A Generalization of Vanishing Theorems for Weakly 1-Complete Manifolds

By

Kensho TAKEGOSHI*

§ 1. Introduction

Let $X$ be a connected complex manifold of complex dimension $n$. $X$ is called weakly 1-complete if there exists an exhaustion function $\Phi$ on $X$ which is $C^\infty$ and plurisubharmonic. In [9] S. Nakano established the following.

**Theorem 1.** Let $B$ be a positive line bundle on a weakly 1-complete manifold $X$, then

$$H^p(X, \mathcal{O}^q(B)) = 0 \quad \text{for} \quad p + q > n.$$

Recently, O. Abdelkader obtained

**Theorem 2** (cf. [1]). *Let $B$ be a semi-positive line bundle over a weakly 1-complete Kähler manifold $X$ and assume that the curvature form of $B$ has at least $n - k + 1$ positive eigenvalues, then

$$H^p(X_c, \mathcal{O}^q(B)) = 0 \quad \text{for any real number} \ c \ \text{with} \ p + q \geq n + k,$$

where $X_c = \{x \in X | \Phi(x) < c\}$.***

In these theorems, the positivity of eigenvalues of the curvature form of $B$ is assumed on the whole space $X$. In this paper, we shall prove that these vanishing theorems still hold, if the positivity of eigenvalues of the curvature admits a compact exceptional subset $K \subseteq X$. We shall prove the following.

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* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.
Main Theorem. Let $B$ be a semi-positive line bundle over a connected weakly 1-complete Kähler manifold $X$ with a metric along the fibres such that its curvature form has at least $n-q+1$ positive eigenvalues on $X\setminus K$, where $K$ is a proper compact subset of $X$. Then

$$H^p(X, \mathcal{O}(B\otimes K_X)) = 0$$

for any $p\geq q$, where $K_X$ is the canonical line bundle of $X$.

In particular, when $q=1$, we obtain

Corollary. Let $X$ be a connected weakly 1-complete Kähler manifold and let $B$ be a semi-positive line bundle on $X$ which is positive on $X\setminus K$ for some proper compact subset $K$ of $X$. Then

$$H^p(X, \mathcal{O}(B\otimes K_X)) = 0$$

for any $p\geq 1$.

Since a positive line bundle over a complex manifold induces a Kähler metric on it, this is not only a direct generalization of Theorem 1 for $q=n$ but also a generalization of the vanishing theorems for the semi-positive line bundle on 1-convex Kähler manifolds and compact Kähler manifolds by Grauert and Riemenschneider (cf. [4], [11]).

This work is inspired by Ohsawa’s article [10] and the author would like to express his hearty thanks to Dr. A. Fujiki and Professor S. Nakano for their kind advices and encouragement during the preparation of this paper.

§ 2. Notations and Definitions

We denote by $X$ a connected paracompact complex manifold of dimension $n$. Let $\pi: F\to X$ be a holomorphic line bundle over $X$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of $X$ by coordinate neighborhoods such that on each $U_i$, $F|_{U_i}$ is isomorphic to the trivial line bundle. We denote local coordinates on $U_i$ by $(z^1, \cdots, z^n)$. If $\Phi_i: U_i \times \mathbb{C} \to F|_{U_i}$ ($i \in I$) are these trivializations of $F$, we denote by $f_{ij}: U_i \cap U_j \to \mathbb{C}^*$ the system of transition functions defined by the conditions:

$$\Phi_j^{-1} \circ \Phi_i(z_i, \xi_i) = (z_i, f_{ij}(z_i) \xi_j)$$
where \( \xi_i \) denotes the fibre coordinates over \( U_i \).

An \( F \)-valued differential form \( \varphi \) on \( X \) is a system \( \{ \varphi_i \}_{i \in I} \) of differential forms defined on \( U_i \), satisfying \( \varphi_i = f_{ij} \varphi_j \) in \( U_i \cap U_j \). We denote by \( C^{p,q}(X,F) \) the space of \( F \)-valued differential forms on \( X \), of class \( C^\infty \) and of type \( (p,q) \), and by \( C^{p,q}_c(X,F) \) the space of the forms in \( C^{p,q}(X,F) \) with compact supports.

Let \( ds^2 = \sum_{a,\beta=1}^n g_{i,a\beta} \, dz^a_i \cdot dz^\beta_i \) be a hermitian metric on \( X \) and let \( \{a_i\} \) be a hermitian metric along the fibres of \( F \), that is, a system of positive valued function \( a_i \) in \( U_i \) satisfying \( |f_{ij}| = a_i \cdot a_j^{-1} \) in \( U_i \cap U_j \).

**Remark.** In this paper, we use the notation of a system of metrics along the fibres in the sense of Kodaira [7], page 1268, (1).

For \( \varphi, \psi \in C^{p,q}(X,F) \), we set

\[
\langle \varphi, \psi \rangle = a_{-1}^{-1} \sum_{\alpha, \beta} \varphi_{i\alpha, j\beta} \cdot \overline{\psi^i_{i\alpha, j\beta}},
\]

where \( \varphi_i = \sum_{\alpha, \beta} \varphi_{i\alpha, j\beta} \, dz^i_{\alpha} \cdot dz^\beta_{i} \) and \( A_p = (\alpha_1, \ldots, \alpha_p) \) and \( B_q = (\beta_1, \ldots, \beta_q) \) run through the sets of multi-indices with \( 1 \leq \alpha_i \leq \cdots \leq \alpha_p \leq n \) and \( 1 \leq \beta_1 < \cdots < \beta_q \leq n \) respectively. Then

\[
a^{-1}_i \varphi_i \wedge \overline{\psi_i} = \langle \varphi, \psi \rangle \, dV
\]

where \( \ast \) is the star operator and \( dV \) is the volume element with respect to the metric \( ds^2 \).

If either \( \varphi \) or \( \psi \in C^{p,q}_c(X,F) \), we define

\[
(\varphi, \psi)_r = \int_X \langle \varphi, \psi \rangle e^{-r} \, dV
\]

for any real-valued \( C^\infty \)-function \( \Psi \).

In particular we set

\[
(\varphi, \psi)_0 = (\varphi, \psi)_{r}
\]

and

\[
\| \varphi \|_r^2 = (\varphi, \varphi)_r
\]

\[
\| \varphi \|^2 = (\varphi, \psi).
\]

We have the operator \( \overline{\partial} : C^{p,q}(X,F) \rightarrow C^{p,q+1}(X,F) \) defined by \( (\overline{\partial} \varphi)_i \),
With respect to (2.1) and (2.2), the formal adjoint operator of \( \overline{\partial} \) are defined, we denote them by \( \partial^* \) and \( \overline{\partial}^* \) respectively. We denote by \( L^{p,q}(X,F,\mathcal{F}) \) (resp. \( L^{p,q}(X,F) \)) the space of the measurable \( F \)-valued forms \( \varphi \) of type \((p,q)\), square integrable in the sense that \( \|\varphi\|_{p,q}^2<\infty \) (resp. \( \|\varphi\|^2<\infty \)). Then, they are Hilbert spaces with respect to the inner product \((\varphi,\psi)\) (resp. \((\varphi,\psi)\)). We denote again by \( \overline{\partial} \) the operator from \( L^{p,q}(X,F,\mathcal{F}) \) to \( L^{p,q+1}(X,F,\mathcal{F}) \) extending the original \( \overline{\partial} \); thus a form \( \varphi \in L^{p,q}(X,F,\mathcal{F}) \) is in the domain of \( \overline{\partial} \) if and only if \( \overline{\partial}\varphi \), defined in the sense of distribution, belongs to \( L^{p,q-1}(X,F,\mathcal{F}) \). Then \( \overline{\partial} \) is a closed, densely defined operator, so the adjoint operator \( \overline{\partial}^* \) (resp. \( \partial^* \)) can be defined. We denote the domain, range and nullity of \( \overline{\partial} \) in \( L^{p,q}(X,F,\mathcal{F}) \) by \( D_{\overline{\partial}}^{p,q} \), \( R_{\overline{\partial}}^{p,q} \) and \( N_{\overline{\partial}}^{p,q} \) respectively. \( D_{\partial}^{p,q} \), \( R_{\partial}^{p,q} \) and \( N_{\partial}^{p,q} \) are defined similarly.

**Definition 2.1.** \( X \) is called weakly 1-complete if there exists a \( C^\infty \)-plurisubharmonic function \( \Phi \) on \( X \) such that for any real number \( c \), \( X_c = \{ x \in X | \Phi(x) < c \} \) is relatively compact in \( X \).

**Remark 2.1.** Let \( \lambda(t) : (\infty, \infty) \rightarrow (-\infty, \infty) \) be a \( C^\infty \)-increasing convex function such that \( \lambda(t) = 0 \) for \( t \leq 0 \), then the composition \( \lambda(\Phi) \) is again \( C^\infty \)-plurisubharmonic and exhausts \( X \). So we may assume that \( \Phi \) is non-negative on \( X \). Then, for any \( c \in (0, \infty) \), \( X_c = \{ x \in X | \Phi(x) < c \} \) is weakly 1-complete with respect to the exhaustion function \( \frac{1}{c - \Phi} \).

**Remark 2.2.** Any connected compact complex manifold is weakly 1-complete, any real constant function being taken as the exhaustion function.

**Definition 2.2.** A holomorphic line bundle \( \pi : F \rightarrow X \) is said to be positive (resp. semi-positive) on a subset \( Y \subset X \), if there exist a coordinate cover \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( X \) such that \( \pi^{-1}(U_i) \) are trivial and a metric \( \{ a_i \} \) along the fibres of \( F \) such that

\[
(2.4) \quad \left( \frac{\partial^4 \log a_i}{\partial \bar{z}_i^a \partial z_i^a} \right) > 0 \text{ (resp. } \geq 0) \text{ on } U_i \cap Y \text{ for every } i \in I.
\]
Definition 2.3. A holomorphic line bundle \( \pi: F \rightarrow X \) is said to be \( q \)-semi-positive \( (1 \leq q \leq n) \) on a subset \( Y \subset X \), if \( F \) is semi-positive on \( Y \) and the hermitian matrix (2.4) has at least \( n - q + 1 \) positive eigenvalues at each point of \( Y \).

§ 3. A Formulation of \( L^2 \)-Estimates and Existence Theorems for the \( \bar{\partial} \) Operator

Let \( X \) be a paracompact complex manifold of dimension \( n \) which is not necessarily connected.

Theorem 3.1. Let \( F \) be a holomorphic line bundle over \( X \). If there exist in the degree \( (p, q) \)

\( (3.1) \) a complete hermitian metric \( ds^2 \) on \( X \),

\( (3.2) \) a hermitian metric \( \{a_i\} \) along the fibres of \( F \),

\( (3.3) \) a constant \( C_1 > 0 \)

and

\( (3.4) \) a compact subset \( K \) of \( X \) which does not contain any connected component of \( X \), such that

\( (3.5) \) \[ \|\varphi\|_{C^0}^2 \leq C_1 \left\{ \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 \right\} \quad \text{for any } \varphi \in D_{p,q} \cap D_{q,p}^*. \]

Then, there exists a constant \( C_2 > 0 \) such that

\( (3.6) \) \[ \|\varphi\|^2 \leq C_2 \left\{ \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 \right\} \quad \text{for any } \varphi \in D_{p,q}^* \cap D_{q,p}^*. \]

Proof. Take any sequence \( \{\varphi_m\} \) such that \( \varphi_m \in D_{p,q}^* \cap D_{q,p}^* \), \( \|\varphi_m\|^2 \leq 1 \), \( \lim_m \|\bar{\partial}\varphi_m\|^2 = 0 \) and \( \lim_m \|\bar{\partial}^*\varphi_m\|^2 = 0 \). Then we assert that there exists a subsequence \( \{\varphi_{m_i}\} \) of \( \{\varphi_m\} \) which converges strongly on \( X \). Since \( ds^2 \) is complete, \( C_{k,q}^*(X, F) \) is dense in \( D_{p,q}^* \cap D_{q,p}^* \) with respect to the norm

\[ (\bar{\partial}\varphi, \bar{\partial}\varphi) + (\bar{\partial}^*\varphi, \bar{\partial}^*\varphi) + (\varphi, \varphi) \]

([12], Theorem 1.1). Hence we may assume \( \varphi_m \in C_{k,q}^*(X, F) \). Therefore we obtain that

\[ (\bar{\partial}\varphi_m, \bar{\partial}\varphi_m) + (\bar{\partial}^*\varphi_m, \bar{\partial}^*\varphi_m) \leq (\varphi_m, \varphi_m) \]

\[ = (\bar{\partial}\varphi + \bar{\partial}\varphi)(\varphi_m, \varphi_m) + (\varphi_m, \varphi_m) \]
is bounded by the assumption. Since $\bar{\partial} + \partial \bar{\partial}$ is an elliptic differential operator of order 2, this means that $(\varphi_n)_i$ and their first derivatives with respect to the coordinate of $U_i$ are bounded in the sense of the integral $\| \cdot \|_{K'}$, where $K'$ is a compact subset of $X$ with $K \subset \text{Int } K'$ (see for example [3], (2.2.1) Theorem). Combining this with Rellich's lemma (see for example [3], Appendix), it follows that $(\varphi_n)$ has a subsequence $(\varphi_{n_k})$ which is strongly convergent on compact subsets. By (3.5), we conclude that $(\varphi_{n_k})$ converges strongly on $X$. Therefore, by Hörmander [5] Theorem 1.1.2 and Theorem 1.1.3, there exists a positive constant $C_1$ such that

$$(3.7) \quad \| \varphi \|^2 \leq C_1 \{ \| \bar{\partial} \varphi \|^2 + \| \partial \varphi \|^2 \}$$

for any $\varphi \in D^p_q \cap D^p_q$ with $\varphi \perp N^p_q = N^p_q \cap N^p_q$.

By the same theorems we obtain the following strong orthogonal decomposition:

$$(3.8) \quad L^p_q(X,F) = R^p_q \oplus N^p_q \oplus R^p_q.$$ 

Each element $\varphi$ in $N^p_q$ is a solution of the Laplace-Beltrami operator $\Box = \bar{\partial} + \partial \bar{\partial}$ with respect to (3.1) and (3.2). Now we refer to the unique continuation theorem for harmonic forms with values in a hermitian vector bundle.

**Theorem 3.2** (Aronszajn [2], Riemenschneider [11]). *Let $E$ be a hermitian vector bundle over a connected complex hermitian manifold $X$. Then a harmonic form $\varphi \in \mathcal{H}^p_q(E)$ vanishes identically on $X$ if it vanishes on a non-empty open subset $U$ of $X$.*

Any form $\varphi$ in $N^p_q$ vanishes on the open subset $X \setminus K$ by (3.5). Since each connected component of $X$ is not contained in $K$ by the assumption, from Theorem 3.2, $\varphi$ vanishes identically on each connected component. Hence $\varphi$ vanishes identically on $X$. Therefore $N^p_q$ is the null space. Combining this with (3.7), our theorem follows. q.e.d.

From the above theorem, we obtain (cf. [5], Theorem 1.1.4)

**Corollary 3.1.** *Let $X$, $F$ and others be as above. Let $\varphi \in L^p_q$
(X, F) satisfy the equation \( \bar{\partial} \varphi = 0 \), then there exists a \( \varphi \in L^{p,q-1}(X, F) \) such that \( \bar{\partial} \varphi = \varphi \). Moreover, if \( \varphi \in C^{p,q}(X, F) \), then \( \varphi \) can be taken from \( C^{p,q-1}(X, F) \) (cf. [6], p. 115, Theorem 5.2.5).

§ 4. The Basic Estimate

Let \( X \) be a connected paracompact complex manifold of dimension \( n \) and let \( \pi : B \to X \) be a holomorphic line bundle over \( X \). Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be a coordinate cover of \( X \) such that \( \pi^{-1}(U_i) \) are trivial and let \( \{ a_i \} \) be a hermitian metric along the fibres of \( B \) with respect to \( \mathcal{U} \). We set

\[
(4.1) \quad \Gamma_{i,a} = \frac{\partial^2 \log a_i}{\partial \bar{z}_i^a \partial z_i^a}.
\]

We assume that \( X \) is provided with a Kähler metric

\[
(4.2) \quad ds^2 = \sum_{i,j=1}^{n} g_{i,a} d\bar{z}_i^a \cdot dz_j^b.
\]

The canonical line bundle \( K_X \) of \( X \) is defined by a system of transition functions \( \{ K_x, \varphi \} \) on \( U_i \cap U_j \), where \( K_x, \varphi = \frac{\partial (z_j^1, \ldots, z_j^n)}{\partial (z_i^1, \ldots, z_i^n)} \). Then we see that

\[
(4.3) \quad |K_x, \varphi|^2 = g_i \cdot g_j^{-1} \quad \text{on} \quad U_i \cap U_j,
\]

where

\[
(4.4) \quad g_i = \det (g_{i,a}^a).
\]

Hence \( \{ g_i \} \) determines a metric along the fibres of \( K_X \). Then \( \{ A_i \} \) defined by

\[
(4.5) \quad A_i = a_i \cdot g_i
\]

determines a metric of \( B \otimes K_X \).

With the notations (2.1), (2.2) and (2.3), the following inequality has been shown by K. Kodaira (cf. [7], pp. 1269-1270).

\[
(4.6) \quad \int_X \frac{1}{A_i} \sum_{j=1}^{n} \sum_{x_{l-1}}^{n} \Gamma_{i,x_{l-1}} \varphi_i^{x_{l-1}} \cdot \varphi_j^{x_{l-1}} \cdot dV \leq \| \bar{\partial} \varphi \| ^2 + \| \bar{\partial} \varphi \| ^2
\]

for any \( \varphi \in C^p_q (X, B \otimes K_X) \) with \( p \geq 1 \).

From now on, we let \( X \) be a connected Kähler manifold, weakly 1-complete with respect to an exhaustion function \( \varphi \) and let \( \pi : B \to X \) be
a holomorphic line bundle which is semi-positive on $X$ and $q$-semi-positive on $X \setminus K$ for some proper compact subset $K$ of $X$. We fix a constant $c > c_\phi = \sup_{x \in K} \phi(x)$. Then $X_c = \{ x \in X : \phi(x) < c \}$ is weakly $1$-complete with respect to the exhaustion function $\frac{1}{c - \phi}$.

We take a Kähler metric

$$d\sigma_0^2 = \sum_{a, \bar{b}, \alpha} g_{i_a \bar{a}, \beta} dz_i^a \cdot d\bar{z}_{i}^{\bar{b}}$$
on $X$. We set

$$G_{i,0} = (g_{i, a \bar{b}, 0}).$$

Let $\{a_{i,0}\}$ be a fibre metric of $B$ which corresponds to the assumption and we set

$$\Gamma_{i,0} = (\Gamma_{i, a \bar{b}, 0}) \quad \text{where} \quad \Gamma_{i, a \bar{b}, 0} = \frac{\partial^q \log a_{i,0}}{\partial \bar{z}_i^a \partial z_i^b}.$$ We can assume that $\inf_{x \in X} \phi(x) = 0$. Then we take a $C^\infty$ increasing convex function $\lambda(t)$ such that

$$\lambda(t) : ( -\infty, \infty ) \rightarrow ( -\infty, \infty ),$$

i) $\lambda(t) = 0$ if $t \leq \frac{1}{c}$,

ii) $\lambda(t) > 0$ if $t > \frac{1}{c}$,

iii) $\int_0^{+\infty} \sqrt{\lambda''(t)} \, dt = +\infty$.

We replace the metric along the fibers of $B$ by

$$a_i = a_{i,0} \cdot \exp (\Psi) \quad \text{where} \quad \Psi = \lambda \left( \frac{1}{c - \phi} \right).$$

We set

$$\Gamma_i = (\Gamma_{i, a \bar{b}}) \quad \text{where} \quad \Gamma_{i, a \bar{b}} = \frac{\partial^q \log a_i}{\partial \bar{z}_i^a \partial z_i^b}.$$ Then we have

$$\Gamma_i \geq \Gamma_{i,0}.$$
We define a Kähler metric $ds^2$ by

$$ds^2 = \sum_{\alpha, \beta=1}^{\infty} (g_{\alpha, \beta, 0} + \Gamma_{\alpha, \beta}) \, dz_\alpha \cdot dz_\beta.$$  

**Remark.** By the choice of $\lambda$ as in (4.9) iii), $ds^2$ is a complete Kähler metric on $X_c$ (cf. [8], Proposition 1).

We set

$$G_i = (g_{\alpha, \beta}) \quad \text{where} \quad g_{\alpha, \beta} = g_{\alpha, \beta, 0} + \Gamma_{\alpha, \beta}.$$  

We replace the metric along the fibres of $B \otimes K_X$ by

$$A_i = a_i \cdot g_i \quad \text{where} \quad g_i = \det G_i.$$  

We replace (4.1), (4.2) and (4.5) by (4.11), (4.13) and (4.14), then from (4.6) we obtain

$$\int_{X_c} \frac{1}{A_i} \sum_{\beta, \gamma=1}^{\infty} \sum_{\tau=1}^{n} (\sum_{\beta=1}^{\infty} g_{\alpha, \beta} \cdot \Gamma_{\alpha, \beta}) \varphi_{\alpha, \beta} \cdot \varphi_{\beta, \gamma} \cdot dV \leq \|\bar{\partial} \varphi\|^2 + \|\partial^\star \varphi\|^2$$

for any $\varphi \in C_c^\infty(X_c, B \otimes K_X)$ with $p \geq 1$.

We rewrite the left hand side as

$$\int_{X_c} \frac{1}{A_i} \sum_{\beta, \gamma=1}^{\infty} \sum_{\tau=1}^{n} (\sum_{\beta=1}^{\infty} g_{\alpha, \beta} \cdot \Gamma_{\alpha, \beta}) \varphi_{\alpha, \beta} \cdot \varphi_{\beta, \gamma} \cdot dV.$$  

We can choose a matrix $T_i$ which depends, together with $T_i^{-1}$, differentiably on $x \in U_i$, satisfying $G_i, 0 = \Gamma_{i, 0}$. Since $G_i = G_i, 0 + \Gamma_i$, we have $G_i = T_i \{E + \Gamma_{i, 1} \cdot \Gamma_i, \cdot T_i^{-1} \} \cdot T_i$. The eigenvalues of the hermitian matrix $\Gamma_{i, 1} \cdot \Gamma_i, \cdot T_i^{-1}$ (resp. $\Gamma_{i, 0} \cdot \Gamma_i, \cdot T_i^{-1}$) are continuous functions on $X_c$ (resp. $X$). From (4.12), we have

$$\Gamma_{i, 1} \cdot \Gamma_i, \cdot T_i^{-1} \geq \Gamma_{i, 0} \cdot \Gamma_i, \cdot T_i^{-1} \quad \text{on} \quad X_c \cap U_i.$$  

Let $K'$ be a compact subset of $X_c$ with $K \subset \text{Int} \ K' \subset K' \subset X_c$. Since the closure of $X_c$ is compact, (4.17) implies that the first $n - q + 1$ eigenvalues of the matrix $\Gamma_{i, 1} \cdot \Gamma_i, \cdot T_i^{-1}$ taken in the order of decreasing magnitude, are positive and stay away from zero on $X_c \setminus K'$. Let $x_0 \in X_c \setminus K'$ and choose a system of local coordinates $(z_1, \ldots, z_n)$ around $x_0$ as follows:

$$G_i, 0 (x_0) = (\delta_{a, \beta}) \quad \text{and} \quad \Gamma_i (x_0) = (v_a, \delta_{a, \beta}),$$

where $\{v_a\}_{1 \leq a \leq n}$ are eigenvalues of the matrix $\Gamma_{i, 1} \cdot \Gamma_i, \cdot T_i^{-1}$ at $x_0$ and...
satisfy $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n-q+1} > 0$ and $\nu_{n-q+2} \geq \cdots \geq \nu_n \geq 0$. Then there exists a positive constant $\varepsilon$, independent of the choice of $x_0 \in X_\varepsilon \setminus K'$, such that $\nu_{n-q+1} > \varepsilon > 0$. Therefore we have

$$G_t(x_0)^{-1} \cdot \Gamma_t(x_0) = \begin{pmatrix} \nu_1 & & \cdots & & \nu_n \\ 1+\nu_1 & & & & \\ & \ddots & & & \\ & & \nu_n & & \\ & & & 1+\nu_n \end{pmatrix} \geq \varepsilon' \begin{pmatrix} E_{n-q+1} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\varepsilon' = \frac{\varepsilon}{1+\varepsilon}$ and $E_{n-q+1}$ is the $(n-q+1, n-q+1)$ unit matrix.

We apply (4.19) to (4.16), then at $x_0$

$$\sum_{B_{p-1}} \sum_{a, \beta=1}^{n} (\sum_{\beta'=1}^{n} t_{i, \beta'} \cdot \Gamma_{t, \beta'} \varphi_{i, a B_{p-1}} \cdot \varphi_{i, \beta B_{p-1}}) \geq \varepsilon' \sum_{B_{p-1}} \sum_{\beta = 1}^{n-q+1} \varphi_{i, \beta B_{p-1}} \cdot \varphi_{i, \beta B_{p-1}}.$$

If $p \geq q$, then $p+n-q+1 \geq n+1$, thus any block $B_p$ of $p$ indices taken from $\{1, 2, \cdots, n\}$ must contain one of the indices $\{1, 2, \cdots, n-q+1\}$, i.e. one of the indices corresponding to the positive eigenvalues $\nu_1, \nu_2, \cdots, \nu_{n-q+1}$. It follows that

$$\sum_{B_{p-1}} \sum_{\beta = 1}^{n-q+1} \varphi_{i, \beta B_{p-1}} \cdot \varphi_{i, \beta B_{p-1}} \geq \sum_{\beta_1 < \cdots < \beta_p} \varphi_{i, B_{\beta_1} \cdots B_{\beta_p}} \cdot \varphi_{i, B_{\beta_1} \cdots B_{\beta_p}}.$$

Since the matrix $G_t^{-1} \cdot \Gamma_t$ is positive semi-definite on $X_c$, from (4.15), (4.16), (4.20) and (4.21) we have

$$\|\varphi\|^2_{X_c, K'} \leq C_1 \{\|\overline{\delta}\varphi\|^2 + \|\overline{\delta^*}\varphi\|^2\} \left( C_1 = \frac{1+\varepsilon}{\varepsilon} \right)$$

for any $\varphi \in C^0_p(X, B \hat{\otimes} K_\varepsilon)$ with $p \geq q$.

§ 5. Proof of the Main Theorem

Step 1. Vanishing Theorems on Each Sublevel Set $X_c$. Let the situations be as above. By Remark in Section 4, our base metric $d s^2$ is complete. Hence, by the same argument as in the proof of Theorem 3.1 and (4.22), we have

$$\|\varphi\|^2_{X_c, K'} \leq C_1 \{\|\overline{\delta}\varphi\|^2 + \|\overline{\delta^*}\varphi\|^2\}$$
for any $\varphi \in D^p_0 \cap D^q_0$ with $p \geq q$.

Any connected compact complex manifold $X$ is weakly 1-complete with respect to the real constant functions. Then we have $X = X$. If $X$ is non-compact, $X = \{ x \in X | \phi(x) < c \}$ has countable connected components. If one of them is contained in the compact subset $K'$, it must be a compact connected component (or manifold) of $X$. Since $X$ is connected, this is a contradiction. Therefore, in our situation, the conditions of Theorem 3.1 are satisfied. Hence there exists a constant $C_2 > 0$ such that

$$\| \varphi \|^2 \leq C_2 \{ \| \bar{\partial} \varphi \|^2 + \| \bar{\partial}^* \varphi \|^2 \}$$

for any $\varphi \in D^p_0 \cap D^q_0$ with $p \geq q$.

Take any $\varphi \in C^0(X, B \otimes K_x)$ with $\bar{\partial} \varphi = 0$. We can choose a $C^\infty$ function $\lambda$ with the condition (4.9) such that

i) the Kähler metric $ds^2$ induced by (4.13) is complete,

ii) $(\varphi, \varphi) < +\infty$ (cf. [9], § 2, Proof of Theorem 1).

Hence, by Corollary 3.1, we have $\varphi = \bar{\partial} \varphi$ for some $\varphi \in C^0(X', B \otimes K_x)$. Therefore we have proved that for any $c > c_* = \sup_{x \in K} \phi(x)$,

$$H^p(X_c, \mathcal{O}(B \otimes K_x)) = 0 \quad \text{for any } p \geq q.$$

**Step 2. Approximation Lemmas.** We fix two constants $d$ and $e$ such that

$$d > e > c_*,$$

i) the boundary $\partial X_e$ of $\{ x \in X | \phi(x) \leq e \}$ is smooth.

We take a $C^\infty$-increasing convex function $\tau(t)$ such that

$$\tau(t) : (-\infty, \infty) \to (-\infty, \infty),$$

i) $\tau(t) \leq \frac{1}{d - e}$

ii) $\tau(t) > \frac{1}{d - e}$

iii) $\int_0^{+\infty} \sqrt{\tau''(t)} \, dt = +\infty$.

We set
\[ \mathcal{Y} = \tau \left( \frac{1}{d - \phi} \right). \]

We define the metrics of \( B \) on \( X_d \) by

(5.6) \[ \begin{align*}
\text{i) } & a_i = a_{i,0} \cdot \exp (\mathcal{Y}), \\
\text{ii) } & a_{m,i} = a_i \cdot \exp (m \mathcal{Y}) \quad \text{for any } m \geq 0.
\end{align*} \]

We set

\[ \begin{align*}
\text{i) } & \Gamma_i = (\Gamma_{i,a\bar{a}}) \quad \text{where } \Gamma_{i,a\bar{a}} = \frac{\partial^2 \log a_i}{\partial z^a \partial \bar{z}^\bar{a}}, \\
\text{ii) } & \Gamma_{m,i} = (\Gamma_{m,i,a\bar{a}}) \quad \text{where } \Gamma_{m,i,a\bar{a}} = \frac{\partial^2 \log a_{m,i}}{\partial z^a \partial \bar{z}^\bar{a}} \quad \text{for any } m \geq 0.
\end{align*} \]

We define a Kähler metric \( ds^2 \) on \( X_d \) by

(5.7) \[ ds^2 = \sum_{a, \bar{a} = 1}^n (g_{i,a\bar{a}} + \Gamma_{i,a\bar{a}}) dz^a \cdot d\bar{z}^{\bar{a}}. \]

Remark. By the choice (5.5), \( ds^2 \) is a complete Kähler metric as in Remark in Section 4.

We set

\[ G_i = (g_{i,a\bar{a}}) \quad \text{where } g_{i,a\bar{a}} = g_{i,a\bar{a},0} + \Gamma_{i,a\bar{a}}. \]

Using (5.6), we define the metrics of \( B \otimes K_X \) on \( X_d \):

(5.8) \[ \begin{align*}
\text{i) } & A_i = a_i \cdot g_i, \\
\text{ii) } & A_{m,i} = a_{m,i} \cdot g_i \quad \text{for any } m \geq 0, \quad \text{where } g_i = \det G_i.
\end{align*} \]

For any integer \( m \geq 0 \), we define

(5.9) \[ \begin{align*}
(\varphi, \psi)_m = (\varphi, \psi)_m^\mathcal{Y} \\
\| \varphi \|^2_m = (\varphi, \varphi)_m
\end{align*} \]

for any \( \varphi, \psi \in L^{a,\bar{a}}(X_d, B \otimes K_X, m \mathcal{Y}) \). We denote the formal adjoint of \( \overline{\partial} \) with respect to the inner product \( (\varphi, \psi)_m \) by \( \overline{\partial}_m \) and the adjoint operator in \( L^{a,\bar{a}}(X_d, B \otimes K_X, m \mathcal{Y}) \) by \( \overline{\partial}_m^\mathcal{Y} \).

Now we have
Hence by the same argument as in Section 4, we have, for any $m \geq 0$,
\begin{equation}
\|\varphi\|_{m, X_d \setminus K'}^2 \leq C_1 \left( \|\bar{\partial}\varphi\|_m^2 + \|\bar{\partial}^\#\varphi\|_m^2 \right)
\end{equation}
for any $\varphi \in D_{\delta}^{b,p} \cap D_{\delta}^{\#b,p}$ with $p \geq q$, where $C_1 > 0$ is independent of $m$ and $K'$ is a compact subset with $K \subset \text{Int} K' \subset K' \subset X_d$. Then, for each $m$, we have a positive constant such that (5.2) holds. In general, this constant depends on $m$. The basic idea of the following lemma is due to Hörmander [5]. (Compare with [10], Proposition 4.2.)

**Lemma 5.1.** There exists $m_0$ and $C_0 > 0$ such that for any $m \geq m_0$ and $p \geq q$,
\begin{equation}
\|\varphi\|_{m}^2 \leq C_0 \left( \|\bar{\partial}\varphi\|_m^2 + \|\bar{\partial}^\#\varphi\|_m^2 \right),
\end{equation}
provided $\varphi \in D_{\delta}^{b,p} \cap D_{\delta}^{\#b,p} \subset L^{b,p}(X_d, B \otimes K_X, m\mathcal{F})$.

**Proof.** Assume that the assertion is false. There would be a sequence $\{\varphi_k\}$ such that
\begin{equation}
\text{i) } \varphi_k \in D_{\delta}^{b,p} \cap D_{\delta}^{\#b,p} \subset L^{b,p}(X_d, B \otimes K_X, k\mathcal{F}),
\end{equation}
\begin{equation}
\text{ii) } \|\varphi_k\|_k = 1,
\end{equation}
\begin{equation}
\text{iii) } \|\bar{\partial}\varphi_k\|_k, \|\bar{\partial}^\#\varphi_k\|_k \to 0 \text{ as } k \to +\infty.
\end{equation}

Let $g_k = e^{-kr} \cdot \varphi_k$, then we have
\begin{equation}
\text{i) } \bar{\partial}^* g_k = e^{-kr} \bar{\partial}^*_k \varphi_k,
\end{equation}
\begin{equation}
\text{ii) } \|\bar{\partial}^* g_k\|_{-k} = \|\bar{\partial}^*_k \varphi_k\|_k.
\end{equation}

By (5.11), we have
\begin{equation}
\|g_k\| \leq \|g_k\|_{-k} - \|\varphi_k\|_k = 1.
\end{equation}

Therefore choosing a subsequence if necessary, we may assume that $\{g_k\}$ has a weak limit $g$ in $L^{b,p}(X, B \otimes K_X)$. On the other hand, it follows that
\begin{equation}
\|g_k\|_{X_d \setminus K'}^2 \leq \|\varphi_k\|_{X_d \setminus K'}^2 \leq C_1 \left( \|\bar{\partial}\varphi_k\|_k^2 + \|\bar{\partial}^\#\varphi_k\|_k^2 \right).
\end{equation}

By (5.11)
\[
\lim_{k \to +\infty} \|g_k\|^2_{X^{p+1}} = 0.
\]

Hence we have \(g|_{X_t \cup K'} = 0\). Then it follows that

\[(5.13) \quad \text{supp } g \subseteq K'.\]

From (5.11), (5.12) and (5.13), we have \(\overline{\partial} g = 0\) and \(\overline{\partial}^* g = 0\) in \(L^{b,p+1}(X_\varepsilon, B \otimes K_X)\) and \(L^{b,p-1}(X_\varepsilon, B \otimes K_X)\) respectively. Since any connected component of \(X_\varepsilon\) is not contained in \(K'\), by Theorem 3.2, we have

\[(5.14) \quad g = 0.\]

By (5.11), we may assume that \(\{g_k\}\) is strongly convergent on \(K_r\).

(5.14) implies that the limit is zero on \(K'\). From (5.10) and (5.11), we obtain a contradiction.

q.e.d.

**Lemma 5.2.** If \(\phi \in L^{b,p}(X_\varepsilon, B \otimes K_X)\) with \(p \geq q - 1\) and \(\overline{\partial} \phi = 0\), then for any \(\varepsilon > 0\), there exists \(\tilde{\phi} \in L^{b,p}(X_\varepsilon, B \otimes K_X)\) such that \(\|\tilde{\phi} - \phi\|^2_{X_\varepsilon} < \varepsilon\) and \(\overline{\partial} \tilde{\phi} = 0\).

**Proof.** It suffices to show that if \(u \in L^{b,p}(X_\varepsilon, B \otimes K_X)\) and

\[(5.15) \quad \int_{X_\varepsilon} \langle f, u \rangle dV = 0\]

for any \(f \in L^{b,p}(X_d, B \otimes K_X)\) with \(\overline{\partial} f = 0\), then we have

\[(5.16) \quad \int_{X_\varepsilon} \langle g, u \rangle dV = 0\]

if \(g \in L^{b,p}(X_\varepsilon, B \otimes K_X)\) and \(\overline{\partial} g = 0\).

Extend the definition of \(u\) by setting \(u = 0\) on \(X_d \setminus X_\varepsilon\). We denote it by \(u'\). Then (5.15) implies that \(u'\) is orthogonal to \(N_0^{b,p} \subset L^{b,p}(X_d, B \otimes K_X, m\Psi)\) for any \(m\), we have \(u' \in \overline{\bigcap_{m=0}^{\infty} N_0}^{b,p} \subset L^{b,p}(X_d, B \otimes K_X, m\Psi)\). The condition \(R^{b,p}_m = \overline{R^{b,p}_m} \subset L^{b,p}(X_d, B \otimes K_X, m\Psi)\) is equivalent to \(R^{b,p+1}_m = \overline{R^{b,p+1}_m}\) (cf. [5], Theorem 1.1.1). By (5.10), we have \(R^{b,p+1}_m = \overline{R^{b,p+1}_m} \subset L^{b,p+1}(X_d, B \otimes K_X, m\Psi)\) for \(m \geq 0\) and \(p \geq q - 1\). Hence, from Lemma 5.1, for any \(m \geq m_0\) we have

\[(5.17) \quad u' = \overline{\partial}^*_m v_m\]

for some \(v_m \in L^{b,p-1}(X_d, B \otimes K_X, m\Psi)\) with \(\|v_m\|^2_m \leq C_0 \cdot \|u'\|^2\).

We set
where \( m \geq m_0 \), then

\[
\| w_m \|_{L^{p-1}} \leq C \cdot \| u' \|^{2}.
\]

Hence \( \{w_m\} \) has a subsequence which is weakly convergent in \( L^{0,p-1}(X_d, B \otimes K_x) \). Let the weak limit be \( w \). On the other hand, for every \( \varepsilon > 0 \)

\[
\int_{\{x \in X_d | \mathcal{V}(x) > \varepsilon\}} e^{m\tau} \langle w_m, w_m \rangle dV \leq C \| u' \|
\]

and we have

\[
e^{m\tau} \int_{\{x \in X_d | \mathcal{V}(x) \geq \varepsilon\}} \langle w_m, w_m \rangle dV \leq C \| u' \|.
\]

It follows that \( \int_{\{x \in X_d | \mathcal{V}(x) \geq \varepsilon\}} \langle w_m, w_m \rangle dV \) tends to zero, and hence \( w_m \to 0 \) almost everywhere in \( \{x \in X_d | \mathcal{V}(x) \geq \varepsilon\} \). Hence \( w = 0 \) on \( \{x \in X_d | \mathcal{V}(x) \geq \varepsilon\} \) for every \( \varepsilon > 0 \). Therefore we have

\[
(5.18) \quad \text{supp } w \subseteq X_d \quad \text{and} \quad \bar{\partial}^* w = u'.
\]

Since \( \bar{X} \) is compact and \( \partial \bar{X} \) is smooth, from [5] Proposition 1.2.3, there exists a sequence \( \{w^k\} \subset C^{0,p+1}_c(X, B \otimes K_x) \) and \( \|w^k - w\|_{C^{0}_c} \), \( \|\bar{\partial}^* w^k - \bar{\partial}^* w\|_{C^{0}_c} \to 0 \) as \( k \to +\infty \).

We have, for any \( v \in D^{0,p}_c \subset L^{0,p}(X, B \otimes K_x) \),

\[
(\bar{\partial} v, w|_{X})_{x} = \lim_{k \to +\infty} (\bar{\partial} v, w^k|_{X})_{x} = \lim_{k \to +\infty} (v, \bar{\partial}^* w^k|_{X})_{x} = (v, \bar{\partial}^* (w|_{X})|_{X}.
\]

Hence

\[
(5.19) \quad \bar{\partial}^* (w|_{X}) = u.
\]

Therefore, if \( g \in L^{0,p}(X, B \otimes K_x) \) and \( \bar{\partial} g = 0 \), we have

\[
\int_{X} \langle g, u \rangle dV = \int_{X} \langle \bar{\partial} g, w \rangle dV = 0.
\]

If in particular \( q = 1 \), replacing \( L^{0,p}(X_d, B \otimes K_x) \) (resp. \( L^{0,p}(X, B \otimes K_x) \)) by \( \Gamma'(X_d, \partial (B \otimes K_x)) \) (resp. \( \Gamma'(X, \partial (B \otimes K_x)) \)), we can prove the following in the same way as we proved Lemma 5.2.

**Lemma 5.3.** Let \( X_d \) and \( X_e \) be as above and let a holomorphic
line bundle $B$ be positive on $X \setminus K$ and semi-positive on $X$. Then for any holomorphic section $\varphi \in \Gamma(X_\delta, \mathcal{O}(B \otimes K_X))$, $X_\delta$ being the closure of $X_\varepsilon$ in $X$, and for any $\varepsilon > 0$, there exists a section $\tilde{\varphi} \in \Gamma(X_\delta, \mathcal{O}(B \otimes K_X))$ such that $\|\tilde{\varphi} - \varphi\|_{X_\delta} < \varepsilon$.

Let $C$ be a compact subset of $X_\delta$. We set $|\varphi|_c = \sup_{x \in C} \sqrt{\langle \varphi, \varphi \rangle (x)}$ for $\varphi \in \Gamma(X_\delta, \mathcal{O}(B \otimes K_X))$, where $\langle \varphi, \varphi \rangle = A_i^{-1}|\varphi_i|^2$ (see (5.8)). Then, using Cauchy's integral formula in each local coordinate $U_i$ with $U_i \cap C \neq \emptyset$, we can find a positive constant $M$ such that

$|\varphi|_c \leq M \|\varphi\|_c$.

Hence we obtain the following.

**Lemma 5.4.** Let $X_\delta$ and $X_\varepsilon$ be as above. Let a holomorphic line bundle $B$ be positive on $X \setminus K$ and semi-positive on $X$. Then for any holomorphic section $\varphi \in \Gamma(X_\delta, \mathcal{O}(B \otimes K_X))$ and for any $\varepsilon > 0$, there exists a section $\tilde{\varphi} \in \Gamma(X_\delta, \mathcal{O}(B \otimes K_X))$ such that $|\tilde{\varphi} - \varphi|_{X_\delta} < \varepsilon$.

**Step 3. Global Vanishing Theorems.** By Sard's theorem, we can choose a sequence $\{c_\nu\}_{\nu=0}^{\infty}$ of real numbers such that

\begin{align}
\text{(5.20)} & \quad \text{i) } c_0 > c_\infty, \\
& \quad \text{ii) } c_{\nu+1} > c_\nu \text{ and } c_\nu \to \infty \text{ as } \nu \to +\infty, \\
& \quad \text{iii) the boundary } \partial X_{c_\nu} \text{ of } \{x \in X | \Phi(x) \leq c_\nu\} \text{ is smooth for any } \nu \geq 0.
\end{align}

For any pair $(c_{\nu+2}, c_\nu) (\nu \geq 0)$, we choose a $C^\infty$ increasing convex function $\tau_{\nu+2}$ such that

\begin{align}
\text{(5.21)} & \quad \text{i) } \tau_{\nu+2}(t) : (-\infty, \infty) \to (-\infty, \infty), \\
& \quad \text{ii) } \tau_{\nu+2}(t) = \begin{cases} 
0 & \text{if } t \leq \frac{1}{c_{\nu+2} - c_\nu} \\
\geq 0 & \text{if } t > \frac{1}{c_{\nu+2} - c_\nu}, 
\end{cases} \\
& \quad \text{iii) } \int_0^{+\infty} \sqrt{\tau_{\nu+2}'(t)} \, dt = +\infty.
\end{align}
We set
\[ X_\nu = \{ x \in X \mid \phi(x) < \epsilon \}, \]
\[ \tau_{\nu+2} = \frac{1}{\epsilon_{\nu+2} - \phi} \]
for any \( \nu \geq 0 \). Then, for any pair \((c_{\nu+2}, c_{\nu})\), Lemma 5.2 and Lemma 5.4 hold.

The case \( q = 1 \). \( \mathcal{X} = \{ X_\nu \}_{\nu \geq 0} \) is a covering of \( X \). For any \( \nu \geq 1 \), we set \( \mathcal{X}_\nu = \{ X_\mu \}_{\mu \leq \nu} \), then \( \mathcal{X}_\nu \) is a covering of \( X_\nu \). By (5.3), \( \mathcal{X} \) (resp. \( \mathcal{X}_\nu \)) is a Leray covering for the sheaf \( \mathcal{O}(B \otimes K_X) \) on \( X \) (resp. \( X_\nu \)). Then we have, for any \( i \geq 1 \) and \( \nu \geq 1 \),
\[ H^i(X, \mathcal{O}(B \otimes K_X)) = H^i(\mathcal{X}, \mathcal{O}(B \otimes K_X)) \]
and
\[ H^i(\mathcal{X}_\nu, \mathcal{O}(B \otimes K_X)) = H^i(X_\nu, \mathcal{O}(B \otimes K_X)) = 0. \]

Let \( \sigma \in Z^i(\mathcal{X}, \mathcal{O}(B \otimes K_X)), i \geq 1 \). Let \( \sigma_\nu \) be the restriction of \( \sigma \) to \( X_\nu \). Then \( \sigma_\nu \in Z^i(\mathcal{X}_\nu, \mathcal{O}(B \otimes K_X)) \) so there is an \( \alpha_\nu \in C^{i-1}(\mathcal{X}_\nu, \mathcal{O}(B \otimes K_X)) \) such that \( \partial \alpha_\nu = \sigma_\nu \). As an element of \( C^{i-2}(\mathcal{X}_\nu, \mathcal{O}(B \otimes K_X)) \), \( \partial \alpha_\nu = \partial \alpha_{\nu-1} \), and thus \( \alpha_\nu - \alpha_{\nu-1} \in Z^{i-1}(\mathcal{X}_{\nu-1}, \mathcal{O}(B \otimes K_X)) \).

When \( i > 1 \). Since \( \alpha_\nu - \alpha_{\nu-1} \in Z^{i-1}(\mathcal{X}_{\nu-1}, \mathcal{O}(B \otimes K_X)), \) there is a \( \beta_{\nu-1} \in C^{i-2}(\mathcal{X}_{\nu-1}, \mathcal{O}(B \otimes K_X)) \) such that \( \partial \beta_{\nu-1} = \alpha_\nu - \alpha_{\nu-1} \) on \( X_{\nu-1} \). Define \( \alpha \in C^{i-1}(\mathcal{X}, \mathcal{O}(B \otimes K_X)) \) as follows:
\[ \alpha = \alpha_\nu - \partial (\sum_{\mu < \nu} \beta_\mu) \] on \( X_\nu \).
It is easily verified that \( \alpha \) is well defined. Finally, for any \( \nu \), \( \partial \alpha = \partial \alpha_\nu - \partial \partial (\sum_{\mu < \nu} \beta_\mu) = \partial \alpha_\nu = \sigma_\nu \). Hence we have \( \partial \alpha = \sigma \).

When \( i = 1 \). Since \( \alpha_\nu - \alpha_{\nu-1} \in \Gamma(X_{\nu-1}, \mathcal{O}(B \otimes K_X)), \) by Lemma 5.4 we can find, for any \( \varepsilon > 0 \), a \( \gamma \in \Gamma(X_\nu, \mathcal{O}(B \otimes K_X)) \) such that \( |\alpha_\nu - \alpha_{\nu-1} - \gamma|_{X_{\nu-1}} < \varepsilon \). Therefore, inductively, we have a sequence \( \{ \lambda_\nu \}_{\nu \geq 1} \) so that
(5.22) i) \( \lambda_\nu \in C^0(\mathcal{X}_\nu, \mathcal{O}(B \otimes K_X)) \) and \( \lambda_1 = \alpha_1 \),
ii) \( \partial \lambda_\nu = \sigma_\nu \),
iii) \( |\lambda_{\nu+1} - \lambda_\nu|_{X_{\nu-1}} < 2^{-\nu} \).
For any \( \nu \), \( \lim_{v \to \infty} \lambda_v \) defines an element of \( C^q(\mathcal{X}_v, \mathcal{O}(B \otimes K_X)) \) and clearly this limit is the same as the restriction of \( \lim_{v \to \infty} \lambda_v \) for any \( \nu \geq \nu + 1 \). Thus we can define an element \( \lambda \) of \( C^q(\mathcal{X}, \mathcal{O}(B \otimes K_X)) \) by \( \lambda = \lim_{v \to \infty} \lambda_v \). For any \( \nu \), \( \delta (\lim_{v \to \infty} \lambda_v) = \lim_{v \to \infty} \delta \lambda_v = \sigma_v \). Hence we have \( \delta \lambda = \sigma \).

**The case \( q \geq 1 \).** We denote by \( L^6_{\text{loc}}(X, B \otimes K_X) \) the set of the locally square integrable \((0, p)\) forms on \( X \) with values in \( B \otimes K_X \). For \( p \geq 1 \), there is a natural isomorphism

\[
H^p(X, \mathcal{O}(B \otimes K_X)) \cong \left\{ f \in L^6_{\text{loc}}(X, B \otimes K_X) ; \bar{\partial} f = 0 \right\} / \left\{ f \in L^6_{\text{loc}}(X, B \otimes K_X) ; f = \bar{\partial} g \text{ for some } g \in L^6_{\text{loc}}(X, B \otimes K_X) \right\}.
\]

Therefore, for \( p \geq q \), it suffices to show that for any \( \varphi \in L^6_{\text{loc}}(X, B \otimes K_X) \) with \( \bar{\partial} \varphi = 0 \), there exists a \( \psi \in L^6_{\text{loc}}(X, B \otimes K_X) \) such that \( \bar{\partial} \psi = \varphi \).

In this proof, for any \( \nu \geq 0 \), we set

\[
(5.24) \quad \begin{align*}
\text{i) } & \varphi_v = \varphi |_{x_v}, \\
\text{ii) } & L^6_{\text{loc}}(X_{v+5}, B \otimes K_X, \mathcal{P}_{v+5}) = L^6_{\text{loc}}(X_{v+5}, B \otimes K_X), \\
\text{iii) } & L^6_{\text{loc}}(X_v, B \otimes K_X, 0) = L^6_{\text{loc}}(X_v, B \otimes K_X, \mathcal{P}_{v+5}), \\
\text{iv) } & \|f\|_{v+2}^2 = \int_{X_{v+1}} \left\langle f, f \right\rangle e^{-r_{v+1}} dV
\end{align*}
\]

where \( \left\langle f, f \right\rangle = (a_{i,0} \cdot g_i)^{-(\nu+1)} \sum g_{i,0} \cdot f_{i,0}^{p} \).

Then \( \varphi_v \in L^6_{\text{loc}}(X_v, B \otimes K_X, \mathcal{P}_v) \) and \( \bar{\partial} \varphi_v = 0 \ (\nu \geq 2) \). Hence there exists a \( \psi_v \in L^6_{\text{loc}}(X_v, B \otimes K_X, \mathcal{P}_v) \) such that \( \bar{\partial} \psi_v = \varphi_v \) for any \( \nu \geq 2 \). We now choose, by induction, a sequence \( \{\psi_v\}_{v \geq 1} \) so that

\[
(5.25) \quad \begin{align*}
\text{i) } & \psi_v \in L^6_{\text{loc}}(X, B \otimes K_X) \\
\text{ii) } & \bar{\partial} \psi_v = \varphi_v \text{ on } X_v \\
\text{iii) } & \|\psi_{v+1} - \psi_v\|_{v+5, X_v}^2 < 2^{-v}
\end{align*}
\]

We set

\[
\psi_1 = \begin{cases} 
\psi_v |_{X_1} & \text{on } X_1 \\
0 & \text{on } X \setminus X_1.
\end{cases}
\]
Since \( \psi'_n \in D^0 \psi^{-1} \subset L^{6, p-1} (X, B \otimes K_x, \mathcal{F}_x) \), we have \( \psi_1 \in D^0 \psi^{-1} \subset L^{6, p-1} (X, B \otimes K_x, 0) \) and \( \overline{\partial} \psi_1 = \varphi_1 \). Suppose \( \varphi_1, \ldots, \varphi_{p-1} \) are chosen. Then
\[
(\psi_{p+1} - \psi_{p-1}) |_{X_{p-1}} \in L^{6, p-1} (X_{p-1}, B \otimes K_x, 0)
\]
and
\[
\overline{\partial} (\psi_{p+1} - \psi_{p-1}) |_{X_{p-1}} = 0.
\]
By Lemma 5.2, there exists a \( g \in L^{6, p-1} (X_{p+1}, B \otimes K_x, \mathcal{F}_{p+1}) \) such that
\[
\| g - (\psi_{p+1} - \psi_{p-1}) \|_{p+1, X_{p+1}} < 2^{-r_{p+1}} \text{ and } \overline{\partial} g = 0.
\]
We set
\[
\psi_n = \begin{cases} 
\psi_{p+1} |_{X_{n}} - g |_{X_{n}} & \text{on } X_n \\
0 & \text{on } X \setminus X_n.
\end{cases}
\]
Then we have
\[
(5.26) \quad \begin{array}{ll}
i) & \psi_n \in D^0 \psi^{-1} \subset L^{6, p-1} (X, B \otimes K_x, 0) \\
ii) & \overline{\partial} \psi_n = \varphi_n \\
iii) & \| \psi_n - \psi_{n-1} \|_{n+1, X_{n+1}} < 2^{-r_{n+1}}.
\end{array}
\]
From (5.26), for any \( \nu \), \( \{\psi_{\nu}\} \) converges with respect to the norm \( \| \|_\nu \), and clearly the limit is the same as the restriction of \( \lim \psi_{n} \) for any \( \nu \geq \nu + 1 \). Thus we can define an element \( \psi \) of \( L^{6, p-1} (X, B \otimes K_x) \) by
\[
\psi = \lim_{\nu \to +\infty} \psi_{\nu}.
\]
For every \( \nu \geq 1 \),
\[
(5.27) \quad \begin{array}{ll}
i) & \lim_{\nu \to \infty} \psi_{\nu} = \psi \text{ in } L^{6, p-1} (X, B \otimes K_x, 0), \\
ii) & \lim_{\nu \to \infty} \overline{\partial} \psi_{\nu} |_{X_{\nu}} = \varphi_\nu \text{ in } L^{6, p} (X, B \otimes K_x, 0).
\end{array}
\]
Since \( \overline{\partial} \) is a closed operator in \( L^{6, p-1} (X, B \otimes K_x, 0) \) for every \( \nu \geq 1 \), we have, for any \( \nu \geq 1 \),
\[
\overline{\partial} \psi = \varphi_\nu \text{ in } L^{6, p} (X, B \otimes K_x, 0).
\]
Hence we have
\[
\overline{\partial} \psi = \phi \text{ in } L^{6, p} (X, B \otimes K_x), \quad \text{q.e.d.}
\]
References


*Added in proof:* The author and T. Ohsawa have proved that the global vanishing theorem of Theorem 2 holds i.e. $H^p(X, \mathcal{O}^q(B)) = 0$ for $p+q \geq n+k$. See "A vanishing theorem for $H^p(X, \mathcal{O}^q(B))$ on weakly 1-complete manifolds", forthcoming.