Group splittings and integrality of traces

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Abstract. In this paper, we elaborate on Connes’ proof of the integrality of the trace conjecture for free groups, in order to show that any action of a group $G$ on a tree leads to a similar integrality assertion concerning the trace on the group algebra $\mathbb{C}G$, which is associated with the set of group elements that stabilize a vertex.

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Introduction

Given a torsion-free group $G$, the integrality of the trace conjecture is the assertion that the image of the additive map

$$\tau_*: K_0(C_r^*G) \longrightarrow \mathbb{C},$$

which is induced by the canonical trace $\tau$ on the reduced $C^*$-algebra $C_r^*G$ of $G$, is the group $\mathbb{Z}$ of integers. Some evidence for the validity of that conjecture is provided by Zalesskii’s theorem [18], which states that for any group $G$ (possibly with torsion) the values of $\tau_*$ on K-theory classes that come from the group algebra $\mathbb{C}G$ are rational. By a standard argument, the integrality of the trace conjecture can be shown to imply the triviality of idempotents in $C_r^*G$. In the case where $G$ is a free group, Connes proved in [5, §IV.5] the integrality of the trace conjecture by using a free action of $G$ on a tree and the associated representations of $C_r^*G$ on the Hilbert space $\ell^2V$, where $V$ is the set of vertices of the tree. We also note that, in the case of a torsion-free abelian group $G$, the integrality of the trace conjecture is an immediate consequence of the connectedness of the dual group $\hat{G}$ (cf. [16, Theorem 2]).

In the case where the group $G$ has non-trivial torsion elements, Baum and Connes had conjectured in [3] that the image of $\tau_*$ is the subgroup of $\mathbb{Q}$ generated by the inverses of the orders of the finite subgroups of $G$. This latter conjecture was disproved by Roy [14]. Subsequently, Lück [10] formulated a modified version of that

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conjecture, according to which the image of \( \tau_* \) is contained in the subring of \( \mathbb{Q} \) generated by the inverses of the orders of the finite subgroups of \( G \), and showed that this is indeed the case if the so-called Baum–Connes assembly map is surjective.

In this paper, we are interested in traces defined on the group algebra \( \mathbb{C}G \) and examine integrality properties of the induced additive maps on the K-theory group \( K_0(\mathbb{C}G) \). If \( S \subseteq G \) is a subset closed under conjugation, then the linear map (partial augmentation)

\[
\tau_S : \mathbb{C}G \to \mathbb{C},
\]

which is defined by letting \( \tau_S \left( \sum_{g \in G} a_g g \right) = \sum_{g \in S} a_g \) for any element \( \sum_{g \in G} a_g g \in \mathbb{C}G \), is a trace. As such, it induces an additive map

\[
(\tau_S)_* : K_0(\mathbb{C}G) \to \mathbb{C}.
\]

In the special case where \( S = G \), the additive map \( (\tau_S)_* \) is that induced by the augmentation homomorphism \( \mathbb{C}G \to \mathbb{C} \). The map \( (\tau_S)_* \) is then referred to as the homological (or naive) rank and its image is the group \( \mathbb{Z} \) of integers (cf. [4, Chapter IX, Exercise 2.5]). On the other hand, if \( S = \{1\} \), then the map \( (\tau_S)_* \) is the Kaplansky rank, whose values are rational in view of Zaleskii’s theorem [18]. In fact, if \( G \) is torsion-free then a weak version of Bass’ trace conjecture [2] asserts that the Kaplansky rank coincides with the homological rank; if this is true, then we must have \( \text{im}(\tau_S)_* = \mathbb{Z} \) in this case as well.

We can now state our main result.

**Theorem.** Let \( G \) be the fundamental group of a connected graph of groups with vertex groups \((G_v)_{v} \) and edge groups \((G_e)_{e} \), and consider the subset \( S \subseteq G \) which consists of the conjugates of all elements of the set \( \bigcup_v G_v \). Then, \( \text{im}(\tau_S)_* = \mathbb{Z} \).

Equivalently, we may state the result above in terms of the universal trace defined by Hattori and Stallings, as follows: If \( x \in K_0(\mathbb{C}G) \) is a K-theory class and \( r_{[S]}(x) \in \mathbb{C} \) the coefficient of the Hattori–Stallings rank \( r_{HS}(x) \) that corresponds to the conjugacy class \([g] \) of an element \( g \in G \), then \( \sum_{[g] \in [S]} r_{[S]}(x) \in \mathbb{Z} \), where \([S] \) is the set of conjugacy classes of the elements of \( S \).

We observe that our integrality result would follow immediately if one could show that the additive map

\[
\bigoplus_v K_0(\mathbb{C}G_v) \to K_0(\mathbb{C}G),
\]

which is induced by the inclusion of the vertex groups \( G_v \) into \( G \), is surjective. Indeed, for any vertex \( v \) the composition

\[
K_0(\mathbb{C}G_v) \to K_0(\mathbb{C}G) \xrightarrow{(\tau_S)_*} \mathbb{C}
\]

is the additive map induced by the restriction of the trace \( \tau_S \) on \( \mathbb{C}G_v \) (cf. Remark 1.1 (ii) below). But \( S \) contains \( G_v \) and hence the restriction of \( \tau_S \) on \( \mathbb{C}G_v \).
is the augmentation homomorphism $\mathbb{C}G_v \rightarrow \mathbb{C}$. Therefore, the composition (2) is the homological rank associated with $G_v$; in particular, its image is the group $\mathbb{Z}$ of integers. In view of the assumed surjectivity of (1), we conclude that the image of $(\tau_S)_*$ is the group $\mathbb{Z}$ as well.

We note that if $G$ is the fundamental group of a graph of groups as above, then any finite subgroup $H \subseteq G$ is contained in a conjugate of $G_v$ for some vertex $v$ of the graph (cf. [15, Chapitre I, Exemple 6.3.1]). Since conjugation by any element of $G$ induces the identity map on $K_0(\mathbb{C}G)$, we conclude that the map (1) is surjective if this is the case for the additive map

$$\bigoplus_H K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}G).$$

Here, the direct sum is over all finite subgroups $H$ of $G$ and the map is induced by the inclusion of the $H$’s into $G$. In particular, it follows that the map (1) is surjective if the so-called isomorphism conjecture for $K_0(\mathbb{C}G)$ holds (cf. [11, Conjecture 9.40]). On the other hand, we may consider the special case where the graph has one edge $e$ and two distinct vertices $v_1$ and $v_2$. In that case, $G$ is the amalgamated free product $G_{v_1} * G_e G_{v_2}$, and a sufficient condition that guarantees the surjectivity of (1) has been given by Waldhausen (see the discussion following [17, Corollary 11.5]).

From the point of view presented above, there is a formal resemblance between our result and those described in [1] and [12], where it is proved that a certain statement is true for $G$ if it is true for all $G_v$’s.

We also observe that our integrality result would follow if one could show that the group $G$ satisfies (the strong version of) Bass’ trace conjecture [2]. Indeed, the latter conjecture asserts that for any $x \in K_0(\mathbb{C}G)$ the coefficient $r_{\text{HS}}(x)$ of the Hattori–Stallings rank $r_{\text{HS}}(x)$ vanishes if $g \in G$ is an element of infinite order. Since $S$ contains all group elements of finite order (as we have already noted above), this would imply that $(\tau_S)_*(x)$ is the homological rank of $x$; in particular, it would follow that $(\tau_S)_*(x) \in \mathbb{Z}$.

The contents of the paper are as follows: In Section 1, we recall certain well-known facts concerning traces on an algebra and the induced additive maps on the $K_0$-group. In the following section, we consider a group $G$ acting on a tree and examine certain representations of the group algebra $\mathbb{C}G$. In Section 3, we prove our main result and explicit the integrality assertions in the special cases where $G$ is an amalgamated free product or an HNN extension. Finally, in the last section, we examine whether our integrality result can be extended from the group algebra $\mathbb{C}G$ to the reduced group $\mathbb{C}^*$-algebra $\mathbb{C}^* G$ of $G$.

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1. Traces and the $K_0$-group

Let $R$ be a unital ring, $V$ an abelian group and $\tau: R \to V$ a trace, i.e. an additive map which vanishes on the commutators $xy - yx$ for all $x, y \in R$. Then, for any positive integer $n$ the map

$$\tau_n: M_n(R) \to V,$$

which is defined by letting $\tau_n(A) = \sum_{i=1}^{n} \tau(a_{ii})$ for any matrix $A = (a_{ij})_{i,j} \in M_n(R)$, is a trace as well. These traces induce an additive map

$$\tau_*: K_0(R) \to V,$$

by mapping the K-theory class of any idempotent matrix $E \in M_n(R)$ onto $\tau_n(E)$.

Remarks 1.1. (i) Let $R$ be a ring and $f: V \to V'$ an abelian group homomorphism. We consider a trace $\tau: R \to V$ and the $V'$-valued trace $f \circ \tau$ on $R$. Then, the induced additive map $(f \circ \tau)_*: K_0(R) \to V'$ is the composition

$$K_0(R) \xrightarrow{\tau_*} V \xrightarrow{f} V',$$

where $\tau_*: K_0(R) \to V$ is the additive map induced by the trace $\tau$.

(ii) Let $\varphi: R \to S$ be a ring homomorphism and $V$ an abelian group. We consider a trace $\tau: S \to V$ and the $V$-valued trace $\tau \circ \varphi$ on $R$. Then, the induced additive map $(\tau \circ \varphi)_*: K_0(R) \to V$ is the composition

$$K_0(S) \xrightarrow{\tau_*} K_0(R) \xrightarrow{\varphi} V,$$

where $\tau_*: K_0(S) \to V$ is the additive map induced by the trace $\tau$.

(iii) Let $R$ be a ring and $[R, R]$ the additive subgroup of it generated by the commutators $xy - yx$, $x, y \in R$. Then, the quotient map $p: R \to R/[R, R]$ is the universal trace defined on $R$ and induces the Hattori–Stallings rank map

$$r_{\text{HS}}: K_0(R) \to R/[R, R];$$

see also [4, Chapter IX, §2].

(iv) In the special case where $R = CG$ is the group algebra of a group $G$, the quotient group $R/[R, R] = CG/[CG, CG]$ is a complex vector space with basis the set $\mathcal{C}(G)$ of conjugacy classes of elements of $G$. If $[g] \in \mathcal{C}(G)$ is the conjugacy class of an element $g \in G$, then the linear functional (partial augmentation) $\sum_{h \in G} a_h h \mapsto \sum_{h \in [g]} a_h$, $\sum_{h \in G} a_h h \in CG$, is a trace and hence induces an additive map

$$r_{[g]}: K_0(CG) \to \mathbb{C}.$$

These maps determine the Hattori–Stallings rank of any K-theory class $x \in K_0(CG)$, since we have $r_{\text{HS}}(x) = \sum_{[g] \in \mathcal{C}(G)} r_{[g]}(x)[g] \in CG/[CG, CG]$. 
We now consider a non-unital ring $I$ and let $I^+$ be the associated unital ring. Here, $I^+ = I \oplus \mathbb{Z}$ as an abelian group, whereas the product of any two elements $(x, n), (y, m) \in I^+$ is equal to $(xy + n y + mx, nm) \in I^+$. The $K_0$-group of $I$ is defined by means of the split extension

$$0 \longrightarrow I \longrightarrow I^+ \overset{\pi}{\longrightarrow} \mathbb{Z} \longrightarrow 0,$$

where $\pi$ is the projection $(x, n) \mapsto n, (x, n) \in I^+$. More precisely, $K_0(I)$ is the kernel of the induced additive map $K_0(\pi): K_0(I^+) \rightarrow K_0(\mathbb{Z})$. Let $\tau: I \rightarrow V$ be a $V$-valued trace on $I$; by this, we mean that $\tau$ is an additive map which vanishes on the commutators $xy - yx$ for all $x, y \in I$. Then, $\tau$ extends to an additive map $\tau^+$ on the associated unital ring $I^+$, by letting $\tau^+(0, 1) = 0$; in fact, $\tau^+$ is a trace. The induced additive map $\tau_*: K_0(I) \rightarrow V$ is defined as the restriction of $(\tau^+)_*: K_0(I^+) \rightarrow V$ to the subgroup $K_0(I) \subseteq K_0(I^+)$. 

**Example 1.2.** Let $U$ be a complex vector space with basis $(\xi_i)_i$ and $L(U)$ the algebra of linear endomorphisms of $U$. We consider the ideal $\mathfrak{F} \subseteq L(U)$ consisting of those endomorphisms of $U$ that have finite rank. Then, for any $a \in \mathfrak{F}$ the family of complex numbers $(a(\xi_i), \xi_i^*)_i$ has finite support. Here, we denote for all $i$ by $\xi_i^*$ the linear functional on $U$ which maps $\xi_i$ onto 1 and vanishes on $\xi_j$ for $j \neq i$, whereas $[ \ , ]$ denotes the standard pairing between $U$ and its dual. Moreover, the map

$$\text{Tr}: \mathfrak{F} \longrightarrow \mathbb{C},$$

which is defined by letting $\text{Tr}(a) = \sum_i [a(\xi_i), \xi_i^*]$ for all $a \in \mathfrak{F}$, does not depend upon the choice of the basis $(\xi_i)_i$, and vanishes on the elements of the form $ab - ba, a \in \mathfrak{F}, b \in L(U)$. In particular, $\text{Tr}$ is a trace on $\mathfrak{F}$. In view of the Morita invariance and the continuity of the functor $K_0$ (cf. [13, Chapter 1, §2]), the induced additive map

$$\text{Tr}_*: K_0(\mathfrak{F}) \longrightarrow \mathbb{C}$$

identifies $K_0(\mathfrak{F})$ with the subgroup $\mathbb{Z} \subseteq \mathbb{C}$. More generally, let $R$ be a complex algebra and consider a linear trace $\tau$ on $R$ with values in a complex vector space $V$. Then, the linear map

$$\text{Tr} \otimes \tau: \mathfrak{F} \otimes R \longrightarrow \mathbb{C} \otimes V \simeq V,$$

which is defined by letting $a \otimes x \mapsto \text{Tr}(a)\tau(x)$ for all elementary tensors $a \otimes x \in \mathfrak{F} \otimes R$, is also a trace. Moreover, the induced additive map

$$(\text{Tr} \otimes \tau)_*: K_0(\mathfrak{F} \otimes R) \longrightarrow V$$

is identified with the additive map

$$\tau_*: K_0(R) \longrightarrow V,$$

which is induced by $\tau$, in view of the Morita isomorphism $K_0(\mathfrak{F} \otimes R) \simeq K_0(R)$. 


A proof of the following result may be found in [9, Proposition 1.44].

**Proposition 1.3.** Let \( \psi, \varphi : A \to B \) be two homomorphisms of non-unital rings and \( I \subseteq B \) an ideal such that \( \psi(a) - \varphi(a) \in I \) for all \( a \in A \). We consider an abelian group \( V \) and an additive map \( \tau : I \to V \) that vanishes on elements of the form \( xy - yx \) for all \( x \in I \) and \( y \in B \); in particular, \( \tau \) is a trace on \( I \). Let \( t : A \to V \) be the additive map which is defined by letting \( t(a) = \tau(\psi(a) - \varphi(a)) \) for all \( a \in A \). Then:

(i) The map \( t \) is a trace on \( A \).

(ii) The image of the additive map \( t : K_0(A) \to V \) is contained in the image of the additive map \( \tau : K_0(I) \to V \). \( \square \)

2. Trees and group actions

Let \( X \) be a graph and denote by \( V, E \) or the set of vertices and oriented edges of it respectively. A path on \( X \) is a finite sequence \( e_1, \ldots, e_n \) of oriented edges such that the terminus \( v_i \) of \( e_i \) is the origin of \( e_{i+1} \) for all \( i = 1, \ldots, n - 1 \). We say that a path as above has origin \( v_0 \) equal to the origin of \( e_1 \), terminus \( v_n \) equal to the terminus of \( e_n \) and passes through the vertices \( v_1, \ldots, v_{n-1} \). The path is reduced if there is no \( i \) such that \( e_{i+1} \) is equal to the reverse edge of \( e_i \). The graph \( X \) is a tree if for any two vertices \( v, v_0 \in V \) with \( v \neq v_0 \) there is a unique reduced path with origin \( v \) and terminus \( v_0 \); this path, denoted by \( [v, v_0] \), is called the geodesic joining \( v \) and \( v_0 \).

Let \( X \) be a tree and denote by \( E \) the corresponding set of un-oriented edges. It is well known that the number of vertices exceeds the number of un-oriented edges by one. More precisely, having fixed a vertex \( v_0 \in V \), we consider for any \( v \in V \setminus \{v_0\} \) the geodesic \( [v_0, v] = (e_1, \ldots, e_n) \) and define the map

\[
\lambda : V \setminus \{v_0\} \to E,
\]

by letting \( \lambda(v) \) be the un-oriented edge associated with the oriented edge \( e_n \). The proof of the next result is straightforward.

**Lemma 2.1.** Let \( X \) be a tree and fix a vertex \( v_0 \in V \).

(i) The map \( \lambda \) defined above is bijective.

(ii) For another vertex \( v'_0 \in V \) consider the corresponding map \( \lambda' : V \setminus \{v'_0\} \to E \).

If the geodesic \( [v_0, v'_0] \) passes through the vertices \( v_1, \ldots, v_{n-1} \), then we have \( \lambda(v) = \lambda'(v) \) for all vertices \( v \in V \setminus \{v_0, v_1, \ldots, v_{n-1}, v'_0\} \). \( \square \)

Let \( \alpha \) be an automorphism of the tree \( X \) and denote by \( \alpha V \) (resp. \( \alpha E \)) the corresponding bijection of the set of vertices (resp. edges) of \( X \). We fix a vertex \( v_1 \in V \)
We now assume that \( v_2 = \alpha_V(v_1) \in V \) and the associated bijection \( \lambda_2 : V \setminus \{v_2\} \to E \). Then, it is easily seen that
\[
\alpha_E \circ \lambda_1 = \lambda_2 \circ \alpha'_V,
\]
where \( \alpha'_V \) denotes the restriction of \( \alpha_V \) to the subset \( V \setminus \{v_1\} \subseteq V \). The automorphism \( \alpha \) is said to have no inversions if there is no edge \( e \in E^\circ \) such that \( \alpha(e) \) is the reverse edge of \( e \).

**Proposition 2.2.** Let \( X \) be a tree, \( v_0 \in V \) a vertex and \( \lambda : V \setminus \{v_0\} \to E \) the associated bijection. We consider a group \( G \) acting on \( X \) and fix an element \( g \in G \).

(i) If \( g \cdot v_0 = v_0 \), then we have \( g \cdot \lambda(v) = \lambda(g \cdot v) \) for all \( v \in V \setminus \{v_0\} \).

(ii) If \( g \cdot v_0 \neq v_0 \) and the geodesic \( [v_0, g^{-1} \cdot v_0] \) passes through the vertices \( v_1, \ldots, v_{n-1} \), then \( g \cdot \lambda(v) = \lambda(g \cdot v) \) for all \( v \in V \setminus \{v_0, v_1, \ldots, v_{n-1}, g^{-1} \cdot v_0\} \).

**Proof.** We consider the vertex \( g^{-1} \cdot v_0 \) and let
\[
\lambda' : V \setminus \{g^{-1} \cdot v_0\} \to E
\]
be the associated bijection. The element \( g \in G \) induces an automorphism of the tree \( X \), which maps the vertex \( g^{-1} \cdot v_0 \) onto \( v_0 \), and hence Equation (3) above implies that \( g \cdot \lambda'(v) = \lambda(g \cdot v) \) for all \( v \in V \setminus \{g^{-1} \cdot v_0\} \). This completes the proof in the case where \( g \cdot v_0 = v_0 \), since we then have \( \lambda = \lambda' \). If \( g \cdot v_0 \neq v_0 \), the proof is finished by invoking Lemma 2.1 (ii), which implies that \( \lambda(v) = \lambda'(v) \) for all vertices \( v \in V \setminus \{v_0, v_1, \ldots, v_{n-1}, g^{-1} \cdot v_0\} \). \( \square \)

Let \( X \) be a graph and consider the sets \( V, E \) of vertices and (un-oriented) edges of \( X \) respectively and the complex vector spaces \( \mathbb{C}(V) = \bigoplus_{v \in V} \mathbb{C} \cdot \xi_v \) and \( \mathbb{C}(E) = \bigoplus_{e \in E} \mathbb{C} \cdot \xi_e \). If \( G \) is a group acting on \( X \), then for any element \( g \in G \) we denote by
\[
\varrho_V(g) : \mathbb{C}(V) \to \mathbb{C}(V) \quad \text{and} \quad \varrho_E(g) : \mathbb{C}(E) \to \mathbb{C}(E)
\]
the linear maps which are defined by letting \( \xi_v \mapsto \xi_{g \cdot v} \) for all \( v \in V \) and \( \xi_e \mapsto \xi_{g \cdot e} \) for all \( e \in E \). These linear maps induce algebra homomorphisms
\[
\varrho_V : \mathbb{C}G \to L(\mathbb{C}(V)) \quad \text{and} \quad \varrho_E : \mathbb{C}G \to L(\mathbb{C}(E)).
\]
We now assume that \( X \) is a tree and fix a vertex \( v_0 \in V \). Then, using the associated bijection \( \lambda : V \setminus \{v_0\} \to E \), we may define the linear maps
\[
p : \mathbb{C}(V) \to \mathbb{C}(E) \quad \text{and} \quad q : \mathbb{C}(E) \to \mathbb{C}(V),
\]
by letting \( p(\xi_v) = \xi_{\lambda(v)} \) for all \( v \in V \setminus \{v_0\} \), \( p(\xi_{v_0}) = 0 \) and \( q(\xi_e) = \xi_{\lambda^{-1}(e)} \) for all \( e \in E \). It is easily seen that \( p \circ q = 1 \in L(\mathbb{C}(E)) \) and \( q \circ p = 1 - p_0 \in L(\mathbb{C}(V)) \),
where \( p_0 \in L(C(V)) \) is the projection onto the 1-dimensional subspace \( C \cdot \xi_{v_0} \), which vanishes on \( \xi_v \) for all \( v \in V \setminus \{v_0\} \). In particular, the linear map

\[
\widehat{\varphi}_E : C G \rightarrow L(C(V)),
\]

which is defined by letting \( \widehat{\varphi}_E(g) = q \circ \varphi_E(g) \circ p \) for all \( g \in G \), is a homomorphism of non-unital algebras.

We say that the group \( G \) acts without inversions on the tree \( X \) if the automorphism induced by the element \( g \in G \) on \( X \) has no inversions for all \( g \in G \).

**Proposition 2.3.** Let \( G \) be a group acting on a tree \( X \) without inversions, fix a vertex \( v_0 \in V \) and consider the algebra homomorphisms \( \varphi_V \) and \( \widehat{\varphi}_E \) defined above. Then, for any element \( g \in G \) the operator \( \varphi_V(g) - \widehat{\varphi}_E(g) \in L(C(V)) \) is of finite rank, whereas its trace (cf. Example 1.2) is equal to 1 if \( g \) stabilizes some vertex of the tree and vanishes otherwise.

**Proof.** First of all, we note that \( \varphi_V(g)(\xi_v) = \xi_{g \cdot v} \) for all \( v \in V \) and \( \widehat{\varphi}_E(g)(\xi_v) = \xi_v \), where \( \lambda = \lambda^{-1}(g \cdot \lambda(v)) \) for all \( v \in V \setminus \{v_0\} \). Moreover, we have

\[
\lambda^{-1}(g \cdot \lambda(v)) = g \cdot v \iff g \cdot \lambda(v) = \lambda(g \cdot v)
\]

for all \( v \in V \setminus \{v_0, g^{-1} \cdot v_0\} \). Hence, Proposition 2.2(i) shows that if \( g \cdot v_0 = v_0 \) then the operator \( \varphi_V(g) - \widehat{\varphi}_E(g) \) is equal to the projection \( p_0 \) onto the 1-dimensional subspace \( C \cdot \xi_{v_0} \), which vanishes on \( \xi_v \) for all \( v \in V \setminus \{v_0\} \). It follows that \( \varphi_V(g) - \widehat{\varphi}_E(g) \) is of finite rank and \( \text{Tr} [\varphi_V(g) - \widehat{\varphi}_E(g)] = 1 \). We now assume that \( g \cdot v_0 \neq v_0 \). In that case, we let \( (e_1, \ldots, e_n) \) be the geodesic \( [v_0, g^{-1} \cdot v_0] \) and consider for all \( i = 1, \ldots, n \) the terminal vertex \( v_i \) of \( e_i \); in particular, \( v_n = g^{-1} \cdot v_0 \). Then, Proposition 2.2(ii) implies that the operator \( \varphi_V(g) - \widehat{\varphi}_E(g) \) vanishes on \( \xi_v \) for all \( v \in V \setminus \{v_0, v_1, \ldots, v_{n-1}, v_n\} \); in particular, \( \varphi_V(g) - \widehat{\varphi}_E(g) \) is of finite rank. On the other hand, it is easily seen that

\[
(\varphi_V(g) - \widehat{\varphi}_E(g))(\xi_v) = \begin{cases} \\
\xi_{g^{-1} \cdot v_0} & \text{if } i = 0, \\
\xi_{g \cdot v_i} - \xi_{g^{-1} \cdot v_{i-1}} & \text{if } i = 1, \ldots, n.
\end{cases}
\]

Therefore, the final assertion in the statement of the proposition to be proved follows readily from the next lemma. \( \square \)

**Lemma 2.4.** Let \( X \) be a tree and \( \alpha \) an automorphism of \( X \). We consider a vertex \( v_0 \in V \) such that \( v' = \alpha(v_0) \neq v_0 \), and the geodesic \( [v_0, v'] = (e_1, \ldots, e_n) \). We denote by \( v_i \) the terminal vertex of \( e_i \) for all \( i = 1, \ldots, n \); in particular, \( v_n = v' \). Then:

(i) If \( \alpha \) has no inversions, then \( \alpha(v_i) \neq v_{i-1} \) for all \( i = 1, \ldots, n \).

(ii) If \( \alpha \) fixes some vertex \( v \in V \), then there is a unique \( i \in \{1, \ldots, n - 1\} \) such that \( \alpha(v_i) = v_i \). \( \square \)
3. Actions on trees and integrality of traces

We assume that \( G \) is a group acting on a tree \( X \) without inversions. We let \( V \) be the set of vertices of \( X \) and consider for any element \( g \in G \) the fixed point set \( V^g = \{ v \in V : g \cdot v = v \} \). We also consider the subset \( S \subseteq G \), consisting of those elements \( g \in G \) for which the fixed point set \( V^g \) is non-empty. In other words, \( S \) consists of those group elements that stabilize some vertex of the tree, i.e. \( S = \bigcup_{v \in V} \text{Stab}_v \). Since the set \( S \) is closed under conjugation, the linear map (partial augmentation)

\[
\tau_S : \mathbb{C}G \rightarrow \mathbb{C},
\]

which maps any element \( \sum_{g \in G} a_g g \in \mathbb{C}G \) onto the complex number \( \sum_{[g] \in S} a_{[g]} \), is easily seen to be a trace. The trace \( \tau_S \) maps a group element \( g \in G \) onto \( 1 \) (resp. onto \( 0 \)) if \( g \) stabilizes a vertex (resp. if \( g \) does not stabilize any vertex). We consider the subset \([S] \subseteq \mathbb{C}(G)\) which consists of the conjugacy classes of the elements of \( S \), i.e. we let

\[
[S] = \{ [g] \in \mathbb{C}(G) : g \in S \} = \{ [g] \in \mathbb{C}(G) : V^g \neq \emptyset \}.
\]

Then, the trace \( \tau_S \) factors through the quotient \( \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \) as the composition

\[
\mathbb{C}G \xrightarrow{p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\overline{\tau_S}} \mathbb{C}.
\]

Here, \( p \) is the quotient map, whereas \( \overline{\tau_S} \) maps any element \( \sum_{[g] \in \mathbb{C}(G)} a_{[g]} [g] \in \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \) onto the complex number \( \sum_{[g] \in [S]} a_{[g]} \). In view of Remark 1.1 (i), we conclude that the additive map

\[
(\tau_S)_*: K_0(\mathbb{C}G) \rightarrow \mathbb{C},
\]

which is induced by the trace \( \tau_S \), coincides with the composition

\[
K_0(\mathbb{C}G) \xrightarrow{\tau_S} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\overline{\tau_S}} \mathbb{C}.
\]

Therefore, \( (\tau_S)_* \) maps any element \( x \in K_0(\mathbb{C}G) \) with Hattori–Stallings rank \( \sum_{[g] \in \mathbb{C}(G)} r_{[g]}(x) [g] \) onto the complex number \( \sum_{[g] \in [S]} r_{[g]}(x) \). Since the subset \( S \subseteq G \) is obviously closed under \( n \)-th powers for all \( n \geq 1 \), it follows from \([8, \text{Proposition 3.2}]\) that \( \sum_{[g] \in [S]} r_{[g]}(x) \in \mathbb{Q} \). The following result strengthens that assertion, as it states that the above rational number is, in fact, an integer.

**Theorem 3.1.** Let \( G \) be a group acting on a tree \( X \) without inversions and consider the subset \( S \subseteq G \) and the additive map

\[
(\tau_S)_*: K_0(\mathbb{C}G) \rightarrow \mathbb{C}
\]

defined above. Then, \( \text{im}(\tau_S)_* = \mathbb{Z} \subseteq \mathbb{C} \).
Proof. Since \( r_S(1) = 1 \), it follows that \( Z \subseteq \text{im}(r_S)_* \). In order to prove the reverse inclusion, we shall use the following result.

**Theorem 3.2.** Let \( G \) be a group acting on a tree \( X \) without inversions and consider the subset \( S \subseteq G \) defined above. Then, for any \( x \in K_0(\mathbb{C}G) \) there exists a suitable element \( y \in K_0(\mathbb{C}G) \) such that \( r_{[g]}(y) = r_{[g]}(x) \) if \( g \in S \) and \( r_{[g]}(y) = 0 \) if \( g \notin S \).

Proof. We fix a vertex \( v_0 \in V \) and consider the representations

\[
\varrho_V : \mathbb{C}G \to L(\mathbb{C}^V) \quad \text{and} \quad \varrho_E : \mathbb{C}G \to L(\mathbb{C}^V)
\]

which were defined before the statement of Proposition 2.3. Using the Hopf algebra structure of \( \mathbb{C}G \), we now define the algebra homomorphisms

\[
\sigma_V : \mathbb{C}G \to L(\mathbb{C}^V) \otimes \mathbb{C}G \quad \text{and} \quad \sigma_E : \mathbb{C}G \to L(\mathbb{C}^V) \otimes \mathbb{C}G,
\]

by letting \( \sigma_V(g) = \varrho_V(g) \otimes g \) and \( \sigma_E(g) = \varrho_E(g) \otimes g \) for all \( g \in G \). Then, for any \( g \in G \) we have \( \sigma_V(g) - \sigma_E(g) = [\varrho_V(g) - \varrho_E(g)] \otimes g \) and hence Proposition 2.3 implies that \( \sigma(a) - \sigma_E(a) \in \mathbb{K} \otimes \mathbb{C}G \) for all \( a \in \mathbb{C}G \), where \( \mathbb{K} \subseteq L(\mathbb{C}^V) \) is the ideal of finite rank operators on \( \mathbb{C}^V \).

We also consider the trace

\[
\text{Tr} \otimes p : \mathbb{K} \otimes \mathbb{C}G \to \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G],
\]

where \( \text{Tr} \) is the standard trace on \( \mathbb{K} \) and \( p \) the universal trace on \( \mathbb{C}G \) (cf. Remark 1.1 (iii) and Example 1.2), and define the map \( t \) as the composition

\[
\mathbb{C}G \xrightarrow{\sigma_V - \sigma_E} \mathbb{K} \otimes \mathbb{C}G \xrightarrow{\text{Tr} \otimes p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].
\]

Then, \( t \) is a trace as well, in view of Proposition 1.3 (i). Moreover, Proposition 1.3 (ii) implies that the image of the induced additive map

\[
t_* : K_0(\mathbb{C}G) \to \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]
\]

is contained in the image of the additive map

\[
(\text{Tr} \otimes p)_* : K_0(\mathbb{K} \otimes \mathbb{C}G) \to \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].
\]

Hence, in view of the identification of the latter map with the Hattori–Stallings rank map \( r_{\text{HS}} \) on \( K_0(\mathbb{C}G) \) (cf. Remark 1.1 (iii) and Example 1.2), we conclude that \( \text{im}(t)_* \subseteq \text{im}(r_{\text{HS}}) \).

On the other hand, Proposition 2.3 implies that the trace \( t \) maps any group element \( g \in G \) onto \( [g] \) (resp. onto 0) if \( g \in S \) (resp. if \( g \notin S \)). It follows that \( t \) factors as the composition

\[
\mathbb{C}G \xrightarrow{p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{t} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].
\]
where \( p \) is the quotient map and \( \tilde{i} \) maps any element \( \sum_{[g] \in \mathcal{E}(G)} a_{[g]}[g] \) onto the partial sum \( \sum_{[g] \in [S]} a_{[g]}[g] \). Hence, invoking Remark 1.1 (i), we conclude that the additive map \( t_\ast \) coincides with the composition

\[
K_0(\mathbb{C}G) \xrightarrow{\text{Hattori–Stallings}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\tilde{i}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].
\]

It follows that \( t_\ast \) maps any element \( x \in K_0(\mathbb{C}G) \) with Hattori–Stallings rank \( \sum_{[g] \in \mathcal{E}(G)} r_{[g]}(x)[g] \) onto \( \sum_{[g] \in [S]} r_{[g]}(x)[g] \). It follows that the assertion in the statement of Theorem 3.2 is equivalent to the inclusion \( \text{im} t_\ast \subseteq \text{im} r_{\text{HS}} \), that we have already established.

\( \square \)

**Proof of Theorem 3.1 (continued).** We fix a K-theory class \( x \in K_0(\mathbb{C}G) \) and choose \( y \in K_0(\mathbb{C}G) \) according to in the statement of Theorem 3.2. Then,

\[
(\tau_S)_\ast(x) = \sum_{[g] \in [S]} r_{[g]}(x) = \sum_{[g] \in \mathcal{E}(G)} r_{[g]}(y)
\]

is the homological rank of \( y \); in particular, we have \( (\tau_S)_\ast(x) \in \mathbb{Z} \). \( \square \)

At this point, we recall that there is a close relationship between group actions on trees on one hand and group splittings on the other. Using the notion of a graph of groups (cf. [6], [15]), this relationship can be described by the Bass–Serre theory, as follows:

(i) If \( G \) is a group acting without inversions on a tree \( X \), then there is a structure of a graph of groups on the quotient graph \( Y = X/G \) such that the corresponding fundamental group is isomorphic to \( G \).

(ii) Conversely, for any graph of groups on a connected graph \( Y \) with fundamental group \( G \) there is a tree \( X \), the so-called universal tree of the graph, on which \( G \) acts without inversions, in such a way that \( X/G \cong Y \) and the stabilizer of any vertex (resp. edge) of \( X \) is a conjugate in \( G \) of a vertex group (resp. edge group) of the graph of groups.

Hence, we may rephrase Theorem 3.1 as follows: Let \( G \) be the fundamental group of a connected graph of groups with vertex groups \( (G_v)_v \). For any vertex \( v \) of the graph we regard the group \( G_v \) as a subgroup of \( G \) and define

\[
[G_v] = \{[g] \in \mathcal{E}(G) : g \in G_v\}.
\]

Then, for any element \( x \in K_0(\mathbb{C}G) \) with Hattori–Stallings rank \( \sum_{[g] \in \mathcal{E}(G)} r_{[g]}(x)[g] \) the complex number \( \sum_{[g] \in [G_v]} r_{[g]}(x)[g] \) is, in fact, an integer.

In particular, we obtain the following two results concerning amalgamated free products and HNN extensions:
Corollary 3.3. Let $G = A \star_H B$ be the amalgamated free product of two groups $A$ and $B$ along a common subgroup $H$ of theirs and consider an element $x \in K_0(CG)$ with Hattori–Stallings rank $\sum_{[g] \in \mathcal{E}(G)} r_{[g]}(x)[g]$. We view $A$ and $B$ as subgroups of $G$ and define

$$[A] = \{[g] \in \mathcal{E}(G) : g \in A\} \quad \text{and} \quad [B] = \{[g] \in \mathcal{E}(G) : g \in B\}.$$ 

Then, the complex number $\sum_{[g] \in [A] \cup [B]} r_{[g]}(x)$ is, in fact, an integer.

Proof. Let $Y$ be the graph consisting of an edge $e$ and two distinct vertices $v = o(e)$ and $v' = t(e)$. Then, the result follows from Theorem 3.1, in view of the discussion above, by considering the graph of groups on $Y$ which is given by letting $G_e = H$, $G_v = A$ and $G_{v'} = B$ with homomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$ the inclusion maps of $H$ into $A$ and $B$ respectively. \qed

Corollary 3.4. Let $A$ be a group, $H \subseteq A$ a subgroup and $\varphi : H \to A$ a monomorphism. We consider the corresponding HNN extension $G = A \star_{\varphi} B$ and let $x \in K_0(CG)$ be an element with Hattori–Stallings rank $\sum_{[g] \in \mathcal{E}(G)} r_{[g]}(x)[g]$. We view $A$ as a subgroup of $G$ and define

$$[A] = \{[g] \in \mathcal{E}(G) : g \in A\}.$$ 

Then, the complex number $\sum_{[g] \in [A] \cup [B]} r_{[g]}(x)$ is, in fact, an integer.

Proof. Let $Y$ be the graph consisting of an edge $e$ and a vertex $v = o(e)$ and $v' = t(e)$. Then, the result follows from Theorem 3.1, in view of the discussion above, by considering the graph of groups on $Y$ which is given by letting $G_e = H$, $G_v = A$ with homomorphisms $G_e \to G_{o(v)}$ and $G_e \to G_{t(e)}$ the inclusion map of $H$ into $A$ and $\varphi : H \to A$ respectively. \qed

Remark 3.5. The result of Corollary 3.3 admits an alternative homological proof, if the group $H$ therein is trivial. Indeed, let $G = A \star B$ be the free product of two groups $A, B$ and consider an element $g \in G$ which is not conjugate to any element of $A$ nor $B$, i.e. an element $g \in G$ for which $[g] \notin [A] \cup [B]$. Then, the centralizer $C_g$ of $g$ in $G$ is easily seen to be infinite cyclic; this can be proved, for example, by invoking the Bass–Serre theory of groups acting on trees. In particular, the quotient group $N_g = C_g / \langle g \rangle$ is finite and hence one may use the Connes–Karoubi character map from $K_0(CG)$ to the second cyclic homology group of the group algebra $CG$, in order to show that the coefficient $r_{[g]}(x)$ of the Hattori–Stallings rank $r_{HS}(x)$ of any element $x \in K_0(CG)$ vanishes (cf. [7]). In particular, for any $x \in K_0(CG)$ we have $\sum_{[g] \in [A] \cup [B]} r_{[g]}(x) = \sum_{[g] \in \mathcal{E}(G)} r_{[g]}(x)$. Since the right-hand side of the latter equality is the homological rank of $x$, we conclude that $\sum_{[g] \in [A] \cup [B]} r_{[g]}(x) \in \mathbb{Z}$. 


On the other hand, if \( G = A \star B \) then the additive map

\[
K_0(\mathbb{C}A) \oplus K_0(\mathbb{C}B) \rightarrow K_0(\mathbb{C}G),
\]

which is induced by the inclusions of \( A \) and \( B \) into \( G \), is surjective; this follows from the discussion following [17, Corollary 11.5]. As explained in the Introduction, the surjectivity of the above map provides yet another proof of Corollary 3.3 (in the case where \( H = 1 \)).

4. Group actions on trees with finite \( S \)

Our goal in this final section is to examine the extent to which Theorem 3.1 can be generalized to an integrality result concerning a trace defined on the reduced \( C^* \)-algebra of a group. Unfortunately, it will turn out that our approach does not lead to any really new results in that direction.

First of all, we recall that the group \( G \) acts on the Hilbert space \( \ell^2G \) by left translations and denote by \( L: \mathbb{C}G \rightarrow \mathfrak{B}(\ell^2G) \) the induced algebra homomorphism. Then, \( L \) is injective, its image \( L(\mathbb{C}G) \) is a self-adjoint subalgebra of \( \mathfrak{B}(\ell^2G) \) and the reduced \( C^* \)-algebra \( C^*_r G \) of \( G \) is the operator norm closure of \( L(\mathbb{C}G) \) in \( \mathfrak{B}(\ell^2G) \). The linear functional

\[
\tau: C^*_r G \rightarrow \mathbb{C},
\]

which is defined by letting \( \tau(a) = \langle a(\delta_1), \delta_1 \rangle \) for all \( a \in C^*_r G \), is a continuous positive faithful and normalized trace, which is referred to as the canonical trace on \( C^*_r G \). (Here, we denote by \( (\delta_g)_g \) the standard orthonormal basis of \( \ell^2G \).) For later use, we note that for any element \( g \in G \) the linear map \( a \mapsto \tau(L(g)^*a) \), \( a \in C^*_r G \), restricts to the subspace \( \mathbb{C}G \cong L(\mathbb{C}G) \) to the linear map \( \sum_{h \in G} a_h h \mapsto a_g \), \( \sum_{h \in G} a_h h \in \mathbb{C}G \).

In order to extend the trace \( \tau_S \) on the group algebra \( \mathbb{C}G \), which was defined in the beginning of §3, to a trace on \( C^*_r G \), we shall make the following assumption: The group \( G \) acts without inversions on a tree \( X \) in such a way that the subset \( S = \bigcup_{v \in V} \text{Stab}_v \) of \( G \), which consists of those group elements that stabilize a vertex, is finite. We note that, under this assumption, the trace \( \tau_S \) on \( \mathbb{C}G \cong L(\mathbb{C}G) \) extends to a continuous trace

\[
\tau_S: C^*_r G \rightarrow \mathbb{C},
\]

by letting \( \tau_S(a) = \sum_{g \in S} \tau(L(g)^*a) \) for all \( a \in C^*_r G \). Indeed, the set \( S \) being finite, \( \tau_S \) is a continuous linear functional on \( C^*_r G \). In view of the remark made above, that
linear functional restricts to the subspace $\mathbb{C}G \simeq L(\mathbb{C}G)$ to the trace $\tau_S$ on $\mathbb{C}G$. It follows by continuity that $\tau_S$ satisfies the trace property on $C^*_rG$ as well. Since the set $S$ is obviously closed under inverses, we also have $\tau_S(a) = \sum_{g \in S} \tau(L(g)a)$ for all $a \in C^*_rG$.

It turns out that the finiteness assumption on $S$ places some severe restrictions on the group $G$. In fact, we shall prove that $S$ must be a normal subgroup of $G$ such that the quotient $G/S$ is free. Then, the integrality of the trace $\tau_S$ on $C^*_rG$ will be an immediate consequence of Connes’ result [5, §IV.5] that free groups satisfy the integrality of the trace conjecture.

Let us consider the subset (normal subgroup) $G_f \subseteq G$ consisting of those elements that have only finitely many conjugates; in other words, we let

$$G_f = \{ g \in G : \text{the conjugacy class } [g] \text{ is finite} \}.$$ 

We recall that a group is 2-ended if and only if it has an infinite cyclic subgroup of finite index (cf. [6, Chapter IV, Theorem 6.12]).

**Proposition 4.1.** Let $G$ be a group acting without inversions on a tree $X$, in such a way that the subset $S = \bigcup_{v \in V} \text{Stab}_v$ of $G$ is finite. Then:

(i) The stabilizer subgroup $\text{Stab}_v$ is a finite subgroup of $G_f$ for all $v \in V$.

(ii) $S = \{ g \in G : \text{the order of } g \text{ is finite} \} \subseteq G_f$.

(iii) The group $G$ has a free subgroup of finite index.

(iv) If $G$ is not 2-ended, then $S = G_f$ and the quotient group $G/G_f$ is free.

(v) If $G$ is 2-ended, then $S$ is a normal subgroup of $G$ and the quotient group $G/S$ is infinite cyclic.

**Proof.** (i) Let us fix a vertex $v \in V$. Then, the finiteness of $\text{Stab}_v$ is clear, since $\text{Stab}_v \subseteq S$. On the other hand, $S$ is closed under conjugation and hence for any $g \in \text{Stab}_v$ the conjugacy class $[g]$ is contained in $S$; in particular, $[g]$ is a finite set, i.e. $g \in G_f$.

(ii) Since $S = \bigcup_{v \in V} \text{Stab}_v$ is a union of finite subgroups of $G_f$ (in view of (i) above), it is contained itself in $G_f$ and consists of elements of finite order. On the other hand, any torsion element $g \in G$ acts on the tree $X$ by fixing some vertex (cf. [15, Chapitre I, Exemple 6.3.1]); hence, $g \in S$. We conclude that $S = \{ g \in G : \text{the order of } g \text{ is finite} \}$.

(iii) Since the orders of the stabilizer subgroups $\text{Stab}_v, v \in V$, are obviously bounded by some integer, the result follows from [6, Chapter IV, Theorem 1.6].

(iv) We fix a free normal subgroup $N \subseteq G$ of finite index; such a subgroup exists, in view of (iii) above. Since the group $G$ is not 2-ended, the free group $N$ is not infinite cyclic. Hence, all non-identity elements of $N$ have infinitely many conjugates in $N$. 


and, \emph{a fortiori}, in $G$; in particular, $N \cap G_f = 1$. It follows that $G_f$ embeds in $G/N$ and hence $G_f$ is a finite group. As such, $G_f$ is contained in the subset of torsion elements of $G$ and hence $G_f = S$, in view of (ii) above. Since the free group $N$ embeds as a subgroup of finite index in $G/G_f$, we may invoke [6, Chapter IV, Theorem 1.6] once again, in order to conclude that there is a tree $T$ on which $G/G_f$ acts without inversions, in such a way that the vertex stabilizer subgroups are finite (and have orders bounded by some integer). On the other hand, since $G_f$ coincides with the subset of torsion elements of $G$, the group $G/G_f$ is easily seen to be torsion-free. It follows that the action of $G/G_f$ on the tree $T$ must be free. Hence, invoking [15, Chapitre I, §3.3], we conclude that the group $G/G_f$ is free.

(v) It is well known that a 2-ended group $G$ admits a surjective homomorphism with finite kernel onto the infinite cyclic group $\mathbb{Z}$ or else onto the infinite dihedral group $D_{\infty}$. The latter case cannot occur, since $D_{\infty}$ has infinitely many elements of finite order, whereas the corresponding set for $G$ is finite (in view of (ii) above). Therefore, there is a finite normal subgroup $H$ of $G$ such that $G/H \cong \mathbb{Z}$. It is now clear that $H$ coincides with the set of elements of finite order in $G$ and hence the proof is finished. \hfill \square

Let us now consider the group $G$ which acts without inversions on a tree $X$, in such a way that the subset $S \subseteq G$ consisting of those group elements that stabilize a vertex is finite. Then, it follows from Proposition 4.1 that $S$ is a finite normal subgroup of $G$, whereas the quotient group $\overline{G} = G/S$ is free. In view of the finiteness of $S$, the quotient map $G \to \overline{G}$ induces an algebra homomorphism

$$\pi_0 : C G \to C \overline{G},$$

which can be extended to a *-algebra homomorphism

$$\pi : C^* G \to C^* \overline{G}.$$ 

We note that the trace $\tau_S$ on $C^* G$, which was defined in the beginning of this section, coincides with the composition

$$C^* G \xrightarrow{\pi} C^* \overline{G} \xrightarrow{\overline{\tau}} \mathbb{C},$$

where $\overline{\tau}$ is the canonical trace on $C^* \overline{G}$. In order to verify this latter assertion, it suffices (by continuity) to show that the trace $\tau_S$ on $C G$, which was defined in the beginning of §3, coincides with the composition

$$C G \xrightarrow{\pi_0} C \overline{G} \xrightarrow{\overline{\tau}} \mathbb{C},$$

where $\overline{\tau}$ is the linear trace on $C \overline{G}$, which maps $\bar{1} \in \overline{G}$ onto 1 and any element $\bar{g} \in \overline{G} \setminus \{1\}$ onto 0. But this is clear, in view of the definitions. Invoking now
Remark 1.1 (ii), we conclude that the additive map

$$(\tau_S)_*: K_0(C^*_r G) \to C, \quad (4)$$

which is induced by the trace $\tau_S$ on $C^*_r G$, coincides with the composition

$$K_0(C^*_r G) \xrightarrow{K_0(\pi)} K_0(C^*_r \bar{G}) \xrightarrow{\tau_*} C,$$

where $\tau_*$ is the additive map induced by the canonical trace $\tau$ on $C^*_r \bar{G}$. Therefore, the group $\bar{G}$ being free, we may invoke Connes’ result [5, §IV.5] that free groups satisfy the integrality of the trace conjecture, in order to conclude that the image of the additive map (4) is the group $\mathbb{Z}$ of integers.

References


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