A Remark on Almost-Quaternion Substructures on the Sphere

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By

Hideaki ŌSHIMA*

In [4] T. Önder has solved the existence problem of almost- quaternion \( k \)-substructures on the \( n \)-sphere \( S^n \) for all \( n \) and \( k \) except for \( n \equiv 1 \, (\text{mod} \, 4) \geq 5 \) and \( k = (n-1)/4 \). The purpose of this note is to solve it for this exceptional case.

**Theorem 1.** Let \( n \equiv 1 \, (\text{mod} \, 4) \geq 5 \) and \( k = (n-1)/4 \). Then \( S^n \) has an almost- quaternion \( k \)-substructure if and only if \( n = 5 \).

We use natural embeddings for the classical groups (see [2]):

\[
\text{Sp}(m) \rightarrow SU(2m) \quad \text{and} \quad SU(m) \rightarrow U(m) \rightarrow SO(2m).
\]

We embed, respectively, \( SU(m) \) and \( SO(m) \) in \( SU(m+1) \) and \( SO(m+1) \) as the upper left hand blocks. We embed also \( SO(m) \times SO(n) \) in \( SO(m+n) \) by

\[
(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
\]

Let \( n \) and \( k \) be positive integers with \( 4k \leq n \). An almost- quaternion \( k \)-substructure on an orientable \( n \)-manifold \( M \) is defined to be a reduction of the structural group of the tangent bundle \( T(M) \) from \( SO(n) \) to the subgroup \( \text{Sp}(k) \times SO(n-4k) \). Since the principal \( SO(n) \)-bundle associated with \( T(S^n) \) is

\[
SO(n) \rightarrow SO(n+1) \rightarrow SO(n+1)/SO(n) = S^n,
\]

it follows that \( S^n \) has an almost- quaternion \( k \)-substructure if and only if the associated fibration

\[
SO(n)/\text{Sp}(k) \times SO(n-4k) \rightarrow SO(n+1)/\text{Sp}(k) \times SO(n-4k) \rightarrow S^n
\]

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* Department of Mathematics, Osaka City University, Osaka 558, Japan.
has a cross section. Hence Theorem 1 is equivalent to

**Theorem 2.** Let \( n \equiv 1 \pmod{4} \geq 5 \) and \( k = (n-1)/4 \). Then \( p \) has a cross section if and only if \( n = 5 \).

**Proof of Theorem 2.** In our case the fibration takes the form:

\[
SO(n)/Sp(k) \longrightarrow SO(n+1)/Sp(k) \longrightarrow S^n.
\]

We have an inclusion map \( j : S^5 = SU(3)/SU(2) \longrightarrow SO(6)/Sp(1) \) induced by the embeddings \( Sp(1) = SU(2) \longrightarrow SU(3) \longrightarrow SO(6) \). It is easily seen that \( j \) is a cross section of \( \tilde{p} \) for \( n = 5 \).

Thus we will always assume that \( n \geq 9 \). By the covering homotopy property of \( p \), the fibration has a cross section if and only if

\[
\tilde{p} : \pi_*(SO(n+1)/Sp(k)) \longrightarrow \pi_*(S^n) = \mathbb{Z}
\]

is an epimorphism. We will prove that

\[
\text{Image}(\tilde{p}_*) = 2\pi_*(S^n)
\]

so that \( \tilde{p} \) does not have a cross section.

Consider the commutative diagram of the fibrations:

\[
\begin{array}{ccccccc}
& & & & & & \\
\text{Sp}(k) & = & \text{Sp}(k) & & & & \\
& i & & & & & \\
& & SO(n) & \longrightarrow & SO(n+1) & \longrightarrow & S^n \ \\
& & \downarrow & & \downarrow \tilde{p}_1 & & \downarrow = \\
& & SO(n)/Sp(k) & \longrightarrow & SO(n+1)/Sp(k) & \longrightarrow & S^n.
\end{array}
\]

Applying the homotopy functor \( \pi_*(-) \) to this, we obtain a commutative diagram with exact columns and rows:

\[
\begin{array}{ccccccc}
& & & & & & \\
\pi_*(\text{Sp}(k)) & & & & & & \\
& & \pi_*(SO(n)) & \rightarrow & \pi_*(SO(n+1)) & \rightarrow & \pi_*(S^n) \ \\
& & \downarrow & & \downarrow \tilde{p}_2 & & \downarrow = \\
& & \pi_*(SO(n)/Sp(k)) & \rightarrow & \pi_*(SO(n+1)/Sp(k)) & \rightarrow & \pi_*(S^n) = \mathbb{Z} \ \\
& & \downarrow \partial & & \downarrow A & & \\
& & \pi_{n-1}(\text{Sp}(k)) & \rightarrow & \pi_{n-1}(\text{Sp}(k)) & \rightarrow & \\
& & \downarrow i_* & & \downarrow \\
& & \pi_{n-1}(SO(n)) & \rightarrow & \pi_{n-1}(SO(n+1)).
\end{array}
\]
We use the following known results (see [1] and [3]):

(1) \( \pi_{n-1}(U(2k)) = \mathbb{Z}_{(2k)} \);

(2) \( \pi_{n-1}(Sp(k)) = \pi_{n-1}(Sp(\infty)) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{8} \\ \mathbb{Z}_2 & \text{if } n \equiv 5 \pmod{8} \end{cases} \);

(3) \( \pi_n(Sp(k)) = \pi_n(Sp(\infty)) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{8} \\ \mathbb{Z}_2 & \text{if } n \equiv 5 \pmod{8} \end{cases} \);

(4) \( \pi_{n-1}(SO(n+1)) = \pi_{n-1}(SO(\infty)) = \begin{cases} \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{8} \\ 0 & \text{if } n \equiv 5 \pmod{8} \end{cases} \);

(5) \( \pi_{n-1}(SO(n)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{8} \\ \mathbb{Z}_2 & \text{if } n \equiv 5 \pmod{8} \end{cases} \);

(6) \( \pi_n(SO(n)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{8} \\ \mathbb{Z}_2 & \text{if } n \equiv 5 \pmod{8} \end{cases} \);

(7) \( \pi_n(SO(n+1)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{8} \\ \mathbb{Z} & \text{if } n \equiv 5 \pmod{8} \end{cases} \).

By (4) and (5), we have

(8) \( \text{Image}(p_{2*}) = 2\pi_n(S^n) \).

By (2) and (3), \( p_{1*} \) is an isomorphism if \( n \equiv 1 \pmod{8} \). It follows that \( \text{Image}(p_{1*}) = \text{Image}(p_{2*}) = 2\pi_n(S^n) \) and so \( p \) does not have a cross section if \( n \equiv 1 \pmod{8} \).

Let \( n \equiv 5 \pmod{8} \). Note that \( i_* \) is the composition of the canonical homomorphisms:

\[
\pi_{n-1}(Sp(k)) \longrightarrow \pi_{n-1}(U(2k)) \longrightarrow \pi_{n-1}(SO(n-1)) \longrightarrow \pi_{n-1}(SO(n)).
\]

It follows from (1), (2), (5) and an equation \((2k)! \equiv 0 \pmod{4}\) that \( i_* = 0 \). Hence, by (2), (3), (4), (6) and (7), we have a commutative diagram with exact columns:
Choose $x \in \pi_n(SO(n)/Sp(k))$ such that $\partial(x)$ is the generator. Then $t_*(x)$ is of finite order and $\Delta(t_*(x)) = \partial(x)$. Hence the order of $t_*(x)$ is two, so $\Delta$ splits and $p_*$ maps $Z$ isomorphically onto a free summand. Therefore $\text{Image}(p_*) = \text{Image}(p_{2*})$ which equals to $2\pi_n(S^n)$ by (8), so $p$ does not have a cross section. This completes the proof.

References


*Added in proof.* I was informed from T. Önder that he also obtained Theorem 1 by using his methods [5].