Transcendental submanifolds of projective space

Wojciech Kucharz

Abstract. Given integers $m$ and $c$ satisfying $m - 2 \geq c \geq 2$, we explicitly construct a nonsingular $m$-dimensional algebraic subset of $\mathbb{P}^{m+c}(\mathbb{R})$ that is not isotopic to the set of real points of any nonsingular complex algebraic subset of $\mathbb{P}^{m+c}(\mathbb{C})$ defined over $\mathbb{R}$. The first examples of this type were obtained by Akbulut and King in a more complicated and nonconstructive way, and only for certain large integers $m$ and $c$.

Mathematics Subject Classification (2000). 57R55,14P25.

Keywords. Smooth manifold, algebraic set, isotopy.

1. Introduction

Denote by $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ real and complex projective $n$-spaces. We regard $\mathbb{P}^n(\mathbb{R})$ as a subset of $\mathbb{P}^n(\mathbb{C})$. A smooth (of class $C^\infty$) submanifold $M$ of $\mathbb{P}^n(\mathbb{R})$ is said to be of algebraic type if it is isotopic in $\mathbb{P}^n(\mathbb{R})$ to the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over $\mathbb{R}$; otherwise $M$ is said to be transcendental. It is not at all obvious that transcendental submanifolds exist. However, Akbulut and King [2] proved the existence of transcendental submanifolds $M$ of $\mathbb{P}^n(\mathbb{R})$ which can even be realized as nonsingular algebraic subsets of $\mathbb{P}^n(\mathbb{R})$. Their examples are obtained in a nonconstructive way, by a method which requires both $m = \dim M$ and $n - m$ to be large integers satisfying $2m - n \geq 2$. In the present paper we explicitly construct such examples, assuming only $n - m \geq 2$ and $2m - n \geq 2$. Moreover, we verify that $M$ is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$ using only the Barth–Larsen theorem [6, Corollary 6.5] and completely avoiding all results of [1], [2]. More precisely, denote by $S^k$ the unit $k$-sphere,

$$S^k = \{(y_1, \ldots, y_{k+1}) \in \mathbb{R}^{k+1} \mid y_1^2 + \cdots + y_{k+1}^2 = 1\}.$$ 

In Section 3 we prove the following:

*The paper was completed at the Max-Planck-Institut für Mathematik in Bonn, whose support and hospitality are gratefully acknowledged.
Theorem 1.1. Let \( m \) and \( n \) be positive integers satisfying \( n - m \geq 2 \) and \( 2m - n \geq 2 \). Let 

\[
\varphi : \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \to \mathbb{P}^n(\mathbb{R})
\]

be defined by 

\[
\varphi((x_1 : x_2 : x_3), (y_1, \ldots, y_{m-1})) = (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : \sigma y_1 : \ldots : \sigma y_{m-1} : 0 : \ldots : 0),
\]

where 0 is repeated \( n - m - 2 \) times and \( \sigma = x_1^2 + 2x_2^2 + 3x_3^2 \). Then:

(i) The image \( M = \varphi(\mathbb{P}^2(\mathbb{R}) \times S^{m-2}) \) is an \( m \)-dimensional nonsingular algebraic subset of \( \mathbb{P}^n(\mathbb{R}) \).

(ii) \( \varphi : \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \to M \) is a biregular isomorphism.

(iii) \( M \) is a transcendental submanifold of \( \mathbb{P}^n(\mathbb{R}) \).

It follows directly from Theorem 1.1 that for any integers \( m \) and \( c \) satisfying \( m - 2 \geq c \geq 2 \), there is a nonsingular algebraic set \( M \) in \( \mathbb{P}^{m+c}(\mathbb{R}) \) such that \( \dim M = m \) and \( M \) is a transcendental submanifold. In particular, there are transcendental submanifolds of arbitrary dimension \( m \geq 4 \). The existence of transcendental submanifolds of dimension 2 or 3 remains unsettled at this time. There are no transcendental submanifolds of dimension 1 or of codimension 1. The last assertion is a special case of the following well known fact.

Remark 1.2. Let \( M \) be a smooth \( m \)-dimensional submanifold of \( \mathbb{P}^n(\mathbb{R}) \). If either \( n - m = 1 \) or \( 2m + 1 \leq n \), then there exists a smooth embedding \( e : M \to \mathbb{P}^n(\mathbb{R}) \), arbitrarily close in the \( \mathcal{C}^\infty \) topology to the inclusion map \( M \hookrightarrow \mathbb{P}^n(\mathbb{R}) \), such that \( e(M) \) is the set of real points of a nonsingular complex algebraic subset of \( \mathbb{P}^n(\mathbb{C}) \) defined over \( \mathbb{R} \).

If \( n - m = 1 \), the claim is explicitly established for example in [3, Theorem 7.1].

For the second case, consider \( \mathbb{P}^n(\mathbb{R}) \) as a subset of \( \mathbb{P}^k(\mathbb{R}) \), where \( k \) is a large integer. By [8], there exists a smooth embedding \( j : M \to \mathbb{P}^k(\mathbb{R}) \), arbitrarily close in the \( \mathcal{C}^\infty \) topology to the inclusion map \( M \hookrightarrow \mathbb{P}^k(\mathbb{R}) \), such that \( j(M) \) is a nonsingular algebraic subset of \( \mathbb{P}^k(\mathbb{R}) \). Increasing \( k \) if necessary and making use of Hironaka’s resolution of singularities theorem [7], we may assume that the Zariski complex closure of \( j(M) \) in \( \mathbb{P}^k(\mathbb{C}) \) is nonsingular. If \( 2m + 1 \leq n \), we obtain an embedding \( e : M \to \mathbb{P}^n(\mathbb{R}) \) with the required properties by composing \( j \) with an appropriate generic projection onto \( \mathbb{P}^n(\mathbb{R}) \).
2. A criterion for transcendence

First we need some results related to the Picard group. Following the current custom, we state them in the language of schemes.

Let $V$ be a smooth projective scheme over $\mathbb{R}$. Assume that the set $V(\mathbb{R})$ of $\mathbb{R}$-rational points of $V$ is nonempty. We regard $V(\mathbb{R})$ as a compact smooth manifold. Every invertible sheaf $\mathcal{L}$ on $V$ determines a real line bundle on $V(\mathbb{R})$, denoted $\mathcal{L}(\mathbb{R})$. The correspondence which assigns to each invertible sheaf $\mathcal{L}$ on $V$ the first Stiefel–Whitney class $w_1(\mathcal{L}(\mathbb{R}))$ of $\mathcal{L}(\mathbb{R})$ gives rise to a canonical homomorphism

$$w_1: \text{Pic}(V) \to H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

defined on the Picard group $\text{Pic}(V)$ of isomorphism classes of invertible sheaves on $V$. We set

$$H^1_{\text{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = w_1(\text{Pic}(V)).$$

It will be convenient to recall another description of $\text{Pic}(V)$. Consider the scheme $V_C = V \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ and its Picard group $\text{Pic}(V_C)$. The Galois group $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ of $\mathbb{C}$ over $\mathbb{R}$ acts on $\text{Pic}(V_C)$. We denote by $\text{Pic}(V_C)^G$ the subgroup of $\text{Pic}(V_C)$ consisting of the elements fixed by $G$. Given an invertible sheaf $\mathcal{L}$ on $V$, we write $\mathcal{L}_C$ for the corresponding sheaf on $V_C$. The correspondence $\mathcal{L} \to \mathcal{L}_C$ defines a canonical group homomorphism

$$\alpha: \text{Pic}(V) \to \text{Pic}(V_C)^G.$$

It follows from the general theory of descent [4] that $\alpha$ is an isomorphism (a simple treatment of the case under consideration can also be found in [5]).

As usual, we set $\mathbb{P}^n_{\mathbb{R}} = \text{Proj}(\mathbb{R}[T_0, \ldots, T_n])$ and identify $\mathbb{P}^n_{\mathbb{R}}(\mathbb{R})$ with $\mathbb{P}^n(\mathbb{R})$. Thus if $V$ is a subscheme of $\mathbb{P}^n_{\mathbb{R}}$, then $V(\mathbb{R})$ is a subset of $\mathbb{P}^n(\mathbb{R})$.

**Proposition 2.1.** Let $V$ be a closed smooth $m$-dimensional subscheme of $\mathbb{P}^n_{\mathbb{R}}$. If $2m - n \geq 2$, then

$$H^1_{\text{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)),$$

where $i: V(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map.

**Proof.** Let $j: V \hookrightarrow \mathbb{P}^n_{\mathbb{R}}$ and $j_C: V_C \hookrightarrow \mathbb{P}^n_C = \mathbb{P}^n_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ be the inclusion morphisms. By the Barth–Larsen theorem [6, Corollary 6.5], the induced homomorphism

$$j_C^*: \text{Pic}(\mathbb{P}^n_C) \to \text{Pic}(V_C)$$

is an isomorphism. Since $j_C^*$ is $G$-equivariant, the restriction

$$j_C^*: \text{Pic}(\mathbb{P}^n_C)^G \to \text{Pic}(V_C)^G$$

is also an isomorphism. \qed
is an isomorphism. We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Pic}(\mathbb{P}_g^n)^G & \xrightarrow{j_*^c} & \text{Pic}(V_c)^G \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\text{Pic}(\mathbb{P}_R^n) & \xrightarrow{j_*^*} & \text{Pic}(V) \\
\downarrow{\omega_1} & & \downarrow{\omega_1} \\
H^1(\mathbb{P}_R^n, \mathbb{Z}/2) & \xrightarrow{i_*^*} & H^1(V(R), \mathbb{Z}/2).
\end{array}
\]

Since the homomorphisms \(\alpha\) are isomorphisms and

\[H^1(\mathbb{P}_R^n, \mathbb{Z}/2) = H^1_{\text{alg}}(\mathbb{P}_R^n, \mathbb{Z}/2),\]

it follows that

\[H^1_{\text{alg}}(V(R), \mathbb{Z}/2) = i_*^*(H^1(\mathbb{P}_R^n, \mathbb{Z}/2)),\]

as required. \(\square\)

Note that a smooth submanifold of \(\mathbb{P}_R^n\) is of algebraic type if and only if it is isotopic in \(\mathbb{P}_R^n\) to \(V(\mathbb{R})\) for some closed smooth subscheme \(V\) of \(\mathbb{P}_R^n\). Hence Proposition 2.1 yields the following criterion for transcendence.

**Proposition 2.2.** Let \(M\) be a compact smooth \(m\)-dimensional submanifold of \(\mathbb{P}_R^n\). Assume that the inclusion map \(i: M \hookrightarrow \mathbb{P}_R^n\) induces a trivial homomorphism

\[e^*: H^1(\mathbb{P}_R^n, \mathbb{Z}/2) \longrightarrow H^1(M, \mathbb{Z}/2),\]

that is, \(e^* = 0\). If \(M\) is nonorientable and \(2m - n \geq 2\), then \(M\) is a transcendental submanifold of \(\mathbb{P}_R^n\).

**Proof.** Suppose to the contrary that \(M\) is of algebraic type. Let \(V\) be a closed smooth subscheme of \(\mathbb{P}_R^n\) with \(V(\mathbb{R})\) isotopic to \(M\) in \(\mathbb{P}_R^n\). Then the homomorphism

\[i_*^*: H^1(\mathbb{P}_R^n, \mathbb{Z}/2) \longrightarrow H^1(V(\mathbb{R}), \mathbb{Z}/2),\]

induced by the inclusion map \(i: V(\mathbb{R}) \hookrightarrow \mathbb{P}_R^n\), is trivial. Since \(\dim V = m\) and \(2m - n \geq 2\), Proposition 2.1 implies

\[H^1_{\text{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = 0.\]

On the other hand, the first Stiefel–Whitney class \(w_1(V(\mathbb{R}))\) of \(V(\mathbb{R})\) is nonzero, \(V(\mathbb{R})\) being a nonorientable manifold. Moreover, \(w_1(V(\mathbb{R})) = w_1(\mathcal{K}(\mathbb{R}))\), where \(\mathcal{K}\) is the canonical invertible sheaf of \(V\), and hence, \(w_1(V(\mathbb{R}))\) is in \(H^1_{\text{alg}}(V(\mathbb{R}), \mathbb{Z}/2)\). In view of this contradiction, the proof is complete. \(\square\)
3. Transcendental submanifolds

We begin with some preliminary observations. Identify \( \mathbb{R}^n \) with its image under the map
\[ \mathbb{R}^n \rightarrow \mathbb{P}^n(\mathbb{R}), \quad (x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n); \]
thus \( \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R}) \). An algebraic subset \( X \) of \( \mathbb{R}^n \) is said to be \textit{projectively closed} if \( X \) is also an algebraic subset of \( \mathbb{P}^n(\mathbb{R}) \). One readily checks that \( X \) is projectively closed if and only if it can be defined by a real polynomial equation
\[ f(x_1, \ldots, x_n) = 0, \]
where the homogeneous form of top degree in \( f \) vanishes only at 0 in \( \mathbb{R}^n \).

**Lemma 3.1.** Let \( X \) be an algebraic subset of \( \mathbb{R}^k \) contained in the open half-space
\[ H = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k > 0\}. \]
Then the map \( \psi : X \times S^\ell \rightarrow \mathbb{R}^{k+\ell} \) defined by
\[ \psi((x_1, \ldots, x_k), (y_1, \ldots, y_{\ell+1})) = (x_1, \ldots, x_{k-1}, x_k y_1, \ldots, x_k y_{\ell+1}) \]
is an algebraic embedding, that is, the image \( Y = \psi(X \times S^\ell) \) is an algebraic subset of \( \mathbb{R}^{k+\ell} \), and \( \psi : X \times S^\ell \rightarrow Y \) is a birational isomorphism. Moreover, if \( X \) is projectively closed in \( \mathbb{R}^k \), then \( Y \) is projectively closed in \( \mathbb{R}^{k+\ell} \).

**Proof.** Let
\[ f(u, v) = 0 \]
be a real polynomial equation defining \( X \), where \( u = (x_1, \ldots, x_{k-1}) \) and \( v = x_k \). Since
\[ X \subset H, \tag{1} \]
the subset \( Y \) of \( \mathbb{R}^{k+\ell} \) is defined by the equation
\[ f(u, \rho) = 0, \tag{2} \]
where
\[ \rho = (x_k^2 + x_{k+1}^2 + \cdots + x_{k+\ell}^2)^{\frac{1}{2}}. \]
We will now show that (2) can be replaced by a polynomial equation in \( x_1, \ldots, x_{k-1}, x_k, \ldots, x_{k+\ell} \). To this end we write
\[ f(u, v) = g(u, v^2) + vh(u, v^2), \tag{3} \]
where \( g \) and \( h \) are real polynomials in \((u, v)\). Then (2) is equivalent to
\[
g(u, \rho^2) + \rho h(u, \rho^2) = 0, \tag{4}
\]
and in view of (1) also to
\[
(g(u, \rho^2))^2 - \rho^2(h(u, \rho^2))^2 = 0, \tag{5}
\]
which is a polynomial equation, as required. Consequently, \( Y \) is an algebraic subset of \( \mathbb{R}^k+\ell \).

It is clear that \( \psi \) is injective and \( \theta : Y \to X \),
\[
\theta(x_1, \ldots, x_{k-1}, x_k, \ldots, x_{k+\ell}) = \left( x_1, \ldots, x_{k-1}, \frac{x_k}{\rho}, \ldots, \frac{x_{k+\ell}}{\rho} \right),
\]
is the inverse of \( \psi : X \to Y \). By (4),
\[
\rho = -\frac{g(x_1, \ldots, x_{k-1}, x_k^2 + \cdots + x_{k+\ell}^2)}{h(x_1, \ldots, x_{k-1}, x_k^2 + \cdots + x_{k+\ell}^2)},
\]
for \((x_1, \ldots, x_{k-1}, x_k, \ldots, x_{k+\ell})\) in \( Y \), and hence \( \theta \) is a regular map. Thus \( \psi : X \to Y \) is a biregular isomorphism.

Assume now that \( X \) is projectively closed in \( \mathbb{R}^k \). We may also assume that the homogeneous form of top degree in \( f \), denoted \( F \), vanishes only at \( 0 \) in \( \mathbb{R}^k \). Note that \( F(u, \rho^2)F(u, -\rho^2) \) is the homogeneous form of top degree in equation (5). This form vanishes only at \( 0 \) in \( \mathbb{R}^{k+\ell} \), and hence \( Y \) is projectively closed in \( \mathbb{R}^{k+\ell} \).

**Lemma 3.2.** The map \( g : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^4(\mathbb{C}) \),
\[
g((x_1 : x_2 : x_3)) = (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : x_1^2 + 2x_2^2 + 3x_3^2),
\]
is an algebraic embedding. In particular, the restriction \( f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^4(\mathbb{R}) \) of \( g \) is an algebraic embedding.

**Proof.** One readily checks that \( g \) is injective. Moreover, the (complex) differential of \( g \) at each point of \( \mathbb{P}^2(\mathbb{C}) \) is of rank 2. It follows that \( g \) is an algebraic embedding, and hence \( f \) is an algebraic embedding. \( \square \)

**Proof of Theorem 1.1.** Let \( f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^4(\mathbb{R}) \) be the algebraic embedding of Lemma 3.2. Note that the image \( X = f(\mathbb{P}^2(\mathbb{R})) \) is a projectively closed algebraic subset of \( \mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R}) \), contained in the open half-space
\[
\{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \mid u_4 > 0\}.
\]
Let
\[
\psi : X \times S^{m-2} \to \mathbb{R}^{4+(m-2)} = \mathbb{R}^{m+2} \subset \mathbb{P}^{m+2}(\mathbb{R})
\]
be the algebraic embedding of Lemma 3.1 (with $k = 4$ and $\ell = m - 2$). Note that $\psi(X \times S^{m-2})$ is projectively closed in $\mathbb{R}^{m+2}$, and hence is an algebraic subset of $\mathbb{P}^{m+2}(\mathbb{R})$.

Clearly, if $i: S^{m-2} \to S^{m-2}$ is the identity map, then

$$f \times i: \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \to X \times S^{m-2}$$

is a biregular isomorphism. Denoting by $j: \mathbb{P}^{m+2}(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ the standard embedding,

$$j((v_0 : \ldots : v_{m+2})) = (v_0 : \ldots : v_{m+2} : 0 : \ldots : 0),$$

we obtain

$$\varphi = j \circ \psi \circ (f \times i),$$

which implies that $\varphi$ is an algebraic embedding. In other words, conditions (i) and (ii) are satisfied. Moreover, $M \subset \mathbb{R}^n \subseteq \mathbb{P}^n(\mathbb{R})$. Since $M$ is nonorientable and $2m - n \geq 2$, condition (iii) follows from Proposition 2.2. □

References


Received February 28, 2007

Wojciech Kucharz, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131-0001, U.S.A.

E-mail: kucharz@math.unm.edu