The First Cohomology Groups of Infinite Dimensional Lie Algebras

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Introduction

Let $V$ be a finite dimensional vector space. We denote by $D(V)$ the Lie algebra consisting of all formal vector fields over $V$. Let $L$ be a Lie subalgebra of $D(V)$. We are interested in the first cohomology group $H^1(L)$ of a Lie algebra $L$ with adjoint representation.

Let $L$ be an infinite dimensional transitive simple Lie algebra, that is, $L$ is one of $D(V)$, $L_{s1}$, $L_{s2}$, or $L_{ct}$. (For a notation, see §2.) It is known in T. Morimoto [5] that $H^1(D(V)) = H^1(L_{ct}) = 0$, and $\dim H^1(L_{s1}) = \dim H^1(L_{s2}) = 1$.

In this paper we will treat the following two types of infinite dimensional Lie algebras:

1. Infinite dimensional transitive graded Lie algebras $\mathfrak{g} = \sum_{p=-\infty}^{\infty} \mathfrak{g}_p$. (For a precise definition, see §1.)

2. Infinite dimensional intransitive Lie algebras $L[W^\ast]$ whose transitive parts $L$ are infinite and simple. (In this case $W$ is a subspace of $V$.)

In Section 3 and Section 4, we will give two criteria for $H^1(\mathfrak{g})$ to be of finite dimension. More precisely we will prove

**Theorem A.** Let $\mathfrak{g} = \sum_{p=-\infty}^{\infty} \mathfrak{g}_p$ be an infinite transitive graded Lie algebra with a semi-simple linear isotropy algebra $\mathfrak{g}_0$. Then $H^1(\mathfrak{g})$ is finite dimensional.

**Theorem B.** Let $\mathfrak{g} = \sum_{p=-\infty}^{\infty} \mathfrak{g}_p$ be an infinite transitive graded Lie algebra whose linear isotropy algebra $\mathfrak{g}_0$ contains an element $e$ which satisfies $[e, x_p] = \ldots$
$px_p$ for all $x_p \in g_p$. Then $H^1(g)$ is finite dimensional. Furthermore if $g$ is derived from $g_0$, then $H^1(g)$ is isomorphic to $n(g_0)/g_0$, where $n(g_0)$ denotes the normalizer of $g_0$ in $gl(g_{-1})$.

It may well be doubted if every infinite transitive graded Lie algebra $g$ has the finite dimensional cohomology group $H^1(g)$. But unfortunately this presumption is false. In Section 5 we will give an easy condition for $g$ to be $\dim H^1(g) = \infty$. (For such a Lie algebra $g$, we can construct derivations of arbitrarily large negative degree.)

That is, we will prove

**Theorem C.** Let $g = \sum_{p=-1}^{\infty} g_p$ be an infinite transitive graded Lie algebra which satisfies $g^{(2)} = [g^{(1)}, g^{(1)}] = 0$, where $g^{(1)} = [g, g]$. Then $H^1(g)$ is infinite dimensional.

In Section 6 our objects are infinite intransitive Lie algebras $L[W^*]$. Let $V = U + W$ (direct sum). We denote by $S(W^*)$ the ring of formal power series over $W$. Let $L$ be an infinite transitive simple Lie algebra over $U$. Then a Lie algebra $L[W^*]$ is obtained as a topological completion of $L \otimes S(W^*)$. These Lie algebras $L[W^*]$ are obtained as the result of the classification theorem of infinite intransitive Lie algebras [6]. In determining $H^1(L[W^*])$, V. Guillemin's work is essential. Using his results we will prove

**Theorem D.** Let $D(W)$ be a Lie algebra of all formal vector fields over $W$ and let $e$ be a basis of one dimensional center of $gl(U)$. Then we have

$$H^1(L[W^*]) \cong \begin{cases} D(W) & \text{for } L = D(U) \text{ or } L_{et}(U), \\ D(W) + S(W^*) \otimes e & \text{for } L = L_{et}(U) \text{ or } L_{ep}(U). \end{cases}$$

Above results can be considered as a formal version of Y. Kanie [3]. In a forthcoming paper, we will give an example of an infinite intransitive Lie algebra $L$ such that $H^1(L) = 0$.

Throughout this paper, all vector spaces and Lie algebras are assumed to be defined over the field $C$ of complex numbers.

§1. Infinite Transitive Graded Lie Algebras

In this section, we define transitive graded Lie algebras which we will study in the subsequent sections.
Definition 1.1. Let \( g \) be a Lie algebra. Assume that there is given a family \( \{g_p\}_{p \geq -1} \) of subspaces of \( g \) which satisfies the following conditions:

a) \( g = \bigoplus_{p=-1}^{\infty} g_p \) (direct sum);

b) \( \dim g_p < \infty \);

c) \( [g_p, g_q] \subseteq g_{p+q} \);

d) For every non-zero \( x_p \in g_p, p \geq 0 \), there is an element \( x_{-1} \in g_{-1} \) such that \( [x_p, x_{-1}] \neq 0 \). Under these conditions, we say that the direct sum \( g = \sum_{p=-1}^{\infty} g_p \) or simply \( g \) is a transitive graded Lie algebra.

By conditions c) and d), \( g_0 \) is considered as a Lie subalgebra of \( gl(g_{-1}) \). The Lie algebra \( g_0 \) is called the linear isotropy algebra of \( g \). A graded Lie algebra \( g \) is said to be irreducible if the representation of \( g_0 \) on the vector space \( g_{-1} \) given by \( [g_0, g_{-1}] \subseteq g_{-1} \) is irreducible.

Definition 1.2. The space \( g^{(p)} \) which is called the \( p \)-th prolongation of \( g_0 \) is defined by

\[
g^{(p)} = g_0 \otimes S^p(g^*_1) \cap g_{-1} \otimes S^{p+1}(g^*_1),
\]

where \( S^p(g^*_1) \) denotes the \( p \)-times symmetric tensor of the dual space \( g^*_1 \) of \( g_{-1} \).

We say that \( g_0 \) is of finite type if \( g^{(p)}_0 = 0 \) for some (and hence for all larger) \( p \). Otherwise we say that \( g_0 \) is of infinite type. Put \( g^{-1}_0 = g_{-1}, g^{(0)}_0 = g_0 \) and \( g = \sum_{p=-1}^{\infty} g^{(p)}_0 \). Then \( g \) has a Lie algebra structure with respect to a canonical bracket operation. We say that the transitive graded Lie algebra \( \tilde{g} = \sum_{p=-1}^{\infty} g^{(p)}_0 \) thus obtained is derived from \( g_0 \). If \( g \) is an abstract transitive graded Lie algebra with a linear isotropy algebra \( g_0 \), then \( g \) is considered as a graded Lie subalgebra of \( \tilde{g} \). It is clear that if a transitive graded Lie algebra \( g \) is of infinite dimension, its linear isotropy algebra \( g_0 \) must be of infinite type.

Let \( A \) be a Lie algebra. A derivation \( c \) of \( A \) is a linear mapping from \( A \) to itself satisfying \( c[x, y] = [c(x), y] + [x, c(y)] \) for all \( x, y \in A \). We denote by \( \text{Der}(A) \) (resp. \( \text{ad}(A) \)) the derivation algebra (resp. the algebra of inner derivations of \( A \)). Then, by definition, the first cohomology group \( H^1(A) \) of \( A \) with adjoint representation is equal to the space \( \text{Der}(A)/\text{ad}(A) \). A derivation \( c \) of a graded Lie algebra \( g = \sum_{p=-1}^{\infty} g_p \) is said to be of degree \( r \) or \( \deg c = r \) if it satisfies \( c(g_p) \subseteq g_{p+r} \) for all \( p \).
§ 2. Infinite Transitive Simple Lie Algebras

It is well-known that there are the following four classes of infinite transitive simple Lie algebras over $\mathbb{C}$ (see [5]).

(1) $L_{\mathfrak{g}1}(n)$: the Lie algebra of all formal (or better, formal power series) vector fields in $n$-variables $x_1, x_2, \ldots, x_n$.

(2) $L_{\mathfrak{g}1}(n)$: the Lie algebra of formal vector fields in $n$-variables $x_1, x_2, \ldots, x_n$, preserving the volume form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

(3) $L_{\mathfrak{g}2}(2n)$: the Lie algebra of formal vector fields in $2n$-variables $x_1, x_2, \ldots, x_{n}, y_1, y_2, \ldots, y_{n}$ preserving the symplectic form $\sum_{i=1}^{n} dx_i \wedge dy_i$.

(4) $L_{\mathfrak{g}3}(2n+1)$: the Lie algebra of formal vector fields in $(2n+1)$-variables $z, x_1, x_2, \ldots, x_{n}, y_1, y_2, \ldots, y_{n}$, preserving the contact form $dz + \sum_{i=1}^{n} x_i \cdot dy_i - y_i dx_i$, up to functional factors.

We will often write $D(V)$ for $L_{\mathfrak{g}1}(n)$, where $V$ is an $n$-dimensional vector space with a basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Let $L$ be one of Lie algebras $D(V), L_{\mathfrak{g}1}$ and $L_{\mathfrak{g}2}$. Each $L$ has the natural filtration $\{L_p\}_{p \in \mathbb{Z}}$ defined as follows.

- $L_p = L$ for $p \leq -1$;
- $L_0 = \{X \in L; \text{ the value } X(0) \text{ of } X \text{ at the origin } = 0\}$;
- $L_p = \{X \in L_{p-1}; [X, L] \subset L_{p-1}\}$ for $p \geq 1$.

Then the decreasing sequence of subspaces: $L = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$ satisfies

(a) $\bigcap_{p=-1}^{\infty} L_p = 0$;
(b) $[L_p, L_{\mathfrak{q}2}] \subset L_{p+\mathfrak{q}2}$;
(c) $\dim L_p/L_{p+1} < \infty$.

Put $\mathfrak{g}_p(L) = L_p/L_{p+1}$. Then by (b), (c) and the definition of $L_p$, $p \geq 1$, we have the transitive graded Lie algebra $\mathfrak{g}(L) = \sum_{p=-1}^{\infty} \mathfrak{g}_p(L)$. We also have the Lie algebra $L' = \prod_{p=-1}^{\infty} \mathfrak{g}_p(L)$, which is the completion of $\mathfrak{g}(L)$.

Under these notations we will summarize a few useful properties of $L$.

(1) Each $L$ is an infinite transitive irreducible Lie algebra and moreover $L$ is isomorphic to $L'$, where the word "irreducible" means that the action of $\mathfrak{g}_0(L)$ on $\mathfrak{g}_{-1}(L)$ is irreducible.

(2) The linear isotropy algebras $\mathfrak{g}_0(L)$ of $D(V), L_{\mathfrak{g}1}$ and $L_{\mathfrak{g}2}$ are $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{C})$, and $\mathfrak{gl}(2n, \mathbb{C})$, respectively.
sl(n, C) and sp(n, C) respectively, and for \( p \geq 1 \) \( g_p(L) \) is isomorphic to the \( p \)-th prolongation \( g_0(L)^{(p)} \) of \( g_0(L) \).

(3) For \( g_0(L) = \text{sl}(n, C) \) or \( \text{sp}(n, C) \), it holds that

(i) \([g_0(L)^{(r)}, g_0(L)^{(s)}] = g_0(L)^{(r+s)} \) for \( r, s \geq 0 \),

(ii) \( g_0(L) \) acts irreducibly on \( g_0(L)^{(r)} \) for \( r \geq -1 \).

By the classification theorem of Kobayashi-Nagano [4], we know that there are only three classes of transitive simple irreducible Lie algebras of infinite type over \( C \), that is, they are \( D(V) \), \( L_{\text{et}} \) and \( L_{\text{gp}} \).

For the contact Lie algebra \( L_{\text{ct}}(2n + 1) \) (or simply \( L_{\text{ct}} \)), we must define another filtration.

\[
L_p = \begin{cases} 
L_{\text{ct}} & \text{for } p \leq -2; \\
L_{-1} &= \{X \in L_{\text{ct}}; \langle X, \theta \rangle_0 = 0, \text{ where } \theta \text{ is the contact form} \}; \\
L_0 &= \{X \in L_{\text{ct}}; X(0) = 0 \}; \\
L_p &= \{X \in L_{p-1}; [X, L_{p-1}] \subseteq L_{p-1} \} & \text{for } p \geq 1.
\end{cases}
\]

Using this filtration, \( L_{\text{ct}} \) is isomorphic to \( \prod_{p=-2}^{\infty} g_p(L) \). For the subsequent discussion about \( L_{\text{ct}} \), we have only to recall that

(4) \( L_{-1} = [L_{\text{ct}}, L_{1}] \).

In Section 3 and Section 6, we essentially use the following facts which were proved by T. Morimoto [5].

**Theorem 2.1.** Let \( L \) be an infinite transitive simple Lie algebra over \( C \). Then

\[
H^1(L) \cong \begin{cases} 
0 & \text{for } L = D(V) \text{ or } L_{\text{ct}} \\
C & \text{for } L = L_{\text{et}} \text{ or } L_{\text{gp}}.
\end{cases}
\]

**Remark 1.** Let \( L \) be one of Lie algebras \( L_{\text{et}} \) or \( L_{\text{gp}} \). Since \( L \) is isomorphic with the Lie algebra \( L' = \prod_{p=-1}^{\infty} g_p(L) \), their isotropy algebras \( \text{sl}(V) \) and \( \text{sp}(V) \) are considered to be subalgebras of them. Let \( e \) denote a unit matrix in \( \text{gl}(V) \). Then the above theorem asserts that \( \text{ad}(e) \) yields a basis of one dimensional space \( H^1(L) \).

**Remark 2.** Let \( \text{gr}(L) \) be a graded Lie algebra associated with an infinite transitive simple Lie algebra \( L \). Then we also have \( H^1(\text{gr}(L)) = 0 \) for \( L = D(V) \) or \( L_{\text{ct}} \), and \( H^1(\text{gr}(L)) \cong C \) for \( L = L_{\text{et}} \) or \( L_{\text{gp}} \). These facts are particularly used in Section 3.
§3. The First Cohomology Groups of Infinite Transitive
Graded Lie Algebras (I)

Throughout this section, let $g = \sum_{p=-1}^{\infty} g_p$ be an infinite (dimensional) transitive graded Lie algebra over $C$ and let its linear isotropy algebra $g_0$ be semi-simple. Put $g_{-1} = V$. Then $g_0$ is considered as a Lie subalgebra of $\text{gl}(V)$. First we will determine the type of $g$.

Since $g_0$ is semi-simple, we can decompose $V$ into $V = V_1 + V_2 + \cdots + V_k$ (vector space direct sum), where each $V_i$ $(i=1, 2, \ldots, k)$ is a $g_0$-invariant subspace and $g_0$ acts irreducibly on $V_i$. We denote by $h_i$ the Lie algebra of linear transformations of $V_i$ induced by $g_0$. By the natural inclusion, $h_i$ is considered as a Lie subalgebra of $\text{gl}(V)$. We also denote it by the same letter $h_i$ if there is no confusion. Put $n_i = \{t \in g_0; t(V_j) = 0 \text{ for all } j \neq i\}$. Then each $n_i$ is an ideal of $g_0$ and $n_1 + \cdots + n_k$ is a direct sum as Lie algebras. It clearly holds

\begin{equation}
(3.1) \quad n_1 + \cdots + n_k \subset g_0 \subset h_1 + \cdots + h_k.
\end{equation}

Lemma 3.1. For $p \geq 1$, $g_0^{(p)} = n_1^{(p)} + \cdots + n_k^{(p)}$ (direct sum).

Proof. Let $t: V \times \cdots \times V \to V$ be an element of $g_0^{(p)}$. First note that $t(v_1, \ldots, v_{p+1}) = 0$ if $v_i \in V_i$, $v_j \in V_j$ for $i \neq j$. (It is easy to see that $t(v_1, \ldots, v_{p+1}) \in V_i \cap V_j$.) Let $v_1, \ldots, v_{p+1} \in V$ and $v_i = v_i^1 + \cdots + v_i^k$ with $v_i^1 \in V_1$, $v_i^2 \in V_2, \ldots, v_i^k \in V_k$ for $i=1, \ldots, p+1$. Then by the above remark, we have

\begin{equation}
(3.2) \quad t(v_1, \ldots, v_{p+1}) = t(v_1^1, \ldots, v_{p+1}^1) + \cdots + t(v_1^k, \ldots, v_{p+1}^k) = t_1(v_1^1, \ldots, v_{p+1}^1) + \cdots + t_k(v_1^1, \ldots, v_{p+1}^k),
\end{equation}

where $t_i$ denotes an element of $n_i^{(p)}$ induced by $t$. (Since $t_i(\ast, v'_1, \ldots, v'_p) \in n_i$ for $v'_1, \ldots, v'_p \in V$, $t_i$ is an element of $n_i^{(p)}$.) Since $n_1 + \cdots + n_k$ is a direct sum, our assertion is obvious. 

q.e.d.

Since $g$ is infinite dimensional and $g_p$ is a subspace of $g_0^{(p)}$, $g_0$ must be of infinite type by Lemma 3.1. From now on, without loss of generality, we assume that $n_1, \ldots, n_l$ $(l \leq k)$ are of infinite type and $n_{l+1}, \ldots, n_k$ are of finite type.

Lemma 3.2. Let $g_0$ be a linear isotropy algebra of an infinite transitive graded Lie algebra $g$. Then there exists a Lie subalgebra $g_0$ of finite type of $h_{l+1} + \cdots + h_k$ and $g_0$ is written as
(3.3) \( g_0 = n_1 + \cdots + n_l + g_b \) \((\text{Lie algebra direct sum})\),
where each ideal \( n_i \) \((i = 1, \ldots, l)\) is isomorphic to either \( \mathfrak{sl}(V_i) \) or \( \mathfrak{sp}(V_i) \).

**Proof.** Let \( \pi : g_0 \to h_i \) \((i = 1, \ldots, l)\) be a natural projection. Since \( \pi \) is a Lie algebra homomorphism, \( g_0 / \text{Ker} \pi \) is isomorphic to \( h_i \). Recall that the quotient space of a semi-simple Lie algebra is also semi-simple. Thus \( h_i \) is semi-simple and its center is zero. Moreover each \( h_i \) acts irreducibly on \( V_i \) and is of infinite type. Hence by the classification theorem of transitive irreducible Lie algebras of infinite type, we know that \( h_i \) must be equal to either \( \mathfrak{sl}(V_i) \) or \( \mathfrak{sp}(V_i) \).

Since \( n_i \) is an ideal of \( h_i \), we have \( n_i = \mathfrak{sl}(V_i) \) or \( \mathfrak{sp}(V_i) \). (\( \mathfrak{sl}(V_i) \) and \( \mathfrak{sp}(V_i) \) are naturally imbedded in \( \mathfrak{gl}(V_i) \)). Note that \( n_1 = h_1, n_2 = h_2, \ldots, n_l = h_l \). Then we can find a subspace \( g_b \) such that \( n_{i+1} + \cdots + n_k \subseteq g_b \subseteq h_{i+1} + \cdots + h_k \). Considering (3.1), we obtain the expression of \( g_0 \) as (3.3). By Lemma 3.1, we also have \( g_0 = n_{i+1} + \cdots + n_k \). Thus \( g_b \) is of finite type. q.e.d.

Next we will determine the type of \( g_1 \). From (3.3) in Lemma 3.2, we have \( g_0 = n_1 + \cdots + n_l + g_b \), and \( g_1 \) is a subspace of \( g_0 \). Without loss of generality, we assume that \( g_1 \cap n_1 \neq 0, \ldots, g_1 \cap n_l \neq 0 \) and \( g_1 \cap n_{l+1} = 0, \ldots, g_1 \cap n_{m+1} = 0 \). Then we have

**Lemma 3.3.** \( g_1 \) has the following form:

\[
g_1 = n_1 + \cdots + n_m + H_1,
\]

where \( H_1 \) is a subspace of \( g_b \).

**Proof.** For \( i = 1, \ldots, m \), \( g_1 \cap n_i \) is an \( n_i \)-invariant subspace of \( n_i \). By the property (3) (ii) in Section 2, we have \( g_1 \supseteq n_1 + \cdots + n_m \). Hence there exists a subspace \( H_1 \) of \( n_1 + \cdots + n_m + g_b \) such that \( g_1 = n_1 + \cdots + n_m + H_1 \) and \( H_1 \cap n_i = 0 \). For \( j = m+1, \ldots, l \), decompose \( t \in H_1 \) into \( t = t_{m+1} + \cdots + t_l + t_b \) with \( t_m + \cdots + t_l + t_b \in g_b \). Define a subspace \( A_j \) of \( n_j \) by

\[
A_j = \{ t_j \in n_j; t = t_{m+1} + \cdots + t_l + t_b \in H_1 \}.
\]

For all \( x_j \in n_j \) and \( t_j \in A_j \), it holds that \( [x_j, t_j] = [x_j, t] \in n_j \) \& \( g_t = \{ 0 \} \). This means that \( n_j, A_j = 0 \). Using the property (3) (ii), we have \( A_j = 0 \) for \( j = m+1, \ldots, l \). Hence \( H_1 \subseteq g_b \). q.e.d.

Since \( g_p \supseteq A_1, [A_1, \ldots, [A_1, g_1], \ldots \} \) for \( p > 1 \), \( g_p \) contains \( n_1 + \cdots + n_m \) by Lemma 3.3 and the property (3) (i) in Section 2. By the same argument as Lemma 3.3, we get
Lemma 3.4. For \( p > 1 \), \( g_p \) has the following form:
\[
g_p = n_1^{(p)} + \cdots + n_m^{(p)} + H_p,
\]
where \( H_p \) is a subspace of \( g_0^{(p)} \). (For sufficiently large \( p \), \( H_p = 0 \) since \( g_0 \) is of finite type.)

By Lemma 3.3 and Lemma 3.4, we can easily determine the form of the given infinite transitive graded Lie algebra \( g = \sum_{p=1}^{\infty} g_p \). That is, we have

Proposition 3.5. Let \( g = \sum_{p=1}^{\infty} g_p \) be an infinite transitive graded Lie algebra. Then \( g \) has the following form:
\[
g = G_1 + \cdots + G_m + G_{m+1} + \cdots + G_i + G_j \quad \text{(direct sum)},
\]
where \( G_i (i = 1, \ldots, m) \) is of the form \( gr(L_{g_1}(V_i)) \) or \( gr(L_{s_0}(V_i)) \), and \( G_j (j = m+1, \ldots, l) \) is of the form \( V_j + sl(V_j) \) or \( V_j + sp(V_j) \), and \( G_j \) is a finite dimensional Lie algebra. (From now on, we put \( G' = G'_{m+1} + \cdots + G_i + G_j \). Then \( G \) is a finite dimensional ideal of \( g \).)

For computing \( H^1(g) \), we need some lemmas.

Lemma 3.6. Let \( A \) be an abstract Lie algebra and let \( A_i (i = 1, \ldots, k) \) be perfect ideals of \( A \). If \( A = A_1 + \cdots + A_k \) (direct sum), then \( H^1(A) \cong H^1(A_1) + \cdots + H^1(A_k) \) (direct sum).

Proof. Let \( c \in \text{Der}(A) \). We denote by \( c_{ij} \) the \( \text{Hom}(A_i, A_j) \)-component of \( c \). For \( x, y \in A_i \), we have
\[
(3.4) \quad c[x, y] = [c(x), y] + [x, c(y)]
= \sum_{j=1}^k [c_{ij}(x), y] + \sum_{j=1}^k [x, c_{ij}(y)]
= [c_{ij}(x), y] + [x, c_{ij}(y)] \in A_i.
\]
Combined this with \( A_i = [A_i, A_i] \), we obtain \( c_{ij} = 0 \) for \( i \neq j \). Put \( c_{ii} = c_i \). By (3.4), \( c_i \) induces a derivation of \( A_i \). Hence \( \text{Der}(A) = \text{Der}(A_1) + \cdots + \text{Der}(A_k) \) (direct sum). Our assertion is now evident. q.e.d.

Lemma 3.7. Let \( A \) be an abstract Lie algebra such that \( A = A_1 + A_2 \) (direct sum) with \( A_1 = [A_1, A_1] \). Moreover assume that the center of \( A_1 \) is zero. Then \( H^1(A) \cong H^1(A_1) + H^1(A_2) \) (direct sum).

Proof. We can write \( c = c_{11} + c_{12} + c_{21} + c_{22} \) by using same notations as Lemma 3.6. Since \( A_1 \) is perfect, we have \( c_{12} = 0 \). Let \( x \in A_1 \) and \( y \in A_2 \). By the equation \( 0 = c[x, y] = [c(x), y] + [x, c(y)] \), we get \( [x, c_{21}(y)] = 0 \). This
means that $c_{21}(y) \in \{\text{center of } A_1\}$. Since the center of $A_1$ is zero, we have $c_{21}=0$. Now it is easy to verify the assertion.

Combined with Theorem 2.1, we obtain finally the following theorem.

**Theorem 3.8.** Let $g = \sum_{p=-1}^\infty g_p$ be an infinite transitive graded Lie algebra with a semi-simple linear isotropy algebra $g_0$. Then $H^1(g)$ is finite dimensional.

**Proof.** By Proposition 3.5, $g$ has the following form: $g = G_1 + \cdots + G_m + G'$, where $\dim G' < \infty$. Since $G_1 + \cdots + G_m$ is perfect and has no non-trivial center, we have $H^1(g) \cong H^1(G_1 + \cdots + G_m) + H^1(G')$ by Lemma 3.7. On the other hand, $H^1(G_1 + \cdots + G_m) \cong H^1(G_1) + \cdots + H^1(G_m)$ by Lemma 3.6, and $\dim H^1(G_i) = 1$ or 0 for $i = 1, \ldots, m$ by Theorem 2.1. (See also Remark 2, and recall that $G_i = \text{gr}(L_{\pi_i}(V_i))$ or $\text{gr}(L_{\pi_i}(V_i))$. Thus we obtain $\dim H^1(g) < \infty$.

q.e.d.

§ 4. The First Cohomology Groups of Infinite Transitive Graded Lie Algebras (II)

In this section, we assume that the linear isotropy algebra $g_0$ of $g = \sum_{p=-1}^\infty g_p$ contains an element $e$ which satisfies $[e, x_p] = px_p$ for all $x_p \in g_p$. Put $g_{-1} = V$. We can write $c(e) = \sum_{p=-1}^\infty x_p$ with $x_p \in g_p$. For all $v \in V$, we have

$$[c(e), v] + [e, c(v)] = c[e, v] = -c(v).$$

Comparing the $V$-components of this equation, we obtain $[x_0, v] = 0$, and hence $x_0 = 0$ by the transitivity condition of $g$. We now define a new derivation $c'$ derived from $c$ by

$$(4.1) \quad c' = c + \text{ad} \left( \sum_{p \neq 0} \frac{1}{p} x_p \right).$$

It is clear that $c'(e) = 0$.

**Lemma 4.1.** $\deg c' = 0$. (For the definition of "degree" of a derivation, see § 1.)

**Proof.** We must show that $c'(g_p) \subset g_p$ for all $p \geq -1$. Put $c'(x) = \sum_{q=-1}^\infty y_q$ ($y_q \in g_q$) for $x \in g_p$. Then we have

$$c'[e, x] = pc'(x) = p \sum_{q=-1}^\infty y_q = [e, c'(x)] = \sum_{q=-1}^\infty q y_q.$$
Hence $y_q = 0$ for $q \neq p$ and thus $c'(x) = y_p \in g_p$.

**Lemma 4.2.** If $c' = 0$ on $V$, then $c' = 0$ on $g$.

**Proof.** For $x \in g_0$ and $v \in V$, it holds that $[c'(x), v] + [x, c'(v)] = c'[x, v]$. By the assumption of $c'$, we obtain $[c'(x), v] = 0$. Combining $c'(g_0) \subseteq g_0$ with the transitivity of $g$, we obtain $c'(x) = 0$. Repeating this procedure for all $p \geq 1$, we can also obtain that $c'(g_p) = 0$. Hence $c' = 0$ on $g$. q.e.d.

Let $[c] \in H^1(g)$ denote an equivalence class of a derivation $c$ of $g$. Since $c' = c + \text{ad}(\sum_{p \geq 0} \frac{1}{p} x_p)$, we have $[c] = [c']$. By Lemma 4.1, a restriction of $c'$ to $V$ is an element of $\text{gl}(V)$. We denote this linear mapping by $c'|_V$.

**Theorem 4.3.** Let $g = \sum_{p = -1}^{\infty} g_p$ be an infinite transitive graded Lie algebra whose linear isotropy algebra $g_0$ contains an element $e$ which satisfies $[e, x_p] = px_p$ for all $x_p \in g_p$. Then $\dim H^1(g) \leq (\dim V)^2$.

**Proof.** Let $c$ be any derivation of $g$. We define a linear mapping $\psi: \text{Der}(g) \rightarrow \text{gl}(V)$ by $\psi(c) = c'|_V$. By Lemma 4.2, we obtain that if $c$ is contained in $\text{Ker} \psi$, then $c$ is an inner derivation. Hence our assertion is obvious. q.e.d.

In case that $g$ is derived from $g_0$, we can get the more precise result. Let $n(g_0)$ denote the normalizer of $g_0$ in $\text{gl}(V)$. Then we have

**Lemma 4.4.** Let $g$ be a Lie algebra derived from $g_0$. Then for all $x \in n(g_0)$, $\text{ad}(x)$ is a derivation of $g$.

**Proof.** It is sufficient to prove that $\text{ad}(x)(g_0^{(p)}) \subseteq g_0^{(p)}$ for all $p \geq 1$. Let $z \in g_0^{(1)}$ and $v \in V$. With respect to the bracket operation in $D(V)$, we have

$$[\text{ad}(x)(z), v] = \text{ad}(x)[z, v] + [z, [v, x]] \in g_0.$$ 

Hence we have $\text{ad}(x)(z) \in g_0^{(1)}$, that is, $\text{ad}(x)(g_0^{(1)}) \subseteq g_0^{(1)}$. Since $g_0^{(p+1)} = (g_0^{(p)})^{(1)}$, it can be inductively proved that $\text{ad}(x)(g_0^{(p)}) \subseteq g_0^{(p)}$ for all $p \geq 1$. q.e.d.

**Theorem 4.5.** Let $g$ be an infinite transitive graded Lie algebra derived from $g_0$. Moreover assume that $g_0$ contains an element $e$ which satisfies $[e, x_p] = px_p$ for all $x_p \in g_p$. Then $H^1(g)$ is isomorphic to $n(g_0)/g_0$.

**Proof.** By Lemma 4.4, we can define a linear mapping $f: n(g_0) \rightarrow H^1(g)$ by $f(x) = [\text{ad}(x)]$. We prove that $f$ is surjective. Let $c$ be any derivation of $g$. Recall that $[c] = [c']$. Since $c'$ satisfies $c'(V) \subseteq V$, there exists an element $x$ of $\text{gl}(V)$ such that $c' = \text{ad}(x)$ on $V$. Let $v \in V$ and $y \in g_0$. By the Jacobi identity
in \( D(V) \), we have

\[
(4.2) \quad \text{ad} (x)[v, y] = [\text{ad} (x)(v), y] + [v, \text{ad} (x)(y)].
\]

On the other hand, \( c' \) satisfies

\[
(4.3) \quad \text{c'}[v, y] = [\text{c'}(v), y] + [v, \text{c'}(y)].
\]

Note that \( \text{ad} (x)[v, y] = c'[v, y] \) and \( \text{ad} (x)(v) = c'(v) \). From equations (4.2) and (4.3), it holds that \( [v, \text{ad} (x) - c')(y)] = 0 \). By the transitivity condition of \( g \), we obtain that \( \text{ad} (x) = c' \) on \( g_0 \) and hence \( x \in n(g_0) \). By Lemma 4.4, \( \text{ad} (x) \) is a derivation of \( g \), and \( c' - \text{ad} (x) \) vanishes on \( V \). Now by Lemma 4.2, we clearly have \( c' = \text{ad} (x) \) on \( g \). Thus we have proved that \( f \) is surjective. Since \( \text{Ker} f = g_0 \), we obtain that \( H^1(g) \) is isomorphic to \( n(g_0)/g_0 \). q. e. d.

§ 5. Example of Infinite Transitive Graded Lie Algebra \( g \) with \( \dim H^1(g) = \infty \)

As stated in Introduction, we give an example of \( g \) such that \( H^1(g) \) is of infinite dimension. Note that a derivation \( c \) of degree \( \leq -2 \) is necessarily an outer derivation. We define a sequence of derived ideals \( g^{(p)} \) of \( g \) inductively by \( g^{(1)} = [g, g], \ldots, g^{(p)} = [g^{(p-1)}, g^{(p-1)}] \). Then we prove

**Theorem 5.1.** Let \( g = \sum_{p=-1}^{\infty} g_p \) be an infinite transitive graded Lie algebra which satisfies \( g^{(2)} = 0 \). Then \( \dim H^1(g) = \infty \).

**Proof.** Put \( \varphi_k = \text{ad} (v_1) \cdots \text{ad} (v_k) \) for \( v_1, v_2, \ldots, v_k \in g_{-1} \). We show that \( \varphi_k (k \geq 1) \) is a derivation of \( g \) by induction. In the case of \( k = 1 \), \( \varphi_1 = \text{ad} (v_1) \) is an "inner" derivation. Let \( k \geq 1 \). Assume that \( \varphi_k[x, y] = [\varphi_k(x), y] + [x, \varphi_k(y)] \) for any \( x, y \in g \). Put \( \varphi_{k+1} = \varphi_k \circ \text{ad} (v_{k+1}) \) for \( v_{k+1} \in g_{-1} \). Then by the Jacobi identity, we have

\[
\varphi_{k+1}[x, y] = \varphi_k[v_{k+1}, [x, y]] = \varphi_k[[v_{k+1}, x], y] + \varphi_k[x, [v_{k+1}, y]].
\]

By the assumption of induction and by \( g^{(2)} = 0 \), this element is equal to

\[
\varphi_k[[v_{k+1}, x], y] + \varphi_k[x, [v_{k+1}, y]] = [\varphi_k[v_{k+1}, x], y] + [x, \varphi_k[v_{k+1}, y]] = [\varphi_{k+1}(x), y] + [x, \varphi_{k+1}(y)].
\]

Hence \( \varphi_k \) is a derivation for all \( k \geq 1 \). Now if \( \varphi_k = \text{ad} (v_1) \cdots \text{ad} (v_k) = 0 \) on \( g \) for all \( v_1, v_2, \ldots, v_k \in g_{-1} \), we would have \( [g_{-1}, [g_{-1}, \ldots, [g_{-1}, g_k] \cdots] = 0 \).

By the transitivity condition of \( g \), we must have \( g_k = 0 \). This is a contradiction.
Thus there exist $v_1, v_2, \ldots, v_k \in \mathfrak{g}_{-1}$ for arbitrarily large $k$ such that $\varphi_k = \text{ad}(v_1) \cdot \text{ad}(v_2) \cdots \text{ad}(v_k) \neq 0$. Since $\deg \varphi_k = -k$, $\varphi_k$ is a non-trivial outer derivation of $\mathfrak{g}$, and hence $\dim H^1(\mathfrak{g}) = \infty$.

**A typical example.** Let $\mathfrak{g}_{-1}$ be a two dimensional vector space with a basis $\partial/\partial x, \partial/\partial y$, and let $\mathfrak{g}_p$ be a one dimensional vector space with a basis $x^{p+1}\partial/\partial y$ for $p \geq 0$. Then we have an infinite transitive graded Lie algebra $\mathfrak{g} = \sum_{p=-1}^{\infty} \mathfrak{g}_p$, which satisfies $\mathfrak{g}^{(2)} = 0$. In this case, $\varphi_k = \text{ad}(\partial/\partial x) \cdots \text{ad}(\partial/\partial x)$ are non-trivial derivations of $\mathfrak{g}$ for all $k \geq 1$. Hence $H^1(\mathfrak{g})$ is of infinite dimension.

§ 6. The First Cohomology Groups of Infinite Intransitive Lie Algebras $L[W^\ast]$

6.1. First we explain a Lie algebra $L[W^\ast]$ which is a main object in this section. Let $V$ be a finite dimensional vector space with $V = U + W$ (direct sum). We denote by $S(W^\ast)$ the ring of formal power series over $W$. Let $L$ be an infinite transitive simple Lie algebra over $U$. Both $L$ and $S(W^\ast)$ are complete topological vector spaces with respect to their natural topology induced by the filtrations. Then a Lie algebra $L[W^\ast]$ is obtained as a topological completion of $L \otimes S(W^\ast)$. Since $L[W^\ast]$ is a perfect Lie algebra, we know that each derivation $c$ of $L[W^\ast]$ is continuous.

6.2. Let $A$ be an abstract Lie algebra over $\mathbb{C}$. Then the commutator ring of $A$, which we denote by $C_A$, is defined as follows:

$$C_A = \{ \rho \in \text{Hom}_{\mathbb{C}}(A, A) ; \rho \circ \text{ad}(x) = \text{ad}(x) \circ \rho \text{ for all } x \in A \}.$$  

In this sub-section we want to determine the commutator rings $C_L$ and $C_{L[W^\ast]}$.

**Proposition 6.1.** For an infinite transitive simple Lie algebra $L$, it holds that $C_L = \mathbb{C}$.

For the proof of Proposition 6.1, we need three lemmas. First we rewrite the some properties of $L$ stated in Section 2 in the following lemma.

**Lemma 6.2.** (1) $L_0 = [L, L_1]$, for $L = D(U)$, $L_{s1}(U)$ and $L_{s2}(U)$,  
(2) $L_{-1} = [L, L_1]$, for $L = L_{ct}(U)$.

**Lemma 6.3.** (V. Guillemin [1]). $C_L$ is a commutative field which canonically contains the field $\mathbb{C}$.
Proof. For $a \in C$, let $\rho_a$ be a mapping such that $x \mapsto ax$ for $x \in L$. Then it is clear that $\rho_a$ belongs to $C_L$. Through a mapping $a \mapsto \rho_a$, we can consider $C$ is contained in $C_L$. Let $\rho$ be a non-zero element of $C_L$. Since $L$ is simple, we have $\rho(L) = L$ and $\text{Ker} \, \rho = 0$. Hence a non-zero $\rho$ has an inverse. Let $\rho_1$, $\rho_2 \in C_L$. It is clear that $\rho_1 \circ \rho_2 \in C_L$. Now it is sufficient to show that $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$. For all $x, y \in L$ we have $\rho_1 \circ \rho_2[x, y] = [\rho_1(x), \rho_2(y)] = \rho_2 \circ \rho_1[x, y]$. Combining this equation with $L = [L, L]$, we obtain $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1$. q.e.d.

Lemma 6.4. Each $C_L$ has a faithful representation as a ring of endomorphisms as follows:

1. $C_L \subseteq \text{Hom}_C(L/L_0, L/L_0)$ for $L = D(U)$, $L_{s_1}(U)$ and $L_{s_2}(U)$.
2. $C_L \subseteq \text{Hom}_C(L/L_{-1}, L/L_{-1})$ for $L = L_{s_1}(U)$.

Proof. (1) Let $\rho$ be an element of $C_L$. Since the filtration $\{L_p\}$ of $L$ satisfies $L_0 = [L, L_1]$ by Lemma 6.2, we obtain $\rho(L_0) \subseteq L_0$. Hence a linear mapping $\rho \mapsto \bar{\rho} \in \text{Hom}_C(L/L_0, L/L_0)$ is naturally induced. Assume $\bar{\rho} = 0$. Then $\rho(L)$ is an ideal of $L$ contained in $L_0$, and hence $\rho(L) = 0$. Thus a linear mapping $\rho \mapsto \bar{\rho}$ is faithful. The assertion (2) is proved by the same argument as (1). q.e.d.

Proof of Proposition 6.1. First let $L$ be an infinite irreducible transitive Lie algebra. Then by Remark 1 in Section 2, the linear isotropy algebra $g_0$ of $L$ is considered as a Lie subalgebra of $gl(U)$. Recall that $g_0$ of $L = D(U)$, $L_{s_1}(U)$ and $L_{s_2}(U)$ are $gl(n, \mathbb{C})$, $sl(n, \mathbb{C})$ and $sp(n, \mathbb{C})$ respectively. By Lemma 6.3 and Lemma 6.4, $C_L$ can be regarded as an abelian Lie subalgebra of $gl(U)$. We will show that $C_L$ is contained in the centralizer of $g_0$ in $gl(U)$. Let $\rho \in C_L$, $x \in g_0$ and $u \in U$. Then in $D(U)$ we clearly have

$$[[\rho, x], u] = [\rho, [x, u]] - [x, [\rho, u]] = (\rho \circ \text{ad}(x) - \text{ad}(x) \circ \rho)(u) = 0.$$ 

Since $[\rho, x] \in g_0$ and $L$ is transitive, we obtain $[\rho, x] = 0$, and hence $[C_L, g_0] = 0$. Put $\tilde{g}_0 = g_0 + C_L$. Then $\tilde{g}_0$ yields a Lie subalgebra of $gl(U)$ and $C_L$ is contained in the center of $\tilde{g}_0$. Since $g_0$ acts irreducibly on $U$, $\tilde{g}_0$ also acts irreducibly on $U$. Note that $\tilde{g}_0$ is of infinite type. By the classification theorem of Lie algebras of infinite type ([2] or [4]), $\tilde{g}_0$ must be equal to $gl(U)$ or $csp(U)$. Thus we have $C_L = C$.

Next let $L = L_{s_1}(U)$. Put $L/L_{-1} = U'$. Then $U'$ is a one dimensional subspace of $gl(U')$, which contains $C$. Hence $C_L = C$. q.e.d.
Using Proposition 6.1, we can verify the following proposition originally proved by V. Guillemin [1].

**Proposition 6.5.** The commutator ring of $L[W^*]$, i.e., $C_{L[W^*]}$, is isomorphic to $S(W^*)$.

**Outline of proof.** We will regard $L$ as imbedded in $L[W^*]$. Let $\rho$ be an element of $C_{L[W^*]}$. We will denote by $\{f^s\}$ the monomial basis in $S(W^*)$. If $x \in L$, then we can write

$$\rho(x) = \sum_{z=0}^\infty \rho_z(x)f^z,$$

where $\rho_z$ depends linearly on $x$. Since $\rho$ is an element of $C_{L[W^*]}$, we clearly obtain $\rho_z \in C_L$. By Proposition 6.1, $\rho_z$ is an element of $C$. Hence we can write

$$\rho(x) = x \otimes \prod_{z=0}^\infty \rho_z f^z, \quad \text{for all } x \in L.$$

Since $L$ is simple, we have $[L, L[W^*]] = L[W^*]$. Hence if $\rho \in C_{L[W^*]}$, it is determined completely by its restriction to $L$. The isomorphism between $C_{L[W^*]}$ and $S(W^*)$ is given by $\rho \mapsto \prod_{z=0}^\infty \rho_z f^z$. This completes the proof. q.e.d.

By Proposition 6.5, $\text{Der}(C_{L[W^*]})$ is identified with $\text{Der}(S(W^*))$. Now we have a homomorphism: $\iota : \text{Der}(S(W^*)) \to \text{Der}(L[W^*])$. Let $X \in \text{Der}(L[W^*])$ and $\rho \in C_{L[W^*]}$. Then $X \circ \rho - \rho \circ X$ is an element of $C_{L[W^*]}$. We denote this element of $C_{L[W^*]}$ by $L_X \rho$. By an easy consideration, the mapping $X \mapsto L_X$ is a homomorphism of $\text{Der}(L[W^*])$ into $\text{Der}(C_{L[W^*]}) = \text{Der}(S(W^*))$. Hence there is a natural homomorphism

$$L : \text{Der}(L[W^*]) \longrightarrow \text{Der}(S(W^*)).$$

It is easy to see that $L \circ \iota = $ identity, which implies that a homomorphism $L$ is surjective. Since any elements of the kernel of $L$ are $S(W^*)$-linear mappings, the kernel of $L$ is identified with the set of all mappings $c : L \to L[W^*]$ satisfying the identity

$$c[x, y] = [c(x), y] + [x, c(y)] \quad \text{for all } x, y \in L.$$

We denote this set by $\text{Der}(L, L[W^*])$.

Summarizing the above remarks, we have

**Proposition 6.6** (V. Guillemin [1]). *There is a split exact sequence of Lie algebras:*
6.3. In this sub-section, we will determine the first cohomology group \( H^1(L[W^*]) \). By Proposition 6.6, we have a natural isomorphism:

\[
\text{Der}(L[W^*]) \cong \text{Der}(L, L[W^*]) + \text{Der}(S(W^*)) \quad \text{(direct sum)}.
\]

The space \( \text{Der}(S(W^*)) \) is canonically identified with \( D(W) \), the Lie algebra of all formal vector fields over \( W \). Hence it suffices to determine \( \text{Der}(L, L[W^*]) \) for calculating \( \text{Der}(L[W^*]) \).

Let \( x \in L \) and \( c \in \text{Der}(L, L[W^*]) \). We denote by \( f^* \) the basis of \( S(W^*) \) consisting of monomials. Then we can write:

\[
c(x) = \prod_{\alpha=0}^{\infty} x_\alpha \otimes f^*, \quad x_\alpha \in L.
\]

Put \( x_\alpha = c_\alpha(x) \). Then \( c_\alpha \) is a linear mapping of \( L \) into itself. For \( x, y \in L \), we have

\[
c[x, y] = \prod_{\alpha=0}^{\infty} c_\alpha[x, y] \otimes f^* = [c(x), y] + [x, c(y)]
\]

\[
= \prod_{\alpha=0}^{\infty} c_\alpha(x) \otimes f^*, \quad x_\alpha \in L
\]

Hence \( c_\alpha[x, y] = [c_\alpha(x), y] + [x, c_\alpha(y)] \), which implies that \( c_\alpha \) is an element of \( \text{Der}(L) \). By Theorem 2.1, there exists a unique element \( z_\alpha \) of \( L = D(U) \) or \( L_{st}(U) \) (resp. \( L = L_{st}(U) \) or \( L_{st}(U) \)).

Thus we have \( c = \text{ad}(\prod_{\alpha=0}^{\infty} z_\alpha \otimes f^*) \). Here the symbol \( e \) denotes a unit matrix, i.e. a basis of one dimensional center of \( \text{gl}(U) \). Now we easily obtain the following isomorphism:

\[
\text{Der}(L[W^*]) \cong \begin{cases} 
L[W^*] + D(W) & \text{for } L = D(U) \text{ or } L_{st}(U) \\
(L[W^*] + S(W^*)) \otimes e + D(W) & \text{for } L = L_{st}(U) \text{ or } L_{st}(U).
\end{cases}
\]

Since \( L[W^*] \) has no non-trivial center, the space \( \text{ad}(L[W^*]) \) of all inner derivations of \( L[W^*] \) is naturally isomorphic to \( L[W^*] \).

Summarizing the above results, we have proved:

**Theorem 6.7.** Let \( D(W) \) be a Lie algebra of all formal vector fields over \( W \) and let \( e \) be a basis of one dimensional center of \( \text{gl}(U) \). Then we have the following isomorphism:
\[ H^1(L[W^*]) \cong \begin{cases} D(W) & \text{for } L = D(U) \text{ or } L_{\text{ct}}(U) \\ D(W) + S(W^*) \otimes e & \text{for } L = L_{\text{et}}(U) \text{ or } L_{\text{et}}(U). \end{cases} \]

References


