Implicit Pseudo-Runge-Kutta Processes

By

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§ 1. Introduction

The present paper is concerned with the numerical solution of the initial value problem:

\[
\begin{align*}
    y' &= f(x, y), \\
    y(x_0) &= y_0.
\end{align*}
\]  

(1.1)

Of all computational one step methods for the numerical solution of this problem, the easiest formula is Euler's rule.

It is linear and explicit, but it is of low order.

Higher order methods have been achieved by sacrificing the linearity. These methods were first proposed by Runge [13] and subsequently developed by Heun [7] and Kutta [8].

Runge-Kutta methods need some functional evaluation at each step. And one looked for other methods to decrease the functional evaluation. These works were due to Ceschino, Kuntzman [4], Byrne, Lambert [1] and Rosen [12].

Now Byrne and Lambert [1] have defined 2-step Runge-Kutta methods and Costabies [5] has proposed Pseudo-Runge Kutta methods. In [9], the present author has studied Pseudo-Runge Kutta methods and has proposed some new ones.

In [1], [5], [9] and [10], the methods are explicit forms, however, no implicit methods have been proposed as yet.

It is possible to consider implicit Pseudo-Runge Kutta methods, and in this paper, we shall give implicit Pseudo-Runge Kutta methods based on the methods developed in [10].

It will be seen that they are equivalent to certain implicit Runge-
Kutta methods in some special cases.

Their advantage over implicit Runge–Kutta methods lies in the fact they are of less expensive in terms of the functional evaluations for given order.

Let \( m(p) \) be the highest order which can be attained by a \( p \)-stage method. In [3], Butcher proved, on his method, the following results

\[
m(p) = 2p \quad (p = 1, 2, 3, \ldots),
\]

whereas our methods have

\[
m(p) = p + 3 \quad (p = 2, 3).
\]

In order to apply implicit Runge–Kutta methods, it is necessary, at each step, to solve non-linear equations by some means. By making clear the algorithm of our methods of \( r \)-stage, we observe that the number of non-linear equations is \( r-1 \).

Let \( n(r) \) be the highest order that can be attained by non-linear equations of order \( r \). Then by our methods

\[
n(r) = r + 4 \quad (r = 1, 2),
\]

and by Runge–Kutta methods

\[
n(r) = 2r \quad (r = 1, 2, \ldots).
\]

§ 2. Derivation of the Methods

We consider \( p \)-stage implicit Pseudo–Runge–Kutta methods:

\[
y_{n+1} = y_n + v(y_{n-1} - y_n) + h\Phi(x_{n-1}, x_n, y_{n-1}, y_n; h),
\]

where

\[
\Phi(x_{n-1}, x_n, y_{n-1}, y_n; h) = \sum_{i=0}^{p} w_i k_i,
\]

\[
k_0 = f(x_{n-1}, y_{n-1}),
\]

\[
k_i = f(x_n + a_i h, (1 + b_i) y_n - b_i y_{n-1} + h \sum_{j=0}^{p} b_{ij} k_j),
\]

\((i = 2, 3, \ldots, p) \quad (0 < a_i \leq 1; i = 2, 3, \ldots, p).\)

In the above formula (2.1), the value \( y_n \) is to be an approximation to the value \( y(x_n) \) of the solution of (1.1) for \( x_n = x_0 + nh \). Coefficients \( a_i, b_i (i = 2, 3, \ldots, p) \) and \( b_{ij} (i = 2, 3, \ldots, p; j = 0, 1, \ldots, p) \) are real constants to be determined.
We consider only two and three stage methods obtained by setting \( p = 2, 3 \) in (2.1). Throughout the paper, the coefficients are constrained by

\[
a_i = b_i + \sum_{j=0}^{p} b_{ij} \quad (i = 2, 3, \ldots, p).
\]

The functions \( k_i (i \geq 2) \) are no longer defined by explicitly but by the set of \( p - 1 \) implicit equations. The special case \( v = w_0 = w_1 = b_i = b_{i0} = 0 \) in (2.1) is implicit Runge-Kutta method. The derivation of the method is rather complicated. The treatment is similar as those in Butcher [3].

Let \( D \) be the differential operator defined by

\[
D = \frac{\partial}{\partial x} + f(x_n, y_n) \frac{\partial}{\partial y},
\]

and put

\[
Df(x_n, y_n) = T^i (i = 1, 2, 3, 5), \quad Df_1(x_n, y_n) = S'(i = 1, 2, 3), \quad (Df_2)^2(x_n, y_n) = P, \quad (Df)^2(x_n, y_n) = Q, \quad Df_{y_2}(x_n, y_n) = R,
\]

\[
f_2(x_n, y_n) = f_{y_1}, \quad f_{yy}(x_n, y_n) = f_{yy}.
\]

We also introduce an abbreviation,

\[
\sum_{i=2}^{p} = \sum, \quad \sum_{k=2}^{p} = \sum_2.
\]

We assume that the solutions of the functions \( k_i (i \geq 2) \) may be expressed in the form:

\[
k_i = f_2 + \sum_{j=1}^{3} c_i h^j + O(h^6),
\]

and

\[
y_{n-1} = y(x_{n-1}).
\]

From (2.1), we have the following equation:

\[
y_{n+1} = y_n + hA_1k_1 + h^2 A_2 T + \frac{1}{2!} h^3 (A_3 f_2 T + A_4 T^2) + \frac{1}{3!} h^4 (B_1 T^3 + B_2 f_2 T^2 + B_3 f_2^2 T + 3B_4 ST) + \frac{1}{4!} h^5 (C_4 T^4 + 6C_2 TS^3 + 4C_3 T^3 S + 3C_4 f_2 Q + 3C_5 T^3 S + 3C_6 f_2 T^2 + 3C_7 f_2^2 T + 3C_8 f_2 TS) + \frac{1}{5!} h^6 (D_1 T^5 + D_2 T^4 S + D_3 T^3 S + D_4 f_2 T + D_5 f_2^2 T^2 T + D_6 f_2 T + D_7 f_2 T^2 + D_8 f_2 T^3) + D_{10} f_2 T^2 + D_{11} f_2 f_2 f_2 + D_{12} f_2 f_2 Q + D_{13} f_2 T S^2 + D_{14} f_2^2 T^2 S + D_{15} f_2 T^2 S).
\]
The constants \([A_i], [B_i], [C_i]\) and \([D_i]\) are

\[
\begin{align*}
A_1 &= -v + w_0 + \sum w_i, & A_2 &= \frac{-v}{2} - w_0 + \sum a_i w_i, \\
A_3 &= -\frac{v}{3} + w_0 + \sum p_{ji} w_i, & A_4 &= -\frac{v}{3} + w_0 + \sum a_i^3 w_i, \\
B_1 &= \frac{v}{4} - w_0 + \sum a_i^4 w_i, & B_2 &= \frac{v}{4} - w_0 + \sum p_{ji} w_i, \\
B_3 &= \frac{v}{4} - w_0 + \sum q_{ji} w_i, & B_4 &= \frac{v}{4} - w_0 + \sum a_i p_{ji} w_i, \\
C_1 &= -\frac{v}{5} + w_0 + \sum a_i^5 w_i, & C_2 &= -\frac{v}{5} + w_0 + \sum a_i p_{ji} w_i, \\
C_3 &= -\frac{v}{5} + w_0 + \sum a_i p_{ji} w_i, & C_4 &= -\frac{v}{5} + w_0 + \sum p_{ji} w_i, \\
C_5 &= -\frac{v}{5} + w_0 + \sum g_{ji} w_i, & C_6 &= -\frac{v}{5} + w_0 + \sum g_{ji} w_i, \\
C_7 &= -\frac{v}{5} + w_0 + \sum r_{ji} w_i, & C_8 &= -\frac{v}{5} + w_0 + \sum q_{ji} w_i, \\
D_1 &= \frac{v}{6} - w_0 + \sum a_i^6 w_i, & D_2 &= 10\left(\frac{v}{6} - w_0 + \sum a_i^6 p_{ji} w_i\right), \\
D_3 &= 10\left(\frac{v}{6} - w_0 + \sum a_i^6 p_{ji} w_i\right), & D_4 &= 5\left(\frac{v}{6} - w_0 + \sum a_i^6 p_{ji} w_i\right), \\
D_5 &= 10\left(\frac{v}{6} - w_0 + \sum p_{ji}^2 w_i\right), & D_6 &= 15\left(\frac{v}{6} - w_0 + \sum a_i p_{ji}^2 w_i\right), \\
D_7 &= 15\left(\frac{v}{6} - w_0 + \sum a_i q_{ji} w_i\right), & D_8 &= 3\left(\frac{v}{6} - w_0 + \sum p_{ji} w_i\right) + 4C_3, \\
D_9 &= \frac{v}{6} - w_0 + \sum g_{ji} w_i, & D_{10} &= \frac{v}{6} - w_0 + \sum r_{ji} w_i, \\
D_{11} &= \frac{v}{6} - w_0 + \sum u_{ji} w_i, & \\
D_{12} &= 3\left(\frac{v}{6} - w_0 + \sum h_{ji} w_i\right) + 10\left(\frac{v}{6} - w_0 + \sum p_{ji} q_{ji} w_i\right), \\
D_{13} &= 16\left(\frac{v}{6} - w_0 + \sum q_{ji} w_i\right), & D_{14} &= 7\left(\frac{v}{6} - w_0 + \sum u_{ji} w_i\right), \\
D_{15} &= 4\left(\frac{v}{6} - w_0 + \sum u_{ji} w_i\right), & \\
p_{ij} &= (i+1)\left(-1\right)^i \frac{1}{(i+1)!} b_{ji} + (-1)^i \frac{1}{i!} b_{ji} + \frac{1}{i!} \sum a_i^2 b_{ji} \right), \\
q_{ij} &= (i+1)\left(-1\right)^i \frac{1}{(i+1)!} b_{ji} + (-1)^i \frac{1}{i!} b_{ji} + \frac{1}{i!} \sum a_i^2 p_{ij} b_{ji} \right),
\end{align*}
\]
The \( p \)-stage method (2.1) has order \( p + 3 \) (\( p = 2, 3 \)) if

\[
(-1)^i \frac{v}{i!} + (-1)^{i-1} \frac{w_0}{(i-1)!} + \sum_{j=2}^{i} \frac{a_j^{i-1}}{(i-1)!} w_j = \frac{1}{i!},
\]

\[
(-1)^k \frac{b_j}{k!} + (-1)^{k-1} \frac{b_{j0}}{(k-1)!} + \sum_{\ell=2}^{k} \frac{a_{\ell}^{k-1} b_{j\ell}}{(k-1)!} = \frac{a_j^{k}}{k!},
\]

\[
(-1)^p \frac{v}{(p+3)!} + (-1)^{p-1} \frac{w_0}{(p+2)!} + \sum_{j=2}^{p} \left( (-1)^{p+1} \frac{b_j}{(p+2)!} + \frac{1}{(p+2)!} \sum_{r=2}^{p} a_r b_{j r} \right) w_j
\]

\[
= \frac{1}{(p+3)!}
\]

\((i=1, 2, \ldots, p+3, j=2, 3, \ldots, p, k=2, 3, \ldots, p+1), (a_i \neq a_j \text{ for } i \neq j).\)

The case \( p = 2 \) in (2.4) gives the following values for the constants in (2.1). It is of order 5,

\[
v = 77 - 12c, \quad w_0 = \frac{1}{4} (45 - 7c), \quad w_1 = \frac{1}{2} (33 - 5c),
\]

\[
w_2 = \frac{1}{4} (201 - 31c), \quad a_2 = \frac{1}{10} (1 + c), \quad b_2 = \frac{1}{250} (-413 + 47c),
\]

\[
b_{j0} = \frac{1}{125} (37 - 3c), \quad b_{21} = \frac{1}{250} (139 + 9c), \quad b_{22} = \frac{1}{10} (9 - c),
\]

\((c = \sqrt{41}).\)

The case \( p = 3 \) in (2.4) gives the following values for the constants in (2.1). It is of order 6,
In order to apply the method (2.1), it is necessary to solve, at each step, non-linear algebraic equations. We seek a solution by an inner iterative procedure and prove the convergence of the method (2.1).

Butcher [3] studied $p$-stage implicit Runge-Kutta method and gave the iterations

$$
(2.6) \quad a_3 = \begin{cases} 
\frac{-\left(-a_2^2 + \frac{2}{3}a_2 + \frac{1}{3}\right) + \sqrt{d_1}}{2(5a_2^2 - a_2 - 2)} & (0 < a_2 \leq (1 + \sqrt{41})/10) \\
\frac{-\left(-a_2^2 + \frac{2}{3}a_2 + \frac{1}{3}\right) - \sqrt{d_1}}{2(5a_2^2 - a_2 - 2)} & ((1 + \sqrt{41})/10 < a_2 \leq 1),
\end{cases}
$$

$$
w_3 = \frac{(5a_2 + 3)\left(\frac{5}{6}a_2^2 - \frac{7}{12}\right) + 5(2a_2 + 1)\left(\frac{7}{12}a_2 - \frac{9}{20}\right)}{a_3(1 + a_3) (a_2 - a_3) \left((5a_2 + 3) + 5a_3(2a_2 + 1)\right)},
$$

$$
v = \left[\frac{a_3}{a_3} (1 + a_3) (a_2 - a_3) w_3 - \left(\frac{7}{12}a_2 - \frac{9}{20}\right)\right] \left(\frac{1}{12}a_2^2 + \frac{1}{20}\right)^{-1},
$$

$$
w_2 = \frac{5 - v - 6a_3(1 + a_3) w_3}{6a_3(1 + a_2)}, \quad w_0 = \frac{1}{2} (v - 1) + \sum a_i w_i,
$$

$$
w_1 = 1 - (w_0 + \sum w_i - v),
$$

$$
b_{12} = \frac{[a_i (a_i + 1)]^2 - 2a_3 (2a_3^2 + 3a_3 + 1) b_{13}}{2a_3 (2a_3^2 + 3a_3 + 1)},
$$

$$
b_{1i} = 6 (a_2 (a_2 + 1) b_{12} + a_3 (1 + a_3) b_{13}) - 2a_2^3 - 3a_3^2,
$$

$$
b_{10} = \frac{1}{2} (-b_1 + 2 \sum a_i b_{1j} - a_i^2),
$$

$$
b_{23} = \frac{1}{30d_2 w_2} \left[1 - v - 6w_0 + 6w_5 (b_5 + 5b_{30} + 5a_2 b_{32} + a_2 b_{33}) - d_3\right],
$$

$$
b_{1i} = a_i - (b_1 + b_{10} + b_{11} + b_{12}) \quad (i = 2, 3),
$$

where

$$
d_1 = \left(-a_2^2 + \frac{2}{3}a_2 + \frac{1}{3}\right)^2 + 4{(5a_2^2 - a_2 - 2) (2a_2^2 - \frac{1}{3}a_2 - 1)},
$$

$$
d_2 = \frac{1}{5} \left[a_3 (5a_2^2 - 7a_2 - 4) - \frac{5a_3^2 - 9a_2 - 4}{2a_2^2 + 3a_2 + 1} a_3 (2a_2^2 + 3a_3 + 1)\right],
$$

$$
d_3 = \frac{1}{5} \left[a_2^2 (3a_2 + 2) + \frac{5a_3 - 9a_2 - 4}{4a_2^2 + 6a_2 + 2} (a_2(a_2 + 1))^2\right].
$$
IMPLICIT PSEUDO-RUNGE-KUTTA PROCESSES

\[ k_r^{(N+1)} = f(x + a_i h, y + h \sum_{j=1}^{r-1} b_{ij} k_r^{(N+1)} + h \sum_{j=r}^p b_{ij} k_j^{(N)}) \quad (i = 1, 2, \ldots, p), \]

where \( k_r^{(N)} \) denotes the number after \( N \)-times iteration for the implicit non-linear equations of \( k_r \). Similarly, we set the iteration in (2.1) by

(3.1) \[ k_r^{(N+1)} = f(x + a_i h, (1 + b_i) y_n - b_i y_{n-1} + h \sum_{j=0}^{r-1} b_{ij} k_j^{(N+1)} + h \sum_{j=r}^p b_{ij} k_j^{(N)}). \]

Then we have the following Theorem.

**Theorem.** The iteration process (3.1) converges provided

\[ h \leq 1 / \{L(U_1 + U_2)\}, \]

where \( L \) is the Lipshitz constant of \( f(x, y) \) with respect to \( y \) and

\[
\begin{align*}
U_1 &= \max \left\{ \sum_{i=1}^{p} |b_{i3}|, \quad \sum_{i=1}^{p} |b_{i2}| \right\}, \\
U_2 &= \max \left\{ \sum_{i=1}^{p} |b_{i2}|, \quad \sum_{i=1}^{p} |b_{i3}| \right\}.
\end{align*}
\]

The proof is done in a similar way as in Butcher [3] and we will omit it.

**§ 4. Stability Analysis**

In this section, we discuss the stability of our methods. We apply our method (2.1) with \( p = 2 \) to the test equation \( y' = \lambda y \) which yields

\[
\begin{align*}
V_1 & = 1 - \frac{1}{10} (9 - \sqrt{41}) \tilde{h}, \\
V_2 & = (-76 + 12\sqrt{41}) + (-5875.75 + 926.25 \sqrt{41}) \lambda \tilde{h} \\
& \quad + (2286.775 - 361.225 \sqrt{41}) \lambda^2 \tilde{h}, \\
V_3 & = 77 - 12\sqrt{41} + (4642.75 - 733.25 \sqrt{41}) \lambda \tilde{h} \\
& \quad + (11.25 - 1.75 \sqrt{41}) \lambda^2 \tilde{h} \quad (\lambda = \lambda \tilde{h}).
\end{align*}
\]

The condition for the stability is that the roots of characteristic equation of this difference equation are all of absolute value less than 1. This yields that, if \( \lambda \) is negative real, the stability bound is

\[-2.9 \leq \tilde{h} \leq 0,\]

if \( \lambda \) is complex, the stability region shown in Figure (1).
Let us consider the method (2.1) with $p=3$. In that case we have

\[(4.1) \quad y_{n+1} - \tilde{V}_1 y_n - \tilde{V}_2 y_{n-1} = 0,\]

where

\[
\tilde{V}_1 = 1 - v + (e_{11} + w_n) \hat{h} + e_{21} \hat{h}^2 + e_{31} \hat{h}^3, \\
\tilde{V}_2 = -v + (e_{21} - w_n) \hat{h} + e_{22} \hat{h}^2 + e_{23} \hat{h}^3, \\
e_{11} = \{(1 + b_2) w_5 + (1 + b_3) w_3\}/L, \\
e_{12} = \{(b_{21} - b_{33} (1 + b_2) + b_{23} (1 + b_3)) w_5 + (b_{21} + b_{23} (1 + b_3) - b_2 (1 + b_3)) w_3\}/L, \\
e_{13} = \{(-b_{21} b_{33} + b_{23} b_{31}) w_5 + (-b_{21} b_{31} + b_{23} b_{33}) w_3\}/L, \\
e_{21} = -\{(b_2 w_5 + b_3 w_3\}/L, \\
e_{22} = \{(b_{30} + b_2 b_{33} - b_3 b_3) w_5 + (b_{30} + b_2 b_3 - b_3 b_2) w_3\}/L, \\
e_{23} = \{-b_2 b_{33} + b_2 b_{30} \} w_5 + (-b_3 b_{30} + b_3 b_{32}) w_3\}/L, \\
L = \{-1 - \hat{h} (b_{22} + b_{33}) + \hat{h}^2 (b_{22} b_{33} - b_{23} b_{32})\}.
\]

Let us consider some specific algorithms in the formula (2.6).

If we set

\[(4.2) \quad b_{33} = -\frac{w_5 (Z_4 + Z_6 Z_{10})}{w_5 Z_6 - w_5 Z_5},\]

where

\[
Z_1 = \frac{(a_2 (a_2 + 1))^2}{2a_2 (2a_2^2 + 3a_2 + 1)}, \\
Z_2 = \frac{(-2a_3 (2a_3^2 + 3a_3 + 1))}{(2a_3 (2a_3^2 + 3a_3 + 1))}, \\
Z_3 = \frac{(a_3 (a_3 + 1))^2}{2a_3 (2a_3^2 + 3a_3 + 1)}, \\
Z_4 = (a_2^2 + a_2^2) Z_3 - (a_2^2 + a_2^2) Z_1, \\
Z_5 = \{(a_2 - 3a_3) Z_3 - (a_2^2 + a_2^2) Z_1\} w_3 + \{(a_2 - 3a_3) Z_3 - 2a_3 - 3a_3^2\} w_2, \\
Z_6 = \{(a_2^2 + a_2^2) Z_2 + (a_2^2 + a_2^2) Z_1\} w_3 - \{(a_2 - 3a_3) Z_3 + a_2 + a_2^2\} w_2, \\
Z_7 = (a_2^2 - 1.8a_2^2 - 0.8a_2) Z_1 + 0.2 (3a_2^3 + 2a_2^2), \\
Z_8 = (a_2^4 - 1.8a_2^3 - 0.8a_2) - \frac{(a_2^3 - 1.8a_2^2 - 0.8a_2) a_2 Z_2}{a_2}, \\
Z_9 = (a_2^4 - 1.8a_2^3 - 0.8a_2) Z_3 + 0.2 (3a_2^3 + 2a_2^2), \\
Z_{10} = \frac{1}{Z_6 Z_2} \left(\frac{1}{30} \frac{w_5 + w_0}{30} + \frac{w_0}{5} - Z_5 w_3\right), \\
Z_{11} = -Z_6 Z_2 Z_5 - Z_5 Z_{10} Z_6 + Z_8 Z_6 Z_{10} - Z_2 Z_9 Z_5 + Z_2 Z_6 Z_{10}.
\]

Then we have
If we set, in addition to the condition (4.2),

\[ a_2 = 0.98, \]

then we have the following stability bound

\[ -55 \leq \hat{h} \leq 0. \]

The stability region is shown in Figure (2).

If we set, in addition to the condition (4.2),

\[ a_2 = 0.57, \]

then we have the following stability bound

\[ -67 \leq \hat{h} \leq 0. \]

The stability region is shown in Figure (3).

Next we consider the case where the coefficient \( b_{33} \) varies.

\[ a_2 = 1, \]

\[ b_{33} = \frac{81(8\sqrt{3} - 14.89) - 27(503.29 - 272\sqrt{3}) \hat{h} - 9(2254.53 - 1254.71\sqrt{3})}{27(33.11\sqrt{3} - 65.34)\hat{h}^2 - (475.2\sqrt{3} + 118.8)\hat{h}^3}, \]

where

\[ \hat{h} = a + bi, \]

\[ \hat{h} = \text{sgn} (a) \sqrt{a^2 + b^2}. \]

In this case, we have the following characteristic roots:

\[ \rho_1 = 0.9, \]

\[ \rho_2 = -\left\{ \frac{s_1 t_1 + s_2 t_2}{t_1^2 + t_2^2} + a \omega_0 \right\} - \left\{ \frac{s_2 t_1 - s_1 t_2}{t_1^2 + t_2^2} + b \omega_0 \right\}, \]

with

\[ s_1 = -\left( -128 + 30\sqrt{3} \right) / 11 + e_{22}(a^2 - b^2) + e_{22}a + e_{23}(a(a^2 - b^2) - 2ab^2), \]

\[ s_2 = e_{22}a + 2e_{23}(b(a^2 - b^2) + 2a^2b), \]

\[ t_1 = 1 - (b_{22} + b_{33})a + (b_{22}b_{33} - b_{23}b_{32})(a^2 - b^2), \]

\[ t_2 = -(b_{22} + b_{33}) + 2(b_{22}b_{33} - b_{23}b_{32})ab. \]

Similarly, we look for the stability region numerically, which is contained in the region:

\[ D = \{ (a, b) : -280 \leq a \leq 0, \ |b| \leq 30 \}. \]
The stability region is shown in Figure (4). In the case (4.3), we have $b_{33} \to \infty$ as $h \to 0$, however the iterations of function $k_i$ are convergent.

The stability intervals for $R-K\, 4$, Lawson’s method (explicit $R-K$ type of order 5) and Huta’s method (explicit $R-K$ type of order 6) are shown in Table (1). The stability regions for these methods are shown in Figure (5).

<table>
<thead>
<tr>
<th>Method</th>
<th>interval of stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R-K, 4$</td>
<td>$(-2, 7, 0)$</td>
</tr>
<tr>
<td>Lawson</td>
<td>$(-5, 7, 0)$</td>
</tr>
<tr>
<td>Huta</td>
<td>$(-3, 7, 0)$</td>
</tr>
</tbody>
</table>

Figure (1). Stability region for the method (2.1) with $p=2$, whose coefficients are given by (2.5). ($\Re(\lambda) \leq 0$).
Figure (2). Stability region for the method (2.1) with $p=3$, whose coefficients are given by (2.6), (4.2) and $a_2=0.98$. ($\Re(\lambda) \leq 0$).

Figure (3). Stability region for the method (2.1) with $p=3$, whose coefficients are given by (2.6), (4.2) and $a_2=0.57$. ($\Re(\lambda) \leq 0$).
Figure (4). Stability region for the method (2.1) with $p=3$, whose coefficients are given by (2.6) and (4.3). ($\Re(\lambda) \leq 0$).

Figure (5). Stability region for R-K method (order 4), Lawson's method (order 5) and Huta's method (order 6). ($\Re(\lambda) \leq 0$).

§ 5. Computational Results

In Table I and II, we present numerical results for the following initial value problems.

\[ I : y' = (y - xy)/x, \; y(1) = e^{-1}, \; y(x) = xe^{-x}, \]
II: \( y' = -y^2(2e^x - 1), \) \( y(0) = 1, \) \( y(x) = 1/(2e^x - x - 1), \)

III: \[
\begin{align*}
y' &= -y + x, \quad y(0) = 1, \quad y(x) = (1 + x)e^{-2x} \\
z' &= -y - 3x, \quad z(0) = 0, \quad z(x) = -xe^{-2x}
\end{align*}
\]

IV: \[
\begin{align*}
y' &= -y + 3z - 8x - 9, \quad y(0) = 6, \quad y(x) = 3e^x + e^{-4x} + x + 2 \\
z' &= 2(y - z) + 4x + 7, \quad z(0) = 5, \quad z(x) = 2e^x - e^{-4x} + 3x + 4
\end{align*}
\]

Computations are done in double precision arithmetic on FACOM M-200 of Kyushu University.

In Table I, the value \( y_1 \) necessary for the evaluations using the formula (2.4) is computed by Runge–Kutta methods of order 4 and in Table II, the value \( y_1 \) necessary for the evaluations using the formula (4.1), (4.2) is computed by Huta's method of order 6.

In order to start the calculations at \((x_n, y_n)\), we need to solve the implicit functions \( k_i(i \geq 2) \). We first calculate the predicted value for the functions by \( k_0 = f(x_n, y_n) \).

**Table I**

Error for the solutions to the Problems I, II, III, and IV. Comparison of errors incurred by using the implicit Runge–Kutta method of order 4 (Im R-K 4), Lawson's method (explicit R-K type of order 5) and the method (2.1) with the coefficients (2.5) (order 5).

**Problem I.** \( h = 1/2^4, M = 5 \) (M: number of iterations),

<table>
<thead>
<tr>
<th>( x )</th>
<th>Lawson's Method</th>
<th>Im R-K 4</th>
<th>method (2.1) with (2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>function evaluations/step</td>
<td>((1+2\times M))</td>
<td>((1+1\times M))</td>
</tr>
<tr>
<td>2</td>
<td>(-0.9970E-10)</td>
<td>0.6963E-08</td>
<td>(-0.1442E-08)</td>
</tr>
<tr>
<td>5</td>
<td>(-0.7074E-11)</td>
<td>0.6034E-09</td>
<td>(-0.1408E-09)</td>
</tr>
<tr>
<td>9</td>
<td>0.7337E-12</td>
<td>0.2277E-10</td>
<td>(-0.6463E-11)</td>
</tr>
<tr>
<td>13</td>
<td>0.5673E-13</td>
<td>0.2142E-11</td>
<td>(-0.2671E-12)</td>
</tr>
</tbody>
</table>

**Problem II.** \( h = 1/2^4, M = 5 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Lawson's Method</th>
<th>Im R-K 4</th>
<th>method (2.1) with (2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4180E-08</td>
<td>(-0.1528E-07)</td>
<td>0.1082E-07</td>
</tr>
<tr>
<td>4</td>
<td>0.1169E-09</td>
<td>(-0.4188E-10)</td>
<td>(-0.2092E-12)</td>
</tr>
<tr>
<td>8</td>
<td>0.1575E-11</td>
<td>(-0.4188E-10)</td>
<td>(-0.2092E-12)</td>
</tr>
<tr>
<td>12</td>
<td>0.2442E-13</td>
<td>(-0.1026E-11)</td>
<td>(-0.2423E-13)</td>
</tr>
</tbody>
</table>
**Problem III.** \( h=1/2^4, \ M=5. \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.5687E-08</td>
<td>0.1114E-07</td>
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<td>0.1385E-06</td>
<td>-0.1478E-07</td>
<td>0.2856E-07</td>
</tr>
<tr>
<td>4</td>
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<td>-0.2280E-08</td>
<td>-0.1366E-08</td>
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<td>-0.1374E-09</td>
</tr>
<tr>
<td>8</td>
<td>0.2163E-12</td>
<td>-0.1800E-12</td>
<td>-0.3980E-11</td>
<td>-0.3368E-11</td>
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<td>-0.4955E-12</td>
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<tr>
<td>12</td>
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<td>-0.2928E-14</td>
<td>-0.5044E-15</td>
<td>-0.4536E-15</td>
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</table>

**Problem IV.** \( h=1/2^4, \ M=5. \)

<table>
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<tr>
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<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
<th>( y_s-y(x_s) )</th>
<th>( z_s-z(x_s) )</th>
</tr>
</thead>
<tbody>
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<td>0.1487E-06</td>
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</tr>
<tr>
<td>4</td>
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<td>0.9168E-06</td>
<td>0.6112E-06</td>
</tr>
<tr>
<td>8</td>
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<td>0.2649E-04</td>
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<td>-0.1012E-02</td>
<td>0.1014E-03</td>
<td>0.6760E-04</td>
</tr>
<tr>
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<td>0.2170E-02</td>
<td>-0.1243E-00</td>
<td>-0.8284E-01</td>
<td>0.8340E-02</td>
<td>0.5560E-02</td>
</tr>
</tbody>
</table>
Table II

Errors for the solutions to the Problems I, II, III, and IV. Comparison of errors incurred by using the implicit Runge–Kutta method of order 6 (Im R-K 6), Huta's method (explicit R-K type of order 6) and the method (2.1) whose coefficients are given by (2.6), (4.2) and (4.3).

**Problem I.** $h=1/2^4, M=5$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Huta's Method</th>
<th>Im R-K 6</th>
<th>method (2.1) (order 6) with (2.6)</th>
<th>method (2.1) (order 6) with (2.6), (4.3)</th>
<th>$k=20$</th>
<th>$k=2 \times 10^3$</th>
<th>$k=2 \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>0.5033E-12</td>
<td>0.5851E-12</td>
<td>0.5229E-12</td>
<td>0.1383E-10</td>
<td>0.1460E-11</td>
<td>0.7655E-12</td>
</tr>
<tr>
<td>5</td>
<td>0.1134E-12</td>
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<td>0.3496E-12</td>
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<td>0.2968E-12</td>
<td>0.2733E-12</td>
<td>0.3389E-12</td>
</tr>
<tr>
<td>9</td>
<td>0.3280E-14</td>
<td>0.1269E-12</td>
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<tr>
<td>13</td>
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<td>0.8996E-14</td>
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</table>

**Problem II.** $h=1/2^4, M=5$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Huta's Method</th>
<th>Im R-K 6</th>
<th>method (2.1) (order 6) with (2.6)</th>
<th>method (2.1) (order 6) with (2.6), (4.3)</th>
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<th>$k=2 \times 10^3$</th>
<th>$k=2 \times 10^4$</th>
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<tbody>
<tr>
<td>1</td>
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<td>0.1125E-09</td>
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<tr>
<td>4</td>
<td>0.9277E-10</td>
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<td>0.4673E-10</td>
<td>0.1833E-10</td>
<td>0.1931E-09</td>
<td>0.5648E-10</td>
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</tr>
<tr>
<td>8</td>
<td>0.8343E-12</td>
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<td>0.5116E-12</td>
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<tr>
<td>12</td>
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<td>0.3404E-13</td>
<td>0.9952E-14</td>
<td>0.8976E-14</td>
</tr>
</tbody>
</table>

Implicit Pseudo-Runge-Kutta Processes
### Problem III. $h=1/2^4, M=5.$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
</tr>
</thead>
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<tr>
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<td>0.2177E-08</td>
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<td>0.3142E-10</td>
</tr>
<tr>
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<td>0.5795E-11</td>
<td>-0.1138E-11</td>
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<td>-0.1657E-13</td>
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<tr>
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<td>-0.4187E-16</td>
<td>0.1900E-16</td>
<td>-0.1699E-16</td>
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</tbody>
</table>

### Method (2.1) with (2.6), (4.2)

<table>
<thead>
<tr>
<th>$x$</th>
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<th>$z_s - z(x_s)$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
<th>$y_s - y(x_s)$</th>
<th>$z_s - z(x_s)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.1724E-08</td>
<td>0.2573E-08</td>
<td>0.1702E-08</td>
<td>0.2538E-08</td>
</tr>
<tr>
<td>4</td>
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<td>-0.2714E-10</td>
<td>0.2038E-10</td>
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<td>0.2017E-10</td>
<td>-0.5413E-11</td>
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<tr>
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<td>0.5794E-13</td>
<td>-0.4721E-13</td>
<td>0.5741E-13</td>
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<td>-0.4642E-16</td>
<td>0.5148E-16</td>
<td>-0.4601E-16</td>
</tr>
</tbody>
</table>
Problem IV. \( h = 1/2^4, \ M = 5. \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_n - y(x_n) )</th>
<th>( z_n - z(x_n) )</th>
<th>( y_n - y(x_n) )</th>
<th>( z_n - z(x_n) )</th>
<th>( y_n - y(x_n) )</th>
<th>( z_n - z(x_n) )</th>
<th>( y_n - y(x_n) )</th>
<th>( z_n - z(x_n) )</th>
</tr>
</thead>
<tbody>
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<tr>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>( \dot{h} = -2 )</th>
<th>( \dot{h} = -20 )</th>
<th>( \dot{h} = -2 \times 10^3 )</th>
<th>( \dot{h} = -2 \times 10^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y_n - y(x_n) )</td>
<td>( z_n - z(x_n) )</td>
<td>( y_n - y(x_n) )</td>
</tr>
<tr>
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<td>4</td>
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<td>-0.1292E-02</td>
<td>0.5115E-03</td>
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</tbody>
</table>
The author wishes to express his hearty thanks to Prof. M. Sakai of Kagoshima University and Prof. S. Huzino of Kyushu University for helpful discussions and suggestions.

He would also like to express his sincere thanks to Prof. S. Hitotumatu of Kyoto University, Prof. M. Tanaka of Yamanashi University and Prof. T. Mitsui of Fukui University for their invaluable suggestions and advice. In the last, we would like to thank the referee for numerous fruitful comments.

References