holomorphic covering map of $M$ to $D_m$ which is unramified over $D_m - A$. If, for a point $x$ of $M$, the point $r(x)$
satisfies the condition (*) with respect to A, then \(x\) is an irreducible point of the fiber \((p \circ r)^{-1}(\{p \circ r\}(x))\).

**Proof.** Suppose that \(r(x)\) satisfies the condition (*) with respect to \(A\). We put \(M_s=(p \circ r)^{-1}(p \circ r)(x)\). Then \(M_s \cap r^{-1}(A)\) is a thin analytic set of \(M_s\) and \(M_s-r^{-1}(A)\) is non-singular. Hence \(x\) is an irreducible point of \(M_s\) if and only if there is a fundamental system of neighborhoods \(\{U'_i\}\) of the point \(x\) in the space \(M_s\) such that \(U'_i-r^{-1}(A)\) is connected.

Take a connected neighborhood \(V\) of \(x\) in the space \(M\) such that \(V \cap M_s\) is sufficiently small and,

(a) the open set \(r(V)\) satisfies the condition (C) with respect to \(A\) at \(r(x)\),

(b) the mapping \(r| V: V \to r(V)\) is proper.

We put \(U'=V \cap M_s\). Then from the above (a) and (b) \(U'-r^{-1}(A)\) is connected. In fact, let \(x_1\) and \(x_2\) be points of \(U'-r^{-1}(A)\). Since \(V\) is connected and normal, we can connect \(x_1\) to \(x_2\) by a curve \(\tilde{C}\) in \(V-r^{-1}(A)\). We put \(C=r(\tilde{C})\). By (a), \(C\) can be deformed to a curve in \(Z_{m-1,r(x)} \cap (r(V)-A)\) through the space \(r(V)-A\), fixing the end points. On the other hand, the map \(r| V\) is a proper unramified covering over \(r(V)-A\). Hence we can deform \(\tilde{C}\) to a curve of \(U'-r^{-1}(A)\) through the space \(V-r^{-1}(A)\), by lifting the deformation of the curve \(C\). Hence \(U'-r^{-1}(A)\) is connected.

**Lemma 3.2.** We put: \(Z_{m-1}=(z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1} | |z_j|<\zeta_j; j=1, \ldots, m-1\), and \(D_{m-1}=T \times Z_{m-1}\) and \(q=\text{the natural projection of } D_m\ to\ D_{m-1}\).

Suppose that \(q| A\) is a proper holomorphic covering map onto \(D_{m-1}\) and it is unramified over \(D_{m-1}-B\), where \(B\) is a thin analytic set of \(D_{m-1}\).

Then, for a point \(x\) of \(D_m\) if \(q(x)\) satisfies the condition (*) with respect to \(B\) then \(x\) also satisfies the condition (*) with respect to \(A\).
Proof. Let $W$ be a neighborhood of $x$. Then we can find a neighborhood $U$ of $x$ having the following properties;

(a) $U \subset W$,

(b) $U$ is of the form $q(U) \times D$, where $D$ is a disk of $\mathbb{C}^1$,

(c) $q(U)$ satisfies the condition (C) at $q(x)$ with respect to $B$, and

(d) $q|_{A \cap U}: A \cap U \to q(U)$ is proper.

Then we can prove that the open set $U$ satisfies the condition (C) at $x$ with respect to $A$ by the same methods as in [11]. We give only an outline of the proof.

Let $C$ be a curve in $U - A$ with the end points $C(0)$ and $C(1)$ such that $p(C(0)) = p(C(1)) = p(x)$. Without loss of generality, we may assume that $q(C(0))$ and $q(C(1))$ do not belong to $B$, because $Z_{m-1, \rho(x)} \cap B \neq Z_{m-1, \rho(x)}$ by above (c) and so we can replace the end points by two suitable points in $U \cap Z_{m, \rho(x)} - (A \cup q^{-1}(B))$ which are connected to $C(0)$ and $C(1)$ by arcs in $U \cap Z_{m, \rho(x)} - A$ respectively. Moreover we may assume that $q(C)$ is disjoint with $B$, because the curve $C$ can be deformed, fixing the end points, to a curve which is sufficiently near to $C$ and whose projection to $q(U)$ is disjoint with $B$ (see [11], §2). Under these assumptions, $q(C)$ can be deformed by the above property (c) to a curve of $q(U) \cap q(Z_{m, \rho(x)})$ though the space $q(U) - B$ with the end points fixed. On the other hand, since $q|_A$ is proper and unramified over $D_{m-1} - B$, we can construct a deformation of $C$ in $U - A$ with the desired properties lying above the deformation of $q(C)$ (see [10], §2 and [11], §2).

Lemma 3.3. We suppose that $A$ is purely $1$-codimensional in $D_m$, and put $D_m^* = \{x \in D_m | x$ satisfies the condition (*) with respect to $A\}$.

Then $p(K_m - D_m^*)$ is nowhere dense in $T$ for any relatively compact subset $K_m$ of $D_m$.

Proof. We prove the lemma by induction on $m$. If $m=0$, it is
trivial. So we suppose that \( m > 0 \) and that the result holds for \( m - 1 \).

We denote the \( \varepsilon \)-neighborhood of the set \( T_0 \) by \( T_0(\varepsilon) \). Then \( p(K_m - D^*_m) \) is nowhere dense in \( T \) if and only if it is nowhere dense in \( T - T_0(\varepsilon) \) for any positive number \( \varepsilon \). We put \( K_\varepsilon(\varepsilon) = K_\varepsilon - p^{-1}(T_\varepsilon(\varepsilon)) \).

Now let \( x \) be a point of \( D_m \). If \( x \in A \), take a neighborhood \( U_\varepsilon(x) \) of \( x \) such that \( U_\varepsilon(x) \cap A = \emptyset \). Then any point of \( U_\varepsilon(x) \) satisfies the condition \((*)\) with respect to \( A \). Next we suppose \( x \in A - p^{-1}(T_\varepsilon) \). Then, since \( A \) is purely codimensional 1, we can find a neighborhood \( V_\varepsilon(x) \) of \( x \) satisfying the following properties:

(a) \( V_\varepsilon(x) \) is the product of two polycylinders \( T(x) \) and \( Y_\varepsilon(x) \), where \( T(x) \) and \( Y_\varepsilon(x) \) are defined as follows;

\[
T(x) = \{(t'_i, \cdots, t'_m) \in \mathbb{C}^m | \ |t'_i| < r'_i; \ i = 1, \cdots, m\},
\]

\[
Y_\varepsilon(x) = \{(y_j, \cdots, y_m) \in \mathbb{C}^n | \ |y_j| < r_j; \ j = 1, \cdots, m\},
\]

where \( t_i = t_i - t_i(p(x)) \) and \( y_j = \sum_{k=1}^{m} c_{jk} z_k + d_j \) such that \( y_j(x) = 0 \) (for any \( j \)) and the matrix \( (c_{jk}) \) is non-singular, and \( r_i \) and \( r_j \) are suitable positive numbers.

(b) Let \( Y_{m-1}(x) = \{(y, \cdots, y_{m-1}) \in \mathbb{C}^{m-1} | \ |y_j| < r_j; \ j = 1, \cdots, m-1\} \) and \( V_{m-1}(x) = T(x) \times Y_{m-1}(x) \) and \( q \) is the natural projection of \( V_\varepsilon(x) \) to \( V_{m-1}(x) \). In this situation, \( q | V_\varepsilon(x) \cap A \) is a proper covering map and unramified over \( V_{m-1}(x) - B \), where \( B \) is an analytic subset of \( V_{m-1}(x) \) purely of codimension 1.

We denote the natural projection of \( V_{m-1}(x) \) to \( T(x) \) by \( p_{m-1} \), and the set \( \{s \in V_{m-1}(x) | s \text{ satisfies the condition } (*) \text{ with respect to } B \} \) by \( V_{m-1}^*(x) \). Let now \( U_{m-1}(x) \) be an arbitrarily fixed relatively compact open neighborhood of \( q(x) \) in \( V_{m-1}(x) \). Then, by the hypothesis of induction, \( p_{m-1}(U_{m-1}(x) - V_{m-1}^*(x)) \) is nowhere dense in \( T(x) \). Hence, by Lemma 3.2, \( p(U_\varepsilon(x) - D^*_\varepsilon) \) is nowhere dense in \( T(x) \), where \( U_\varepsilon(x) \) is the set \( q^{-1}(U_{m-1}(x)) \).

For each point \( x \) of \( K_\varepsilon(\varepsilon) \) we take such an open neighborhood \( U_\varepsilon(x) \) mentioned above. Since \( K_\varepsilon(\varepsilon) \) is compact, it is covered by a finite system of such neighborhoods \( U_\varepsilon(x_i) \) and hence \( p(K_\varepsilon(\varepsilon) - D^*_\varepsilon) \).
is nowhere dense in $T$.

**Proof of Proposition 3.** We may assume that $Y$ is non-singular and $\pi$ is of constant rank. For, our assertion is of local character about $Y$ and the $\pi$-image of the set of degeneracy of $\pi$ is a thin analytic set in $Y$. Moreover we may assume; $Y = \{(t_1, \cdots, t_n) \in \mathbb{C}^n \mid |t_i| < r_i; i = 1, \cdots, n\}$. Then, for each point $x$ of $X$, we can find a connected open neighborhood $U(x)$ such that there is a proper holomorphic covering map $r$ of $U(x)$ to $D_m$, where $D_m$ is a polycylinder which is obtained by replacing $t_i$ by $t_i - t_i(\pi(x))$ in $D_m$ of the beginning of this section.

Let $A$ be a purely one codimensional analytic set in $D_m$ such that $r$ is unramified over $D_m - A$. Further let $W$ be a relatively compact open set of $D_m$ containing $r(x)$ and $V(x)$ be the open set $r^{-1}(W) \cap U(x)$. Then, by Lemma 3.3, $p(W - D_m)$ is nowhere dense in $T(G \subset Y)$ and hence $X_t \cap V(x)$ is locally irreducible by Lemma 3.1 for any point $t$ of $p(W) - p(W - D_m)$.

For each point $x$ of $X$, we take such a neighborhood $V(x)$. Let $Q$ be a relatively compact open set of $Y$. Then the set $\pi^{-1}(Q)$ is compact and so it is covered by a finite system of open sets $V(x_t)$. Hence the set \{\(t \in Y \mid X_t \text{ is not locally irreducible}\)} is nowhere dense in $Y$.

**§4. Meromorphic function fields on fibers**

In this section, we consider a fiber space $X \rightarrow Y$, where $X$ and $Y$ are complex spaces and $\pi$ is a proper surjective holomorphic mapping. We put $\text{dim} Y = n$ and $\text{dim} X = m + n$. Furthermore we assume;

(a) $X$ and $Y$ are normal and connected,

(b) $\pi$ is of constant rank, $n$,

(c) for every $t \in Y$, the fiber $X_t$ is irreducible.

These assumptions imply,

(d) $\pi^{-1}(U)$ is connected for any connected open set $U$ of $Y$.

From now on, we use occasionally a notation $h_t$ instead of $h_t\mid X_t$. 

On some fields of meromorphic functions on fibers
where \( h \) is a meromorphic mapping of \( X \) to a certain complex space and \( t \) is a point of \( Y \) such that \( h\|X_t \) is defined.

**Lemma 4.1.** Let \( f_1, \ldots, f_l \) be meromorphic functions on \( X \). We put \( F=f_1 \times \cdots \times f_l \) and \( S(F)=\) the singular set of \( F \). Then the set \( \{ t \in Y \mid X_t \cap S(F) \} \) is a dense open subset of \( Y \).

Let \( t \) be a point of \( Y \) such that \( X_t \cap S(F) \). We suppose that \( \{ f_i, \ldots, f_l \} \) is independent. Then there is an open neighborhood \( U \) of \( t \) such that \( f_i \psi_i(t) \) is defined and \( \{ f_i, \ldots, f_l \} \) is independent for any \( t' \) of \( U \). (In this case, \( r(F \times \pi)=n+l \) and \( (F \times \pi)(X)=P^l \times Y \).)

**Proof.** The first assertion is trivial.

Suppose that \( \{ f_i, \ldots, f_l \} \) is independent. We can find a point \( x \) of \( X \), such that \( x \in S(F) \) and \( r_i(F)=r(F)=l \) (here we consider \( F \) as a holomorphic mapping on a neighborhood of \( x \)). Then \( r_i(F \times \pi)=r(F \times \pi)=n+l \) because \( \dim(F \times \pi)^{-1}((F \times \pi)(x))=\dim F_i^{-1}(F_i(x)) \)
\[ = m-r_i(F_t)=m-l, \] and so \( (F \times \pi)(X)=P^l \times Y \).

Take a neighborhood \( Q \) of \( x \) such that \( Q \cap S(F)=\emptyset \) and \( r_i'(F \times \pi)=n+l \) for any point \( x' \) of \( Q \). Put \( \pi=\pi(Q) \). Since \( \pi \) is of constant rank, \( U \) is an open set and clearly has our desired properties.

**Theorem I.** Let \( t_0 \) be a point of \( Y \) and \( f_1, \ldots, f_l \) be meromorphic functions on \( X \) such that \( f_i, t_0 \) is defined for any \( i \) and the system \( \{ f_1, \ldots, f_l \} \) is independent. We put \( F=f_1 \times \cdots \times f_l, \)
\( \varphi=F \times \pi, G=\) the graph of \( \varphi, \) and \( G_{t_0}=\) the graph of \( F_{t_0}, \) and we denote the normalization of \( G \) by \( \tilde{G} \rightarrow G \).

We suppose that;

1. the complex space \( \tilde{G} \mid X_{t_0} =\) the restriction of \( \tilde{G} \) over \( X_{t_0} \) is locally irreducible.

Then there is an open neighborhood \( U \) of \( t_0 \) such that any meromorphic function defined on \( X_{t_0} \) which is dependent on \( F_{t_0} \) can be extended to a meromorphic function on \( \pi^{-1}(U) \).

**Proof.** Since \( X \) is normal, every fiber of the map \( \tilde{G} \rightarrow X \) is connected, and \( X_t \) is irreducible by the assumption. Hence \( \tilde{G} \mid X_t \) is con-
On some fields of meromorphic functions on fibers

Let \( \widetilde{G} \to H \to P^1 \times Y \) be the Stein factorization of the proper holomorphic mapping \( \tilde{\alpha} \circ \mu_t \), where \( \tilde{\alpha} \) is the natural projection of \( G \) to \( P^1 \times Y \), and \( \widetilde{G}_t \to H_t \to P^1 \) be the Stein factorization \( \tilde{F}_t \circ \mu_t \), where \( \mu_t \) is the normalization map \( \widetilde{G}_t \to G_t \) and \( \tilde{F}_t \) is the natural projection of \( G_t \) to \( P^1 \). From above, \( h_t^{-1}(P^1 \times t) \) is also naturally homeomorphic and bimeromorphic to \( H_t \), so we may identify \( H_t \) with \( h_t^{-1}(P^1 \times t) \).

By proposition 1, we can find a meromorphic function \( g' \) on \( H_t \) such that \( g = g' \circ h_t \circ \mu_t^{-1} \circ \tilde{F}_t^{-1} \), where \( \tilde{F}_t \) is the natural projection of \( G_t \) to \( X_t \).

On the other hand, the map \( h_t : H \to P^1 \times Y \) is proper (and surjective) with discrete fibers. Hence, by Theorem of §2, there is a neighborhood \( U \) of \( t_0 \) and a biholomorphic mapping \( \omega \) of \( h_t^{-1}(P^1 \times U) \) to an analytic subspace \( L_u \) of \( P^1 \times U \times P_N \).

We put \( g'' = g' \circ (\omega \| H_t)^{-1} \) on \( L_t \). By proposition 2, there is a rational function \( g''' \) on \( P^1 \times P_N (\equiv P^1 \times t \times P_N) \) such that \( g''' \| L_t = g'' \). Further we put \( \tilde{g}''' = g''' \circ \tau \), where \( \tau \) is the natural projection of \( P^1 \times U \times P_N \) to \( P^1 \times P_N \), and put \( \tilde{g}'' = \tilde{g}''' \| L_u \). Finally we set \( \tilde{g} = \tilde{g}''' \circ (\omega \circ h_t \circ \mu_t^{-1} \circ \tilde{F}_t^{-1})^{-1} \) on \( \pi^{-1}(U) \). Then \( \tilde{g} \) is a meromorphic function with \( g = \tilde{g} \| X_t \).

**Remark.** By the construction of \( \tilde{g} \), it is easily shown that there is a polynomial \( P(t)(X_0, X_1, \ldots, X_t) \) with holomorphic functions on \( U \) as coefficients (if necessary, replace \( U \) with a smaller neighborhood of \( t_0 \)) such that \( P(t)(g, f_1, \ldots, f_t) = 0 \) on \( \pi^{-1}(U) \) and \( P(t_0)(X_0, X_1, \ldots, X_t) \neq 0 \).

We use the following notations:

- \( K_t \) = the field of all meromorphic functions on the fiber \( X_t \),
- \( K'_t \) = the subfield of \( K_t \) consisting of all elements of \( K_t \), which
can be extended to neighborhoods of \( Y \),
\[
Y(k) = \{ t \in Y \mid \text{there is a neighborhood } U \text{ of } t \text{ in } Y \text{ such that the transcendence degree of } K'_j = k \text{ for any } t_1 \text{ of } U \}, \quad \text{and} \quad Y' = Y(0) \cup Y(1) \cup \cdots \cup Y(m).
\]

**Corollary.** Let \( f_1, \ldots, f_i \) be meromorphic functions on \( X \) such that \( f_1, \ldots, f_i \) is defined for any \( i \) and the system \( \{ f_1, \ldots, f_i \} \) is independent for any \( t \) of \( Y \). We set \( K'_j(f) \) as the algebraic closure of the field \( C(f_1, \ldots, f_i) \) in \( K_j \). Then the set \( \{ t \in Y \mid K'_j(f) \subseteq K'_j \} \) is nowhere dense in \( Y \).

**Proof.** By Proposition 3 there is a nowhere dense set \( Y_0 \) of \( Y \) such that any point of \( Y - Y_0 \) satisfies the condition (I) of Theorem I. Hence our assertion is proved by Theorem I.

**Theorem II.** The set \( Y_i = \{ t \in Y \mid K'_j \text{ is not algebraically closed in } K_i \} \) is nowhere dense in \( Y \).

**Proof.** The assertion is of local character about \( Y \), and \( Y' \) is a dense open set of \( Y \). So we may assume that \( Y = Y(k) \), and there are \( k \) meromorphic functions on \( X \) such as in the above corollary. Hence \( Y_i \) is nowhere dense in \( Y \) by corollary of Theorem I.

Lastly we discuss the case \( Y = Y(m) \) (where \( m \) is the dimension of fibers).

**Lemma 4.2.** Let \( Z \) be a complex space and \( W \) be a compact irreducible analytic subspace of \( Z \) of dimension \( m \), and \( f_1, \ldots, f_n \) be meromorphic functions on \( Z \) such that \( f_i \mid W \) is defined for any \( i \) and \( \{ f_1 \mid W, \ldots, f_n \mid W \} \) is independent. We put \( F = f_1 \times \cdots \times f_n, \quad F_0 = F \mid W, \quad G = \text{the graph of } F, \quad G_0 = \text{the graph of } F_0, \text{ and } G_1 = G \mid W \) (the restriction of \( G \) over \( W \)).

Now \( \lambda \) be the natural projection of \( G_1 \) to \( \mathbb{P}^n \) and \( G_1 \to G_1 \to \mathbb{P}^n \) be the Stein factorization of \( \lambda \). Then the holomorphic mapping \( \lambda_i | G_0 : G_0 \to \lambda_i(G_0) \) is bimeromorphic and \( \lambda_i(G_0) \) is an irreducible component of \( G_1 \).
Proof. The graph $G_0$ of $F_0$ is an irreducible component of $G_1$. Let $G_2$ be the union of all the irreducible components of $G_1$ which are distinct from $G_0$. Since $\{f_i\| W, \ldots, f_w\| W\}$ is independent and $\dim W = m$, the proper holomorphic mapping $\lambda|G_0: G_0 \to \mathbb{P}^n$ is surjective and of rank $m$. From this, it follows that $\lambda(G_0 \cap G_2) = \mathbb{P}^n$, for $\lambda(G_0 \cap G_2) = \mathbb{P}^n$ implies $G_1 \subset G_2$. Hence our assertion is proved.

Theorem III. If the transcendence degree of the field $K_1$ is equal to the (complex) dimension of the fiber, then $K_1 = K_i$.

Proof. By Lemma 4.1, we may assume that $Y = Y(m)$ and that there are $m$ meromorphic functions $f_i, \ldots, f_m$ on $X$ such that $f_i, \ldots, f_m$ is defined $(i = 1, \ldots, m)$ and the system $\{f_i, \ldots, f_m\}$ is independent. We put $F = f_1 \times \cdots \times f_m$, and $G = \text{the graph of } F \times \pi$ and $G_i = \text{the graph of } F_i$, and denote the Stein factorization of $F \times \pi$ by $G \to G' \to \mathbb{P}^n \times Y$. Then, by Lemma 4.2, $G_i$ is bimeromorphically equivalent to an irreducible component of $G' \times X_i$. Hence we can prove this theorem similarly to Theorem I.

REFERENCES
