Finiteness of Numbers of Curves on a Minimal Surface with $\kappa=1$

By

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§ 0. Introduction

In [4], Namikawa proposed the following problem.

Problem. Let $S$ be a minimal compact analytic surface. For an arbitrary curve (i.e. a reduced, irreducible effective divisor) $C$ on $S$, let us define the arithmetic genus of $C$ by

$$\pi(C) = \frac{1}{2} C(C + K_S) + 1,$$

where $K_S$ is the canonical divisor on $S$.

If we fix a non-negative integer $g$, how many algebraic families of curves of genus $g$ are there on $S$?

And he proved the finiteness of numbers of algebraic families of curves with the fixed arithmetic genus modulo $\text{Aut}(S)$ except the case where $\kappa(S)=1$. The purpose of this note is to prove that the above finiteness also holds in the case where $\kappa(S)=1$. We state our main theorem.

Theorem A. Let $S$ be a minimal analytic surface with $\kappa(S)=1$ and $g$ be a fixed non-negative integer. Then the number of algebraic families of curves on $S$ with the arithmetic genus $g$ is finite modulo $\text{Aut}(S)$. In particular, the number of non-singular rational curves on $S$ is finite modulo $\text{Aut}(S)$.

Notations and Convention

By a surface, we mean a compact complex manifold of dimension two. For a compact connected complex manifold $X$, we use the following notation.

$K_X$: the canonical bundle of $X$.

$\kappa(X)$: the Kodaira dimension of $X$.

$\text{Pic}(X) := H^1(X, \mathcal{O}_X^*)$: the Picard group of $X$, which has the natural structure of
a commutative complex Lie group.

NS(X): the Néron-Severi group of X, i.e. the group of algebraic equivalence classes of divisors on X.

\( \mathcal{O}_X \): the structure sheaf of X.

\( \rho(X) := \text{rank}_X NS(X) \): the Picard number of X.

\( \chi(\mathcal{O}_X) \): the Euler-Poincaré characteristic.

The symbol \( \approx (\text{resp. } \sim) \) indicates algebraic equivalence (resp. linear equivalence) of divisors.

§ 1. Preliminaries

In this section, we shall prove Theorem A in the case where S has the structure of the Jacobian fibration (i.e. elliptic fibration with a section). The following theorem is fundamental.

**Theorem 1.1.** (Miyaoka, Umezu [3]) Let S be a surface with \( \kappa(S) \geq 0 \) and fix a non-negative integer g. Then there exists a positive integer \( N = N(S, g) \) which is determined only by S and g such that \( C \cdot K_S < N \) for any curve C on S with the arithmetic genus g.

**Proposition 1.2.** Let \( f : S \to \Delta \) be a relatively minimal elliptic surface over a non-singular complete curve \( \Delta \) with the zero section (\( \emptyset \)) and at least one singular fiber. Assume that \( \kappa(S) = 1 \) and fix a non-negative integer g. Then the number of algebraic families of curves on S with the arithmetic genus g is finite modulo \( \text{Aut}(S) \).

**Corollary 1.3.** Under the same conditions as in Proposition 1.2, if the Mordell-Weil rank of S vanishes, the number of algebraic families of curves on S with the fixed arithmetic genus g is finite.

**Proof of Proposition 1.2.** Fix a non-negative integer g and take an arbitrary algebraic curve C on S with the arithmetic genus g. Since all sections are mapped to the zero section (\( \emptyset \)) by an automorphism of S and each irreducible component of a reducible singular fiber is a non-singular rational curve with self-intersection number -2, we may assume that C is a multi-section, that is, \( f|_C : C \to \Delta \) is a finite covering of degree \( m > 1 \).

Now let \( r \) be the rank of the Mordell-Weil group \( \mathcal{E}(K) \) and take \( r \) generators \( s_1, \ldots, s_r \) of \( \mathcal{E}(K) \) modulo the torsion group \( \mathcal{E}(K)_{\text{tor}} \). \( \mathcal{E}(K)_{\text{tor}} \) is generated by at most two elements \( t_1, t_2 \) of order \( e_1, e_2 \) with \( 1 \leq e_1, e_2 \leq e_1, e_2 \); \( |\mathcal{E}(K)_{\text{tor}}| = e_1 e_2 \). Let \( \{ F_\lambda \}_{\lambda \in \Delta} \) (resp. \( F \)) be the set of all singular fibers of \( f : S \to \Delta \) (resp. generic fiber of \( f \)) and for each \( \lambda \in \Delta \) we denote by \( \Theta_{\lambda, i} \) (0 \leq i \leq m_i - 1) the irreducible component of \( F_\lambda, m_i \) being the number of irreducible components. Then we have
\[ F_i = \theta_{i,i} + \sum_{k=1}^{m_i} \sum_{j=1}^{a_{i,j}} \mu_{i,j} \theta_{j,k}, \quad \mu_{i,j} \geq 0, \]

where \( \theta_{i,i} \) is the unique component of \( F_i \) intersecting the zero section \((o)\). By a theorem of Shioda [5], \( C \) can be written uniquely in the following way:

\[ C \approx \alpha(o) + \beta F + \sum_{i=1}^{r} a_i s_i + \sum_{j=1}^{r} b_j t_j + \sum_{k=1}^{m_i-1} d_{i,k} \theta_{j,k}, \]

\[ 0 \leq b_j < e_j, \text{ where } \alpha, \beta, a_i, b_j \text{ and } d_{i,k} \text{ are all rational integers.} \]

**Claim 1.** \( m := (C, F) \) is bounded above by a constant \( A = A(S, g) \), which is determined only by \( S \) and \( g \).

*Proof of Claim 1.* By the canonical bundle formula of Kodaira [2], we have \( K_S \approx (2\pi(\Delta) - 2 + \chi(O_S))F \). Since \( \kappa(S) = 1 \), we have \( 2\pi(\Delta) - 2 + \chi(O_S) > 0 \). Moreover by Theorem 1.1, there exists a constant \( N = N(S, g) \) depending only on \( S \) and \( g \) such that \( C \cdot K_S < N \). Put \( A(S, g) = \frac{N(S, g)}{2\pi(\Delta) - 2 + \chi(O_S)} \). Since \( C \cdot F = m \), we have \( m < A \). q.e.d.

**Claim 2.** In the above situation, by translation of \( C \) by a suitable automorphism \( \Phi \) of \( S \), we may assume that \( 0 \leq a_i < m \) for all \( 1 \leq i \leq r \).

*Proof of Claim 2.* We can consider \( S \) as a one-dimensional abelian variety \( E \) over the function field \( K \) of \( S \), given with a \( K \)-rational point \( o \). Then each section \( s_i \) (\( 1 \leq i \leq r \)) defines a \( K \)-rational point \( \bar{s}_i \) and let \( \sigma_i \) be a birational mapping of \( S \) induced from a translation of \( E \) by \( \bar{s}_i \) over \( K \). Since \( S \) is relatively minimal, \( \sigma_i \) is an analytic automorphism of \( S \). We choose integers \( c_i \) and \( \delta_i \) with \( a_i = m c_i + \delta_i, 0 \leq \delta_i < m \) and define an automorphism \( \Phi \) of \( S \) by \( \Phi = (\sigma_1)^{c_1} \cdots (\sigma_r)^{c_r} \). Put \( \bar{C} = \Phi(C) \). Clearly \( \bar{C} \) is also a \( m \)-section of \( S \), where we have

\[ m = C \cdot F = \alpha + \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j. \]

Put \( C' := (\alpha + m \sum_{i=1}^{r} c_i)(o) + \beta' F + \sum_{i=1}^{r} \delta_i s_i + \sum_{j=1}^{s} b_j t_j + \sum_{k=1}^{m_i-1} d_{i,k} \theta_{j,k}. \) The divisor \( \bar{C} \) (resp. \( C' \)) cuts out on the generic fiber \( F \) divisors of degree \( m \) and the sum \( \bar{S}(\bar{C}) \) (resp. \( S(C') \)) of points in \( \bar{C} \) (resp. \( C' \)) under the group operation on the abelian group \( F \) gives a \( K \)-rational point. By our construction, we get \( \bar{S}(\bar{C}) = S(C') \). With the aid of Abel's theorem on an elliptic curve, the divisor \( \bar{C} \mid F \) is linearly equivalent to \( C' \mid F \) on \( F \). Hence the divisor \( \bar{C} - C' \) is linearly equivalent to a divisor contained in fibers of \( f \) and \( \bar{C} \) can be written in the form:

\[ \bar{C} \approx (\alpha + m \sum_{i=1}^{r} c_i)(o) + \beta F + \sum_{i=1}^{r} \delta_i s_i + \sum_{j=1}^{s} b_j t_j + \sum_{k=1}^{m_i-1} d_{i,k} \theta_{j,k}, \]

where \( 0 \leq \delta_i < m \) for all \( 1 \leq i \leq r \). q.e.d.
Now, we continue the proof of Proposition 1.2. Since $m = C \cdot F = \alpha + \sum_{i=1}^{r} a_i + \sum_{j=1}^{\delta} b_j$, it follows from Claim 1 and Claim 2 that $\alpha$, $a_i(1 \leq i \leq r)$ and $b_j(1 \leq j \leq 2)$ are all bounded by constants determined only by $S$ and $g$. Since $C$ is a multi-section, we have $0 \leq C \cdot \Theta_{\lambda, k} \leq C \cdot F = m = A(S, g)$ for each $\lambda$ and $1 \leq k \leq m_2 - 1$. We have

$$(C, \Theta_{\lambda, p}) = \sum_{i=1}^{r} a_i(s_i, \Theta_{\lambda, p}) + \sum_{j=1}^{\delta} b_j(t_j, \Theta_{\lambda, p}) + \sum_{k=1}^{m_2 - 1} d_{\lambda, k}(\Theta_{\lambda, k}, \Theta_{\lambda, p}).$$

Since the intersection matrix $((\Theta_{\lambda, k}, \Theta_{\lambda, p}))_{1 \leq k, p \leq m_2 - 1}$ is negative definite and $(C, \Theta_{\lambda, p})$, $a_i$ and $b_j$ are bounded by constants determined only by $S$ and $g$, it follows from (*) that $d_{\lambda, k}$'s are uniquely determined and bounded by constants depending on $S$ and $g$. Setting $\Gamma = \alpha(\alpha) + \sum_{i=1}^{r} a_i s_i + \sum_{j=1}^{\delta} b_j t_j + \sum_{k=1}^{m_2 - 1} d_{\lambda, k} \Theta_{\lambda, k}$, we have $\langle \Gamma, F \rangle = m$ and $C^* = (\Gamma + \beta F)^* = \Gamma^* + 2\beta m$. Since $(C, K_S)$ and $\Gamma^*$ are bounded by constants determined by $S$ and $g$, we infer from the above equality that $\beta$ is also bounded by constants determined only by $S$ and $g$. q.e.d.

§ 2. Proof of Theorem A

In this section, we shall prove Theorem A in the case where $S$ has the structure of an elliptic surface with multiple fibers. First, we need the following lemma.

**Lemma 2.1.** Let $g : S \to C$ be a relatively minimal algebraic elliptic surface with multiple fibers. Then there exists a curve $s_0$ with $g(s_0) = C$ such that $(s_0, F)$ divides $(D, F)$ for general fiber $F$ of $g$ and for all curve $D$ with $g(D) = C$.

**Proof.** Put $m_0 = \min \{D \cdot F | D \text{ is a curve on } S \text{ with } g(D) = C\}$ and fix a curve $s_0$ with $s_0 \cdot F = m_0$ and $g(s_0) = C$ arbitrarily. For any curve $D$ on $S$ with $g(D) = C$, we choose integers $t, r$ with $D \cdot F = m_0 + r$, $0 \leq r < m_0$. Then $0 \leq (D - m_0 s_0, F) = r < m_0$ and $g_* \mathcal{O}_S(D - m_0 s_0)$ is a coherent $\mathcal{O}_C$-module of rank $r$. By Serre's theorem, we have $H^n(C, g_* \mathcal{O}_S(D - m_0 s_0) \otimes \mathcal{O}_C(n \mathcal{P})) = 0$, $\mathcal{P} \subset C$ and hence $H^n(S, \mathcal{O}_S(D - m_0 s_0 + n S_0)) = 0$, $S_0 = g^{-1}(\mathcal{P})$ for a sufficiently positive integer $n$. A general member $\Delta \subset |D - m_0 s_0 + n S_0|$ can be written in the form: $\Delta = H + \Gamma$, where $H$ and $\Gamma$ are effective divisors with $g(*) = \mathcal{C}$ and $\mathcal{C}$ contained in fibers of $g$. Then $0 \leq r = \langle \Delta, F \rangle = \langle H, F \rangle < m_0$ and from the minimality of $(s_0, F)$, we have $H = 0$ and $r = 0$. q.e.d.

Now, we are ready to prove our main Theorem A.

**Proof of Theorem A.** Clearly, we may assume that $S$ is algebraic. Since $\kappa(S) = 1$, $S$ has the unique structure of an elliptic surface $f : S \to C$ with multiple fibers $m_i E_i$ of multiplicity $m_i$ at $p_i \in C$. $(1 \leq i \leq l)$. Then by Kodaira [2], $S$ can
be obtained from the basic member $g : B \to C$ by twisting and successive logarithmic transformations, that is, $S = L_{\rho_1} \cdots L_{\rho_k}(B^\eta)$, where $B^\eta$ is obtained by twisting $g : B \to C$ by $\eta \in H^1(C, \mathcal{O}(B^\eta))$.

First, we assume throughout that $g : B \to C$ is not smooth, i.e., there is at least one singular fiber.

Let $r$ be the rank of the Mordell-Weil group $\text{MW}(B)$ of $g : B \to C$ and take $r$ generators $s_1, \ldots, s_r$ of $\text{MW}(B)$ modulo the torsion group $\text{MW}(B)_{\text{tor}}$. $\text{MW}(B)_{\text{tor}}$ is generated by at most two elements $t_1, t_2$ of order $e_1, e_2$ with $e_2 | e_1$; $|\text{MW}(B)_{\text{tor}}| = e_1 e_2$.

By considering $B$ as an one-dimensional abelian variety $E$ over the function field $K$ of $C$, each section $s_i$ (resp. $t_j$) defines a $K$-rational point $\bar{s}_i$ (resp. $\bar{t}_j$). Since $S$ is isomorphic to $B^\eta$ outside the multiple fibers, $\bar{s}_i$ (resp. $\bar{t}_j$) induces an automorphism of $S|_{C \setminus \{p_i\}}$ by locally translating $B^\eta$ by $\bar{s}_i$ (resp. $\bar{t}_j$). And from the definition of logarithmic transformations, we see easily that it can be extended to an automorphism $g_i$ (resp. $h_j$) of $S$. We fix a multi-section $D_0$ of $f : S \to C$ which enjoys the properties as in Lemma 2.1 and put $D_i = g_i(D_0)$, $T_j = h_j(D_0)$. Clearly $D_i$'s and $T_j$'s also enjoys the property in Lemma 2.1. Let $\{F_\lambda\}_{\lambda \in C}$ (resp. $F$) be the set of all singular fibers of $f : S \to C$ (resp. general fiber of $f$). For each non-multiple singular fiber $F_\lambda$ (resp. multiple singular fiber $m_\lambda E_\lambda$), choose an irreducible component $\Theta_{\lambda,0}$ of $F_\lambda$ (resp. $E_\lambda$) of multiplicity one arbitrarily and fix them. Then we have

$$F_\lambda = \Theta_{\lambda,0} + \sum_{i=1}^{n_\lambda} \mu_{\lambda,i} \Theta_{\lambda,i}, \quad \mu_{\lambda,i} > 0$$

where we denote by $\Theta_{\lambda,i}$ ($0 \leq i \leq n_\lambda - 1$) the irreducible component of $F_\lambda$ (resp. $E_\lambda$), $n_\lambda$ being the number of the irreducible components of $F_\lambda$ (resp. $E_\lambda$).

Claim 1. $[D_0], [F], [D_i]$ (for $i \leq r$) and $[\Theta_{\lambda,i}]$'s ($\lambda \in C$, $1 \leq i \leq n_\lambda-1$) are basis of $\text{NS}(S) \otimes \mathbb{Q}$.

Proof of Claim 1. For an arbitrary divisor $D$ on $S$, put $d = (D, F)$, $d_0 = (D_0, F)$. By Lemma 2.1, there exists a positive integer $k$ with $d = d_0 k$. Since $(D - kD_0, F) = 0$, the divisor $D - kD_0$ cuts out on the generic fiber $F$ a divisor $\delta$ of degree $0$. Since $S$ is isomorphic to $B^\eta$ outside the multiple fibers, the sum $S(\delta)$ of points in $\delta$ under the group operation of $\text{MW}(B)$ over $\mathbb{Q}$ gives a section of $g : B|_{\mathbb{P}^1 \setminus \{p_i\}} \to C\setminus \{p_i\}$. By Kodaira [3], it can be extended to a holomorphic section $s$ of $g : B \to C$ over $C$, since $B$ is a basic member. By the Mordell-Weil theorem, we can write:

$$s = \sum_{i=1}^{r} a_i \bar{s}_i + \sum_{j=1}^{g} b_j \bar{t}_j,$$

where $a_i$ and $0 \leq b_j < e_j$ are integers.

Put $D' = \sum_{i=1}^{r} a_i (D_i - D_0) + \sum_{j=1}^{g} b_j (T_j - T_0)$. Since $S((D_i - D_0) \cdot F) = d_0 s_i$ (resp. $S((T_j - T_0) \cdot F) = d_0 t_j$), we have

$$s = \sum_{i=1}^{r} a_i \bar{s}_i + \sum_{j=1}^{g} b_j \bar{t}_j,$$

where $a_i$ and $0 \leq b_j < e_j$ are integers.
we see that $S(d_0(D - kD_0) \cdot F) = S(D' \cdot F)$ (resp. $S(e_j(T_j - D_0) \cdot F) = 0$) for a generic fiber $F$ of $f$.

By Abel's theorem on an elliptic curve, the divisor $d_0(D - kD_0) |_F$ (resp. $e_j(T_j - D_0) |_F$) is linearly equivalent to $D' |_F$ (resp. $0$) on $F$. Hence the divisor $d_0(D - kD_0) - D'$ (resp. $e_j(T_j - D_0)$) is linearly equivalent to a divisor contained in fibers of $f$. Hence we have:

$$(*) \quad \text{med}_o D \approx \text{med} D_0 + \beta F + \sum_{i=1}^{r} a_i (D_i - D_0) + \sum_{k=1}^{n_0} \sum_{z=1}^{n_k-1} d_{1,k} \Theta_{1,k},$$

where $m$ (resp. $a$) is the least common multiple of $m_1, \ldots, m_i$ (resp. $a_1, a_2$) and $\beta$, $a_i$'s and $d_{1,k}$'s are all integers. And it is easy to see that $D_0, F$, $D_i$'s and $\Theta_{1,k}$'s are linearly independent in $\text{NS}(S) \otimes \mathbb{Q}$. Hence the claim follows. q.e.d.

Now, let $D$ be an arbitrary curve on $S$ with the fixed arithmetic genus $g$. We may assume that $f(D) = C$.

**Claim 2.** $d := (D, F) \in \mathbb{Z}_{\geq 0}$ is bounded above by a constant $A = A(S, g)$, which is determined only by $S$ and $g$.

**Proof of Claim 2.** By the canonical bundle formula of Kodaira [2],

$$K_S \approx (2\pi(C) - 2 + \mathcal{O}_S)F + \sum_{j=1}^{r} (m_j - 1)E_j.$$  

Since $\kappa(S) = 1$,

$$M(S) := 2\pi(C) - 2 + \mathcal{O}_S + \sum_{j=1}^{r} \frac{m_j - 1}{m_j} > 0.$$  

By Theorem 1.1, there exists a constant $N = N(S, g)$ depending only on $S$ and $g$ with $D \cdot K_S < N$. If we put $A = A(S, g) = \frac{N(S, g)}{M(S)}$, we have $d < A$. q.e.d.

**Claim 3.** By translating $D$ by a suitable automorphism $\rho$ of $S$, we may assume that $0 \leq a_i < \text{med}$ for all $i$ in $(*)$, Claim 1.

**Proof of Claim 3.** By the proof of Claim 2, we have:

$$d_0 D \approx dD_0 + \beta' F + \sum_{i=1}^{r} \delta_i c_i + \sum_{i=1}^{r} a'_i (D_i - D_0) + \sum_{j=1}^{r} b'_j (T_j - T_0) + \sum_{k=1}^{n_0} \sum_{z=1}^{n_k-1} d'_{1,k} \Theta_{1,k},$$

where $\beta'$, $\delta_i$, $a'_i$, $b'_j$ and $d'_{1,k}$ are all integers and $0 \leq \beta' < m_i$. We choose integers $c_i$ and $\delta_i$ with $a'_i = d c_i + \delta_i$, $0 \leq \delta_i < d$, and define an automorphism $\rho$ of $S$ by $\rho := (g_1^{-s_1} \cdots g_r^{-s_r})^{c_i}$. Then we have $\langle \rho(D), F \rangle = d$.

Next, put $D' = dD_0 + \beta' F + \sum_{i=1}^{r} \delta c_i (D_i - D_0) + \sum_{j=1}^{r} b'_j (T_j - T_0)$. The divisor $D' - dD_0$ (resp. $d_0 (D - kD_0)$) cuts out on the generic fiber $F$ a divisor $G'$ (resp. $G$) of degree 0 and the sum $S(G')$ (resp. $S(G)$) of points in $G'$ (resp. $G$) under the
group operation gives a section of \( g : B \to C \). Since \( S(D_1 - D_0) = d \delta_s \), we have:

\[
S(G) = \sum_{i=1}^{r} a_i (d \delta_s) - \sum_{i=1}^{r} \delta_i (d \delta s) = d \sum_{i=1}^{r} (a_i - \delta_i d) s_i = \sum_{i=1}^{r} \delta_i (d \delta s) = S(G').
\]

Hence the divisor \( d \delta - D' = (d \delta - dD_0) - (D' - dD_0) \) is algebraically equivalent to a linear combination of \( F, E_i \)'s and \( \Theta_{\lambda, k} \)'s over \( \mathbb{Z} \). Since \( 0 \leq \delta_i < d \), Claim 3 follows immediately by the same argument as in the proof of Claim 1. \( \text{q.e.d.} \)

Now, we continue the proof of Theorem A. We use the same notation as before. By Claim 2 and Claim 3, we have

\[ 0 \leq a_i < meA(S, g) \quad \text{for all} \quad 1 \leq i \leq r. \]

Since \( D_0 \) and \( D \) are multi-sections of \( f \), we have:

\[
0 \leq (D, \Theta_{\lambda, p}) \leq (D, F) = d < A(S, g)
\]

and

\[
0 \leq (D_0, \Theta_{\lambda, p}) \leq (D_0, F) = d_0
\]

for each \( \lambda \) and \( 1 \leq k \leq n_{\lambda} - 1 \). We have:

\[(**): \quad \text{med}(D, \Theta_{\lambda, p}) = \text{med}(D_0, \Theta_{\lambda, p}) + \sum_{i=1}^{r} a_i (D_1 - D_0, \Theta_{\lambda, p}) + \sum_{k=1}^{n_{\lambda} - 1} d_{\lambda, k} (\Theta_{\lambda, k}, \Theta_{\lambda, p}),\]

since the intersection matrix \((\Theta_{\lambda, k}, \Theta_{\lambda, p})_{1 \leq k, p \leq n_{\lambda} - 1}\) is negative definite and \((D, \Theta_{\lambda, p}), (D_0, \Theta_{\lambda, p})\) and \( a_i \)'s \((1 \leq i \leq r)\) are all bounded by constants determined only by \( S \) and \( g \), \( d_{\lambda, k} \)'s are also bounded by constants determined by \( S \) and \( g \).

Setting \( \Gamma = \text{med}D_0 + \sum_{i=1}^{r} a_i (D_1 - D_0) + \sum_{k=1}^{n_{\lambda} - 1} d_{\lambda, k} \Theta_{\lambda, k} \), we have \( \langle \text{med}D \rangle^2 = (\Gamma + \beta F)^s = \Gamma^s + 2\beta \text{med} \). Since \((D, K_S)\) and \( \Gamma^s \) are all bounded by constants determined by \( S \) and \( g \), we infer from the above equality that \( \beta \) is also bounded by constants determined only by \( S \) and \( g \). Hence if we express \( D \) as a linear combination of \( D_0, F, D_i \) \((1 \leq i \leq r)\) and \( \Theta_{\lambda, k} \) \((1 \leq k \leq n_{\lambda} - 1)\) in \( NS(S) \otimes \mathbb{Q} \), the number of the possible values for each coefficient is bounded by \( S \) and \( g \) and Theorem A has been proved.

Next, we consider the case where \( f : S \to C \) is a Seifert fiber space, that is, \( S \) is an elliptic surface with constant moduli which has at most multiple singular fibers. Let \( D \) be an arbitrary curve with the fixed arithmetic genus \( g \). We may assume that \( f(D) = C \). Then by completely the same method as above, we can show that by translating \( D \) by a suitable automorphism \( \rho \) of \( S \), we may assume that:

\[ mD \approx aD_0 + \beta F, \]

where \( m \) is the least common multiple of \( m_i \)'s and \( \alpha \) and \( \beta \)
are integers. Hence we can easily show the boundedness of $\alpha$ and $\beta$ by Miya-
oka's Theorem 1.1.

Thus we have finished the proof of Theorem A.

References

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[4] Namikawa, Y., Finiteness of numbers of curves on a surface, Symposium on alge-