Erdös-Mordell-type inequalities

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The famous Erdös-Mordell inequality states that, if $P$ is a point in the interior of a triangle $ABC$ whose distances are $p$, $q$, $r$ from the vertices of the triangle and $x$, $y$, $z$ from its sides, then

$$p + q + r \geq 2(x + y + z).$$

In the paper by Satnoianu [1], some generalizations of the above inequality were given. His proof depends heavily on the geometry of the triangle $ABC$. In this note, we give a more algebraic proof of the Erdös-Mordell inequality.

**Theorem.** Let $p$, $q$, $r \geq 0$ and let $\alpha + \beta + \gamma = \pi$. Then we have the inequality

$$p + q + r \geq 2\sqrt{qr} \cos \alpha + 2\sqrt{rp} \cos \beta + 2\sqrt{pq} \cos \gamma. \tag{1}$$

**Proof.** We consider the following quadratic function of $x$:

$$x^2 - 2(\sqrt{r} \cos \beta + \sqrt{q} \cos \gamma)x + q + r - 2\sqrt{qr} \cos \alpha. \tag{2}$$

Then a quarter of the discriminant is

$$\frac{1}{4} \Delta = (\sqrt{r} \cos \beta + \sqrt{q} \cos \gamma)^2 - (q + r - 2\sqrt{qr} \cos \alpha).$$

Since $\alpha + \beta + \gamma = \pi$, we have

$$\cos \alpha = -\cos(\beta + \gamma) = -\cos \beta \cos \gamma + \sin \beta \sin \gamma.$$
Using the above identity, the discriminant can be simplified as

$$\Delta = -(\sqrt{r} \sin \beta - \sqrt{q} \sin \gamma)^2 \leq 0.$$ 

Thus the expression (2) is always nonnegative. Letting $x = \sqrt{p}$, we get (1). \hfill \square

**Corollary.** Let $x', y', z'$ be the length of the angle bisectors of $\angle BPC$, $\angle CPA$, and $\angle APB$, respectively. Then we have

$$p + q + r \geq 2(x' + y' + z').$$

**Proof.** We have

$$x' = \frac{2qr}{q + r} \cos \gamma \leq \sqrt{qr} \cos \gamma,$$

$$y' = \frac{2pr}{p + r} \cos \beta \leq \sqrt{pr} \cos \beta,$$

$$z' = \frac{2pq}{p + q} \cos \alpha \leq \sqrt{pq} \cos \alpha.$$

The corollary follows from the theorem. \hfill \square

**Remark.** Since $x' \geq x$, $y' \geq y$ and $z' \geq z$, the corollary implies the Erdős-Mordell inequality

$$p + q + r \geq 2(x + y + z).$$

**References**


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