Corrigendum to “Decay of Solutions of Wave-type Pseudo-differential Equations over \( p \)-adic Fields”

By

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1. The proof of Theorem 4 in [7] contains a gap. This theorem deals with the asymptotic estimation of exponential sums depending on several parameters (or oscillatory integrals depending on several parameters). By using a result due to Cluckers [3, Theorem 6.1], a more general version of Theorem 4 can be proved easily, however, the decay rate obtained is not optimal. With the notation given in [7], the statement and the proof of Theorem 4 should be corrected as follows.

Theorem 4. Let \( \phi(x) \in \mathcal{R}_K[x] \), \( x = (x_1, \ldots, x_{n-1}) \), be a non-constant polynomial such that \( \mathcal{C}_\phi(K) = \{0\} \subset K^{n-1} \). Let \( d_j(\phi) \) be the degree of \( \phi \) with respect the variable \( x_j \), and let \( \beta_\phi := \max_j d_j(\phi) \). Let \( \Theta_S \) be the characteristic function of a compact open set \( S \), let \( Y = \{ x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1}) \} \), and let \( d\mu_Y = \Theta_Sd\sigma_Y \). Then

\[
\left| \hat{d\mu_Y}(\xi) \right| \leq C \|\xi\|_K^{-\beta},
\]

for \( 0 \leq \beta \leq \beta_\phi - \epsilon \), with \( \epsilon > 0 \).

Proof. By passing to a sufficiently fine covering we may suppose that

\[
\hat{d\mu_Y}(\xi) = \int_{(z_n + \pi a R_K)^{n-1}} \Psi(-\xi_n \phi(x) - [x, \xi']) |dx|.
\]

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By applying Theorem 6.1 of [3], we have
\[ \left| \hat{d\mu} (\xi) \right| \leq C \left( \log_q \|\xi\|_K \right)^{n-1} \|\xi\|_K^{-\beta_\phi}, \]
and then
\[ \left| \hat{d\mu} (\xi) \right| \leq C \|\xi\|_K^{-\beta}, \text{ for } 0 \leq \beta \leq \beta_\phi - \epsilon, \epsilon > 0. \]

It is important to mention that Cluckers’ Theorem 6.1 is established only for \( \mathbb{Q}_p \), however this result is valid for any \( p \)-adic field. Indeed, the proof of this result is based on a result of Chubarikov [2, Lemma 3] whose proof uses inductively an estimation for one-dimensional exponential sums due to I. M. Vinogradov (see e.g. [1, Theorem 2.1]). The proof of this last estimation as given in [1] can be adapted to the case of \( p \)-adic fields easily using the notion of dilation as in [8].

The Cluckers’ result does not give an optimal decay rate, and then \( \beta_\phi \) is not optimal (see also [9]).

2. The following remark should be added after Theorem 4.

**Remark 1.** If \( \phi(x) = \sum_{i=1}^{n-1} a_i x_i^2 \), then the phase of \( \hat{d\mu} (\xi) \) around any critical point has the form \( \sum_{i=1}^{n-1} a_i x_i^2 \). By using [7, Theorem 3], one verifies that the decay rate around any critical point is \( \frac{n-1}{2} \), therefore Theorem 4 holds for \( 0 \leq \beta \leq \frac{n-1}{2} = \beta_\phi \). Note that by the principle of stationary phase the contribution of the non-critical points can be neglected (see [7, Theorem 1]). If \( n = 1 \) and \( \phi(x) = x^d, d > 1 \), the phase of \( \hat{d\mu} (\xi) \) around a critical point can take the form \( x^f p(x), 2 \leq f \leq d, p(x) \neq 0 \) locally. By using the fact the real parts of the possible poles of the corresponding local zeta functions have the form \( 1, 2 \leq f \leq d \), and Theorem 8.4.2 in [4], one verifies that Theorem 4 holds for \( 0 \leq \beta \leq \frac{1}{d} = \beta_\phi \).

In the case of real numbers the results described in the previous remark are well-known (see e.g. [6]).

3. The hypothesis “let \( \phi(x) \in K[x], x = (x_1, \ldots, x_{n-1}) \), be a non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(\phi) \)” should be replace by “let \( \phi(x) \in R_K[x], x = (x_1, \ldots, x_{n-1}) \), be a non-constant polynomial such that \( C_{\phi}(K) = \{0\} \subset K^{n-1} \)” in the Theorems 5 and 6. The proofs do not need any modification. The new versions are as follows.

**Theorem 5.** Let \( \phi(x) \in R_K[x], x = (x_1, \ldots, x_{n-1}) \), be a non-constant polynomial such that \( C_{\phi}(K) = \{0\} \subset K^{n-1} \). Let

\[ Y = \{x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1})\} \]
with the measure $d\mu_{Y,S} = \Theta_S d\sigma_Y$, where $\Theta_S$ is the characteristic function of a compact open subset $S$ of $K^n$. Then

$$\left( \int_{Y} |\mathcal{F}g(\xi)|_K^2 d\mu_Y(\xi) \right)^\frac{1}{2} \leq C(Y) \|g\|_{L^p},$$

holds for each $1 \leq \rho < \frac{2(1+\beta_\phi)}{2+\beta_\phi}$.

**Theorem 6 (Main Result).** Let $\phi(\xi) \in R_K[\xi]$, $\xi = (\xi_1, \ldots, \xi_n)$, be a non-constant polynomial such that $C_\phi(K) = \{0\} \subset K^n$. Let

$$(H\Phi)(t,x) = \mathcal{F}^{-1}_{(t,\xi)\rightarrow(x,t)}(|\tau - \phi(\xi)|_K \mathcal{F}_{(t,x)\rightarrow(t,\xi)}\Phi, \Phi \in \mathcal{S}(K^{n+1}),$$

be a pseudo-differential operator with symbol $|\tau - \phi(\xi)|_K$. Let $u(x,t)$ be the solution of the following initial value problem:

$$
\begin{cases}
(Hu)(x,t) = 0, & x \in K^n, \ t \in K, \\
u(x,0) = f_0(x),
\end{cases}
$$

where $f_0(x) \in \mathcal{S}(K^n)$. Then

$$\|u(x,t)\|_{L^\sigma(K^{n+1})} \leq A \|f_0(x)\|_{L^2(K^n),}$$

for $\frac{2(1+\beta_\phi)}{2+\beta_\phi} < \sigma \leq \infty$.

4. The last subsection (Wave-type Equations with Quasi-homogeneous Symbols) should be rewritten as follows.

§ 5.2 Wave-type Equations with Homogeneous Symbols

In the cases $\phi(\xi) = a_1\xi_1^2 + \cdots + a_n\xi_n^2$ and $n = 1$, $\phi(\xi) = \xi^d$ by using Remark 1, we have the following estimations for the solution of Cauchy problem (1).

**Theorem 7.** If $\phi(\xi) = a_1\xi_1^2 + \cdots + a_n\xi_n^2$, then

$$\|u(x,t)\|_{L^\frac{2(a_n+1)}{a_n}(K^{n+1})} \leq C \|f_0(x)\|_{L^2(K^n),}.$$

**Theorem 8.** If $n = 1$ and $\phi(\xi) = \xi^d$, then

$$\|u(x,t)\|_{L^2(d+1)(K^2)} \leq C \|f_0(x)\|_{L^2(K)}.$$
In particular if $d = 3$, then

$$\|u(x, t)\|_{L^8(K^2)} \leq C \|f_0(x)\|_{L^2(K)}.$$  

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References


