
by

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Abstract

Here we provide a correct proof of Proposition 6 of [2]. No other results of the latter paper are affected.

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§1. Necessary results and corrected proof

To make this note self-contained we briefly recall the basic set-up of [2]. Given a \((k + 1)\)-tuple of polynomials \((Q_k(z), Q_{k-1}(z), \ldots, Q_0(z))\) with \(\deg Q_i(z) \leq i\) consider the homogenized spectral pencil of differential operators given by

\[
T_\lambda = \sum_{i=0}^{k} Q_i(z) \lambda^{k-i} \frac{d^i}{dz^i}.
\]

Introduce the algebraic curve \(\Gamma\) associated with \(T_\lambda\) and given by the equation

\[
\sum_{i=0}^{k} Q_i(z) w^i = 0,
\]

where the polynomials \(Q_i(z) = \sum_{j=0}^{i} a_{i,j} z^j\) are the same as in (1.1).

The curve \(\Gamma\) and its associated pencil \(T_\lambda\) are called of general type if the following two nondegeneracy requirements are satisfied:

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(i) \( \deg Q_k(z) = k \) (i.e., \( a_{k,k} \neq 0 \)),

(ii) no two roots of the (characteristic) equation

\[
a_{k,k} + a_{k-1,k-1} t + \ldots + a_{0,0} t^k = 0
\]  

lie on a line through the origin (in particular, 0 is not a root of (1.3)).

The first statement of [2] we need is as follows.

**Proposition 1.** If the characteristic equation (1.3) has \( k \) distinct solutions \( \alpha_1, \ldots, \alpha_k \) and satisfies the above nondegeneracy assumptions (in particular, these imply that \( a_{0,0} \neq 0 \) and \( a_{k,k} \neq 0 \)) then

(i) for all sufficiently large \( n \) there exist exactly \( k \) distinct eigenvalues \( \lambda_{n,j} \), \( j = 1, \ldots, k \), such that the associated spectral pencil \( T_\lambda \) has a polynomial eigenfunction \( p_{n,j}(z) \) of degree exactly \( n \),

(ii) the eigenvalues \( \lambda_{n,j} \) split into \( k \) distinct families labeled by the roots of (1.3) such that the eigenvalues in the \( j \)-th family satisfy

\[
\lim_{n \to \infty} \frac{\lambda_{n,j}}{n} = \alpha_j, \quad j = 1, \ldots, k.
\]

The main result of [2] is given below.

**Theorem 1.** In the notation of Proposition 1, for any pencil \( T_\lambda \) of general type and every \( j = 1, \ldots, k \) there exists a subsequence \( \{n_{i,j}\}, i = 1, 2, \ldots \) such that the limits

\[
\Psi_j(z) := \lim_{i \to \infty} \frac{\mu'_{n_{i,j}}(z)}{\lambda_{n_{i,j}} p_{n_{i,j}}(z)}, \quad j = 1, \ldots, k,
\]

exist almost everywhere in \( \mathbb{C} \) and are analytic functions in some neighborhood of \( \infty \). Each \( \Psi_j(z) \) satisfies equation (1.2), i.e., \( \sum_{i=0}^k Q_i(z) \Psi_i(z) = 0 \) almost everywhere in \( \mathbb{C} \), and the functions \( \Psi_1(z), \ldots, \Psi_k(z) \) are independent sections of \( \Gamma \) considered as a branched covering over \( \mathbb{C}P^1 \) in a sufficiently small neighborhood of \( \infty \).

The proof requires Lemma 1 and Proposition 2 below. (The proof of the latter proposition suggested in [2] was erroneous and is corrected below.)

**Lemma 1** (cf. Lemma 8 of [1]). Let \( \{q_m(z)\} \) be a sequence of polynomials with \( \deg q_m(z) \to \infty \) as \( m \to \infty \). Denote by \( \mu_m \) and \( \mu'_m \) the root-counting measures of \( q_m(z) \) and \( q'_m(z) \), respectively, and assume that there exists a compact set \( K \) containing the supports of all measures \( \mu_m \) and therefore also the supports of all measures \( \mu'_m \). If \( \mu_m \to \mu \) and \( \mu'_m \to \mu' \) as \( m \to \infty \), and \( u \) and \( u' \) are the logarithmic potentials of \( \mu \) and \( \mu' \), respectively, then \( u' \leq u \) in \( \mathbb{C} \) with equality in the unbounded component of \( \mathbb{C} \setminus \text{supp}(\mu) \).
Example 1. Consider the polynomial sequence \( \{ z^m - 1 \} \). The measure \( \mu \) is then the uniform distribution on the unit circle of total mass 1. Its logarithmic potential \( u(z) \) equals \( \log |z| \) if \( |z| \geq 1 \) and 0 in the disk \( |z| \leq 1 \). On the other hand, the sequence of derivatives is given by \( \{ mz^{m-1} \} \) and the corresponding (limiting) logarithmic potential \( u'(z) \) equals \( \log |z| \) in \( \mathbb{C} \setminus \{0\} \). Obviously, \( u(z) = u'(z) \) in \( |z| \geq 1 \) and \( u'(z) < u(z) \) in \( |z| < 1 \).

In the notation of Theorem 1 consider the family of eigenpolynomials \( \{ p_{n,j}(z) \} \) for some arbitrarily fixed value of the index \( j = 1, \ldots, k \). Assume that \( N_j \) is a subsequence of the natural numbers such that

\[
\mu_j^{(i)} := \lim_{n \to \infty, n \in N_j} \mu_{n,j}^{(i)}
\]

exists for \( i = 0, \ldots, k \), where \( \mu_{n,j}^{(i)} \) denotes the root-counting measure of \( p_{n,j}^{(i)}(z) \). The existence of such \( N_j \) follows by Helly’s theorem from the existence of a compact set \( K \) that contains the support of all \( \mu_{n,j}^{(i)} \). Notice that for each \( i \) the logarithmic potential \( u_j^{(i)} \) of \( \mu_j^{(i)} \) satisfies a.e. the identity

\[
u_j^{(i)}(z) - u_j^{(0)}(z) = \lim_{n \to \infty, n \in N_j} \frac{1}{n} \log \left| \frac{p_{n,j}^{(i)}(z)}{n(n-1) \cdots (n-i+1)p_{n,j}(z)} \right|.
\]

The next proposition completes the proof of Theorem 1 and also shows the remarkable property that if one considers a sequence of eigenpolynomials for some spectral pencil then the situation \( u'(z) < u(z) \) seen in Example 1 can never occur. In fact, for the validity of Proposition 2 one only needs two assumptions:

(a) \( \deg Q_k(z) = k \) (i.e., \( a_{k,k} \neq 0 \), so that all \( \alpha_j, j = 1, \ldots, k \), are non-zero) and
(b) \( Q_0 \neq 0 \).

**Proposition 2.** The measures \( \mu_j^{(i)} \), \( i = 0, \ldots, k \), are all equal and the scalar multiple \( \tilde{\Psi}_j = C \mu_j / \alpha_j \) of the Cauchy transform of this common measure \( \mu_j \) satisfies equation (1.2) almost everywhere.

**Proof.** For \( n \in N_j \) one has

\[
\frac{p_{n,j}^{(i+1)}(z)}{(n-i)p_{n,j}^{(i)}(z)} \to C^{(i+1)}(z) := \int_{\mathbb{C}} \frac{dp_j^{(i)}(\zeta)}{z - \zeta} \quad \text{as } n \to \infty
\]

with convergence in \( L^1_{\text{loc}} \). The well-known property of convergence in \( L^1_{\text{loc}} \) implies that passing to a subsequence one can assume that the above convergence is actually the pointwise convergence almost everywhere in \( \mathbb{C} \). It follows that

\[
\frac{p_{n,j}^{(i)}(z)}{n^i p_{n,j}(z)} \to C^{(1)}(z) \cdots C^{(i)}(z),
\]
pointwise almost everywhere in C. We claim that this limit is non-zero a.e. Granted this, consider

\[ u_j^{(k)}(z) - u_j^{(0)}(z) = \lim_{n \to \infty, n \in \mathbb{N}} \frac{1}{n} \log \left| \frac{p_{n,j}^{(k)}(z)}{n(n-1) \ldots (n-k+1)p_{n,j}(z)} \right| = 0 \]

almost everywhere in C. On the other hand, \( u_j^{(0)} \geq u_j^{(1)} \geq \ldots \geq u_j^{(k)} \) by Lemma 1. Hence the potentials \( u_j^{(i)} \) are all equal and the corresponding measures \( \mu_j^{(i)} = \Delta u_j^{(i)}/2\pi \) are equal as well.

It remains to settle the above claim. Recall that \( p_{n,j}(z) \) satisfies the differential equation \( T_{\lambda_{n,j}} p_{n,j}(z) = 0 \), i.e.,

\[ Q_k(z)p_{n,j}(z) + \lambda_{n,j} Q_{k-1}(z)p_{n,j}^{(k-1)}(z) + \cdots + \lambda_{n,j}^k Q_0(z)p_{n,j}(z) = 0. \]

Therefore,

\[ Q_k(z) \frac{p_{n,j}^{(k)}(z)}{n^k p_{n,j}(z)} + \frac{\lambda_{n,j}}{n} Q_{k-1}(z) \frac{p_{n,j}^{(k-1)}(z)}{n^{k-1} p_{n,j}(z)} + \cdots + \frac{\lambda_{n,j}^k}{n^k} Q_0(z) = 0. \]

Using the asymptotics \( \lambda_{n,j} \sim \alpha_j n \) and the pointwise convergence a.e. in (1.5) we get

\[ Q_k(z)C^{(1)}(z) \ldots C^{(k)}(z) + \alpha_j Q_{k-1}(z) C^{(1)}(z) \ldots C^{(k-1)}(z) \]

\[ + \cdots + \alpha_j^k Q_0(z) = 0. \]

Using the assumption that \( Q_0 \neq 0 \neq \alpha_j \), we conclude that \( C^{(1)}(z) \neq 0 \) a.e. To prove that \( C^{(2)}(z) \) is also non-zero a.e., we consider the differential equation satisfied by \( p_{n,j}'(z) \),

\[ Q_k(z)p_{n,j}^{(k+1)}(z) + (Q_k(z) + \lambda_{n,j} Q_{k-1}(z))p_{n,j}^{(k)}(z) \]

\[ + \cdots + (\lambda_{n,j}^k Q_1(z) + \lambda_{n,j}^k Q_0(z))p_{n,j}'(z) = 0, \]

which is obtained by differentiating (1.6). Repeating the previous analysis we get

\[ Q_k(z) \frac{p_{n,j}^{(k+1)}(z)}{n^k p_{n,j}'(z)} + \frac{\lambda_{n,j}}{n} Q_{k-1}(z) \frac{p_{n,j}^{(k)}(z)}{n^{k-1} p_{n,j}'(z)} \]

\[ + \cdots + \frac{\lambda_{n,j}^k}{n^k} Q_1(z) + \lambda_{n,j}^k Q_0(z) = 0. \]

Hence in the limit we obtain

\[ Q_k(z)C^{(2)}(z) \ldots C^{(k+1)}(z) + \alpha_j Q_{k-1}(z) C^{(2)}(z) \ldots C^{(k)}(z) + \cdots + \alpha_j^k Q_0(z) = 0, \]
which implies that $C^{(2)}(z)$ is non-zero a.e. as well. Similarly, $C^{(i)}(z)$, $i \geq 3$, is non-zero a.e., which proves the claim.

The fact that the multiple $\bar{\Psi}_j = C_\mu/\alpha_j$ of the Cauchy transform of this common measure $\mu_j$ satisfies equation (1.2) almost everywhere follows by (1.7), since the equality of the measures implies that $C_\mu = C^{(1)} = C^{(2)} = \ldots$, and thus

$$Q_k(z)(C_\mu(z))^k + \alpha_j Q_{k-1}(z)(C_\mu(z))^{k-1} + \cdots + \alpha_j^k Q_0(z) = 0,$$

which is equivalent to (1.2).

Note that in Example 1, the polynomials $p_n(z) := z^n - 1$ satisfy the differential equation $zp_n''(z) - (n-1)p_n' = 0$. They may thus be thought of as eigenpolynomials of the pencil $z \frac{d^2}{dz^2} - (\lambda - 1) \frac{d}{dz}$ corresponding to positive integer values $n$ of $\lambda$. The corresponding homogenized pencil $z \frac{d^2}{dz^2} - \lambda \frac{d}{dz}$ has $Q_0 = 0$, and so does not satisfy the hypothesis of the proposition.

References
