Linear Radon-Nikodym Theorems for States on a von Neumann Algebra

By

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Abstract

Several linear Radon-Nikodym theorems for states on a von Neumann algebra are obtained in the context of a one parameter family of positive cones introduced by H. Araki. Among other results, we determine when a normal state $\phi$ admits a linear Radon-Nikodym derivative with respect to a distinguished normal faithful state $\phi_0$ in the sense of Sakai, that is, $\phi = h \phi_0 + \phi_0 h$ with a positive $h$ in the algebra.

§ 0. Introduction

We consider a von Neumann algebra on a Hilbert space admitting a cyclic and separating vector. Making use of the associated modular operator, [9], Araki introduced a one parameter family of positive cones. Several Radon-Nikodym theorems are known in the context of positive cones ([1], [4], [5]) in which "Radon-Nikodym derivatives" reduce to the square roots of measure theoretic Radon-Nikodym derivatives provided that the algebra in question is commutative.

In this paper we obtain three linear Radon-Nikodym theorems (Theorem 1.5, 1.6, 1.7). Our proofs are very constructive so that we obtain explicit expressions of linear Radon-Nikodym derivatives.

Our main tools are relative modular operators and the function $\{\cosh (\pi t)\}^{-1}$ which was used by van Daele to obtain a simple proof of the fundamental theorem of the Tomita-Takesaki theory, [12].

§ 1. Notations and Main Results

Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a unit cyclic
and separating vector $\xi_0$ with the vector state $\phi_0 = \omega_{\xi_0}$, and $\Lambda, J$ be the associated modular operator and modular conjugation respectively, [9]. Fixing these throughout, we denote the modular automorphism group $\{\text{Ad } \Lambda^t\}_{t \in \mathbb{R}}$ on $\mathcal{M}$ simply by $\sigma_t$. Let $\mathcal{M}_0$ be a $\sigma$-weakly dense $*$-subalgebra consisting of every $x \in \mathcal{M}$ such that $t \in \mathbb{R} \rightarrow \sigma_t(x) \in \mathcal{M}$ extends to an entire function.

The following one parameter family of positive cones was introduced by Araki:

**Definition 1.1 ([1]).** For each $0 \leq x \leq 1/2$, we denote the closure of the positive cone $\Lambda^x \mathcal{M} + \xi_0$ in $\mathcal{M}$ by $P^x (= P_{\phi_0}^x)$. These cones enjoy the following properties:

**Proposition 1.2 ([1]).** For each $0 \leq x \leq 1/2$, we have

(i) $P^x = J P^{(1/2) - x} = (P^{(1/2) - x})'$ ( = $\{ \xi \in \mathcal{M} ; (\xi | \xi) \geq 0, \xi \in P^{(1/2) - x} \},$ the dual cone),

(ii) $P^x \subseteq \mathcal{D}(\Lambda^{(1/2) - 2x})$ and $\Lambda^{(1/2) - 2x} \xi = \mathcal{J} \xi, \xi \in P^x$.

(iii) the map: $\xi \in P^{1/4} \rightarrow \omega_\xi \in \mathcal{M}_0^+$ is bijective.

By (iii) in the above proposition, each $\phi \in \mathcal{M}_0^+$ admits a unique implementing vector in $P^{1/4}$, the natural cone, which we will denote by $\xi_\phi$, that is, $\phi = \omega_{\xi_\phi}$. Then a positive self-adjoint operator $\Lambda_{\phi_0}$ on $\mathcal{M}$ with a form core $\mathcal{M} \xi_0$ satisfying $J\Lambda_{\phi_0}^{1/2} x \xi_0 = x^* \xi_0, x \in \mathcal{M}$, is known as the relative modular operator (of $\phi$ with respect to $\phi_0$). Also, a partial isometry $\Lambda_{\phi_0}^{it} \Lambda_{\phi_0}^{-it} = (D\phi ; D\phi_0), t \in \mathbb{R}$, in $\mathcal{M}$ is known as the Radon-Nikodym cocycle (of $\phi$ with respect to $\phi_0$). (See [3] or §1 of [5] for full details.)

In the first main theorem (Theorem 1.5), we need the following concept:

**Definition 1.3 (One parameter family of orderings, [3]).** For $\lambda > 0$, we write $\phi \leq \phi_0(\lambda)$ if the map: $t \in \mathbb{R} \rightarrow (D\phi ; D\phi_0), t \in \mathbb{R}$, extends to a bounded $\sigma$-weakly continuous function on $-\lambda \leq \text{Im } z \leq 0$ which is analytic in the interior and $\| (D\phi ; D\phi_0)_{-ik} \| \leq 1$. It is well-known that $\phi \leq \phi_0(1/2)$ is equivalent to $\phi \leq \phi_0$ in the usual ordering in $\mathcal{M}_0$, that is, $\phi(x) \leq \phi_0(x), x \in \mathcal{M}_0^+$. The next lemma is our main tool in the paper.

**Lemma 1.4.** Let $f(z)$ be a bounded continuous function on $0 \leq \text{Re } z \leq 1$ which is analytic in the interior. We then have

$$f(1/2) = \int_{-\infty}^{\infty} \{ f(it) + f(1 + it) \} \left\{ 2 \cosh (\pi t) \right\}^{-1} dt.$$ 

In fact, it is well-known (see p. 208, [8] for example) that the pair $(P_0$
(x + \beta, t) = \sin (\pi x)/2 \{\cosh (\pi t - \beta) - \cos \pi x \}, \quad P_1(x + \beta, t) = \sin (\pi x)/2 \{\cosh (\pi t - \beta) + \cos \pi x \}) gives rise to the harmonic measure for the strip 0 \leq \Re z \leq 1. And both of \( P_0(1/2, t) \) and \( P_1(1/2, t) \) are exactly \( \{2 \cosh (\pi t)\}^{-1} \). Since we use this result repeatedly, we shall denote the function \( \{2 \cosh (\pi t)\}^{-1} \) simply by \( F(t) \) throughout the paper.

We now state our three main results. The first one is a slight strengthening of the result in Section 6, [1]. However, more importantly, we obtain the explicit expression of Radon-Nikodym derivatives.

**Theorem 1.5.** Let \( \phi \) be a normal state on \( \mathcal{M} \) and \( 0 \leq \alpha \leq 1/2 \). If \( \phi \leq l \phi_0 \) (Max \( (\alpha, (1/2) - \alpha) \)) for some \( \lambda > 0 \), the vector \( \\zeta_x \equiv \int_{-\infty}^{\infty} F(t) \{(\mathcal{D}\phi; D\phi_0) = \xi_{(1/2) - \alpha}, \phi = \mathcal{D}\phi_0; D\phi_0) = \xi_{\alpha} \} \) satisfies \( \phi(x) = (x \xi_x | \zeta_x) + (x \xi_x | \zeta_x) \) for \( x \in \mathcal{M} \).

For the special value of \( \alpha = 0 \), the assumption in the theorem is exactly \( \phi \leq l \phi_0 \) in the usual ordering. Furthermore, \( \zeta_0 \in P^0 \) is written as \( \zeta_0 = h \zeta_0 \) with \( h = \int_{-\infty}^{\infty} F(t) \sigma_{\alpha} \{((\mathcal{D}\phi; D\phi_0) - \xi_{1/2})^2 \} dt \in \mathcal{M} \) so that we have \( \phi = h \phi_0 + \phi_0 h \), which is Sakai's linear Radon-Nikodym theorem (Proposition 1, 16/4, [7]). As the second main result, we prove

**Theorem 1.6.** Let \( \phi \) be a normal state on \( \mathcal{M} \). Then \( \phi \) admits a (unique) linear Radon-Nikodym derivative \( \Phi \in \mathcal{M}_+ \), that is, \( \phi = h \phi_0 + \phi_0 h \), if and only if \( \Phi \leq l \phi_0 \) with some \( l > 0 \). Here, \( \Phi \in \mathcal{M}_+ \) is defined by

\[
\Phi(t) = \int_{-\infty}^{\infty} F(t) \sigma_{\alpha} dt.
\]

Furthermore, if this is the case, \( h \) is exactly \( \{(\mathcal{D}\Phi; D\phi_0) - \xi_{1/2} = (\mathcal{D}\Phi; D\phi_0) \xi_{1/2} \cdot (\mathcal{D}\phi_0; D\phi_0) - \xi_{1/2} \}. \) (See also Lemma 4.1.)

In the next result, we further assume that \( \phi_0 \) is periodic in the sense that \( \sigma_T = 1d \) for some \( T > 0 \) ([10]). When \( \phi = h \phi_0 + \phi_0 h \in \mathcal{M}_+ \) (\( h \in \mathcal{M}_+ \)), for each positive \( x \) in the centralizer \( \mathcal{M}_{\phi_0} \) ([10]), we estimate

\[
\phi(x) = \phi_0(xh) + \phi_0(hx) = 2 \phi_0(x^{1/2}hx^{1/2}) \leq 2 \|h\| \phi_0(x).
\]

Our third result asserts that the converse is also true.

**Theorem 1.7.** Let \( \phi \) be a normal state on \( \mathcal{M} \). When the distinguished \( \phi_0 \) is periodic, \( \phi \) admits a linear Radon-Nikodym derivative as in the previous
theorem if and only if $\phi \leq l\phi_0$ on $\mathcal{M}_{\phi_0}$, the centralizer of $\mathcal{M}_{\phi_0}$, for some $l > 0$, that is, $\phi(x) \leq l\phi_0(x)$ for each positive $x \in \mathcal{M}_{\phi_0}$.

The rest of the paper is devoted to the proofs of these three theorems. We denote a generic normal state on $\mathcal{M}$ by $\phi$ and the distinguished $\phi_0$ is supposed to be periodic only in the proof of the last theorem.

§ 2. Proof of Theorem 1.5

In this section, we prove two lemmas from which Theorem 1.5 follows immediately.

**Lemma 2.1.** Let $\zeta$ be a vector in $\mathcal{H}$ satisfying

$$\phi(x) = (\Delta^{(1/2)-s}x\xi_0 | \zeta), \quad x \in \mathcal{M}.$$ 

Then $\zeta$ belongs to $P^s$. If we set

$$\eta = \int_{-\infty}^{\infty} F(t)\Delta^{(1-2s)it}\zeta dt,$$

then $\eta$ belongs to $P^s$ and satisfies

$$\phi(x) = (x\eta | \xi_0) + (x\xi_0 | \eta), \quad x \in \mathcal{M}.$$ 

**Proof.** The cone $\Delta^{(1/2)-s}\mathcal{M} + \xi_0$ being dense in $P^{(1/2)-s}$, $\zeta \in P^s$ follows from the positivity of $\phi$ and Proposition 1.2, (i). Also, since $P^{s}$ is invariant under $\Delta^{(1-2s)it}$ and $F(t)$ is positive for each $-\infty < t < \infty$, $\eta$ belongs to $P^s$ as well.

To prove the final equality, we may and do assume $x \in \mathcal{M}_0$. Firstly we observe

$$(x\xi_0 | \eta) = \int_{-\infty}^{\infty} F(t)(x\xi_0 | \Delta^{-(1-2s)it}\zeta) dt$$

$$= \int_{-\infty}^{\infty} F(t)(\Delta^{(1-2s)it}x\xi_0 | \zeta) dt$$

$$= \int_{-\infty}^{\infty} F(t)(\Delta^{(1/2)-s}(1-2s)_{\mathcal{M}} + i((1/2)-s)(x)\xi_0 | \zeta) dt$$

$$= \int_{-\infty}^{\infty} F(t)\phi(\sigma_{(1-2s)t + i((1/2)-s)}(x)) dt.$$ 

Secondly, by Proposition 1.2, (ii), we observe

$$(x\eta | \xi_0) = (J\Delta^{(1/2)-2s}\eta | x^*\xi_0) = (J\Delta^{(1/2)-2s}\eta | J\Delta^{1/2}x\xi_0)$$

$$= (\Delta^{1/2}x\xi_0 | \Delta^{(1/2)-2s}\eta) = (\Delta^{1-2s}x\xi_0 | \eta)$$

$$= \int_{-\infty}^{\infty} F(t)(\Delta^{1-2s}x\xi_0 | \Delta^{-(1-2s)it}\zeta) dt.$$
Hence, Lemma 1.4 applied to \( f(z) = \phi(\sigma_w(x)) \), \( w = -(1 - 2\alpha)(z - (1/2)) \), yields:

\[
(x \xi_0 | \eta) + (x \eta | \xi_0) = \phi(\sigma_0(x)) = \phi(x). \tag{Q.E.D.}
\]

**Lemma 2.2.** If \( \phi \preceq \langle 1 \rangle (\text{Max} (\alpha, (1/2) - \alpha)) \), then we have

\[
\phi(x) = (A^{(1/2)} - z x \xi_0 | \zeta), \quad x \in \mathcal{M}
\]

with

\[
\zeta = (D \phi; \sigma_0)^{-1}(1/2) - z \phi \xi_0^* \phi.
\]

**Proof.** We simply compute

\[
(A^{(1/2)} - z x \xi_0 | \zeta) = ((D \phi; \sigma_0)^{-1}(1/2) - z D \phi; \sigma_0) x \xi_0^* \phi.
\]

Using the uniqueness of analytic continuation, we can easily prove

\[
(D \phi; \sigma_0)^{-1}(1/2) - z D \phi; \sigma_0) x \xi_0^* \phi = D \phi; \sigma_0 x \xi_0^* \phi.
\]

Since \( (D \phi; \sigma_0)^{-1}(1/2) - z D \phi; \sigma_0^* \xi_0^* \phi \) belongs to \( \mathcal{M} \xi_0^* \subseteq \mathcal{D}(A^{(1/2)} - z) \), we have \( \xi_0^* \in \mathcal{D}(A^{(1/2)} - z) \) and

\[
(A^{(1/2)} - z x \xi_0^* \phi | \phi) = (A^{(1/2)} - z x \xi_0^* \phi | A^{(1/2)} - z x \xi_0^* \phi) = (J x \xi_0^* \phi | \xi_0^* \phi) = \phi(x), \tag{Q.E.D.}
\]

§ 3. **Proof of Theorem 1.6**

The proof of Theorem 1.6 is divided into three lemmas.

**Lemma 3.1.** Assume that \( \phi = h \phi_0 + \phi_0 h \in \mathcal{M}_*^+ \) with a positive \( h \in \mathcal{M} \). Then for each \( x \in \mathcal{M} \) we have

\[
\phi_0(x \sigma_{-i/2}(h)) = \int_{-\infty}^{\infty} F(t) \phi(\sigma_{-t}(x)) dt
\]

\[
= \int_{-\infty}^{\infty} F(t) \phi(\sigma_t(x)) dt
\]

Here, the left hand side makes sense due to the K.M.S. condition, [9].
Proof. By the invariance $\phi_0 \circ \sigma_t = \phi_0$, we compute
\[ \int_{-\infty}^{\infty} F(t) \phi_0(\sigma_{-t}(x)) dt = \int_{-\infty}^{\infty} \{ \phi_0(\sigma_{-t}(x) h) + \phi_0(h \sigma_{-t}(x)) \} F(t) dt \]
\[ = \int_{-\infty}^{\infty} \{ \phi_0(x \sigma_t(h)) + \phi_0(\sigma_t(h) x) \} F(t) dt . \]
Thus, the result follows from Lemma 1.4 and the fact that $\phi_0(x \sigma_z(h))$ is bounded analytic on $-1 \leq \text{Im } z \leq 0$ and $\phi_0(x \sigma_{-i+t}(h)) = \phi_0(\sigma_t(h) x)$ (the K.M.S. condition).

Q. E. D.

Lemma 3.2. For $x, h \in \mathcal{M}_+$, we have
\[ 0 \leq \phi_0(x \sigma_{-i/2}(h)) \leq \| h \| \phi_0(x) . \]

Proof. We notice that
\[ \phi_0(x \sigma_{-i/2}(h)) = (x \Delta^{1/2} h \xi_0 | \xi_0) \]
\[ = (x J h J \xi_0 | \xi_0) \]
due to the positivity of $h$. Since $0 \leq J h J \leq \| h \|$ and they commute with $x \in \mathcal{M}_+$, we have
\[ 0 \leq (x J h J \xi_0 | \xi_0) \leq \| h \| (x \xi_0 | \xi_0) = \| h \| \phi_0(x) . \]

Q. E. D.

Lemma 3.3. When
\[ \mathcal{F} = \int_{-\infty}^{\infty} F(t) \phi_0 \sigma_t dt \leq \| \phi_0 \| , \]
we have $\phi = h \phi_0 + \phi_0 h$ with $h = [D \bar{\phi} ; D \phi_0]_{-i/2} \in \mathcal{M}_+$.

Proof. To prove $\phi(x) = \phi_0(x h) + \phi_0(h x)$, we may and do assume that $x \in \mathcal{M}_0$. At first we notice that
\[ (D \bar{\phi} ; D \phi_0)_{-i/2} \xi_0 = \bar{\xi}_\phi , \]
\[ (D \bar{\phi} ; D \phi_0)_{-i/2} y \xi_0 = (D \bar{\phi} ; D \phi_0)_{-i/2} J \sigma^{-i/2} y* \xi_0 \]
\[ = (D \bar{\phi} ; D \phi_0)_{-i/2} \sigma^{-i/2} (y*) J \xi_0 \]
\[ = J \sigma^{-i/2} (y*) J (D \bar{\phi} ; D \phi_0)_{-i/2} \xi_0 \]
\[ = J \sigma^{-i/2} (y*) \bar{\xi}_\phi \quad (y \in \mathcal{M}_0) . \]

By using these facts, it is easily shown that
\[ \phi_0(h x) + \phi_0(x h) = \bar{\phi}(\sigma_{i/2}(x) + \sigma_{-i/2}(x)) \]
\[ = \int_{-\infty}^{\infty} F(t) \{ \phi(\sigma_{i/2+t}(x)) + \phi(\sigma_{-i/2+t}(x)) \} dt . \]

Thus, the result follows from Lemma 1.4 applied to the function
\[ f(z) = \phi(\sigma_w(x)), \quad w = -i(z - 2^{-1}). \]

Q. E. D.

§ 4. Proof of Theorem 1.7

The next lemma is obtained in [6] in a slightly different setup. However, for the sake of completeness, we present its proof in our context.

**Lemma 4.1.** We have

\[ \tilde{\phi} = \int_{-\infty}^{\infty} F(t)\phi \sigma_t dt \leq l\phi_0 \]

if and only if for each \( \alpha > 0 \) (hence all \( \alpha \)) there exists a positive \( c = c_\alpha \) such that

\[ \int_{-\alpha}^{\alpha} \phi \sigma_t dt \leq c\phi_0. \]

**Proof.** The "only if" part is trivial since \( F(t) \) is a strictly positive even function and monotone decreasing on \([0, \infty)\). To show the "if" part, we assume that

\[ \int_{-\alpha}^{\alpha} \phi \sigma_t dt \leq c\phi_0. \]

We compute that

\[
\begin{align*}
\int_{-\infty}^{\infty} F(t)\phi \sigma_t dt &= \sum_{n=-\infty}^{\infty} \int_{(2n-1)\alpha}^{(2n+1)\alpha} F(t)\phi \sigma_t dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\alpha}^{\alpha} F(t+2n\alpha)\phi \sigma_{t+2n\alpha} dt \\
&\leq \sum_{n=-\infty}^{\infty} c\{\exp(\pi(2|n|+1)\alpha)\}^{-1}\phi_0 \sigma_{2n\alpha} \\
&= \left( \sum_{n=-\infty}^{\infty} c\{\exp(\pi(2|n|+1)\alpha)\}^{-1}\phi_0 \right). 
\end{align*}
\]

Q. E. D.

**Proof of the theorem.** Let \( T > 0 \) be the period of \( \sigma_t \). It is easy to observe that

\[ \int_{-T}^{T} \sigma_t dt = 2T\varepsilon, \]

where \( \varepsilon \) is a normal projection of norm 1 from \( \mathcal{M} \) onto the centralizer \( \mathcal{M}_{\phi_0} \) satisfying \( \phi_0 \circ \varepsilon = \phi_0 \). (See [2], [10], [11].) By the previous lemma and Theorem 1.6, we know that \( \phi = h_0 \phi_0 + \phi_0 h \) if and only if \( \phi \circ \varepsilon \leq l_0 \phi_0 = l_0 \phi_0 \circ \varepsilon \), which is equivalent to \( \phi \leq l_0 \phi_0 \) on \( \mathcal{M}_{\phi_0} \). Q. E. D.
References


